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The Generalized Binomial Transform of the Bivariate Fibonacci and Lucas p-Polynomials

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Abstract

The generalized binomial transforms of the bivariate Fibonacci $p-$polynomials and Lucas $p-$polynomials are introduced in this study. Furthermore, the generating functions of these polynomials are provided. Moreover, some relations are found for them. All results obtained are reduced to the $k-$binomial, falling binomial, rising binomial, and binomial transforms of the Pell, Pell-Lucas, Jacobsthal, Jacobsthal-Lucas, Fibonacci, and Lucas numbers.

Keywords: *Binomial transform, Bivariate Fibonacci $p-$polynomials, Bivariate Lucas $p-$polynomials, Generating function*

AMS Subject Classification (2020): *11B37; 11B39; 11B83; 05A15*

1. Introduction

Integer sequences play a crucial role in many fields, including mathematics. Furthermore, several special integer sequences have been studied in recent years. Especially, Fibonacci and Lucas sequences are among the number sequences widely studied. Their properties are provided in many papers [1, 2].

The generalization of these number sequences is another one of the most studied topics by researchers. Catalini introduced the generalization of bivariate Fibonacci and Lucas polynomials in [3]. Moreover, the bivariate Fibonacci and Lucas $p-$ polynomials are considered, and some properties of them are investigated in [4].

The bivariate Fibonacci $p-$polynomials are defined by the following recurrence relation:

$$F_{p,n}(x,y) = xF_{p,n-1}(x,y) + yF_{p,n-p-1}(x,y) \tag{1.1}$$

where $n > p$. In addition, the initial values are

$$F_{p,0}(x,y) = 0, F_{p,1}(x,y) = 1, F_{p,2}(x,y) = x, F_{p,3}(x,y) = x^2, \dots, F_{p,p}(x,y) = x^{p-1}.$$

Similarly, the bivariate Lucas $p-$polynomials are presented by the following recurrence relation:

$$L_{p,n}(x,y) = xL_{p,n-1}(x,y) + yL_{p,n-p-1}(x,y) \tag{1.2}$$

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where $n > p$. Furthermore, the initial values are

$$L_{p,0}(x,y) = p+1, L_{p,1}(x,y) = x, L_{p,2}(x,y) = x^2, L_{p,3}(x,y) = x^3, \ldots, L_{p,p}(x,y) = x^p.$$

The generating functions of these polynomials are as follows:

$$g_F(t) = \sum_{i=0}^{\infty} F_{p,i}(x,y)t^i = \frac{t}{1-xt-yt^{p+1}} \tag{1.3}$$

and

$$g_L(t) = \sum_{i=0}^{\infty} L_{p,i}(x,y)t^i = \frac{1+p(1-xt)}{1-xt-yt^{p+1}} \tag{1.4}$$

respectively. The relation between them is

$$L_{p,n}(x,y) = F_{p,n+1}(x,y) + pyF_{p,n-p}(x,y)$$

in [4]. Detailed information on Fibonacci numbers, Lucas numbers, and their generalizations can be found in [1–5].

Another topic of study in mathematics is the binomial transformation; using the binomial coefficient, a new sequence can be obtained from a sequence. An integer sequence $\{a_n\}$ has the following binomial transformation.

$$b_n = \sum_{i=0}^{n} \binom{n}{i} a_i$$

in [6]. In addition, a generalization of the binomial transform is

$$b_n = \sum_{i=0}^{n} \binom{n}{i} u^{n-i} s^i a_i \tag{1.5}$$

where u and s are complex numbers. Let $f(t)$ and $h(t)$ be the generating functions of the sequences $\{a_n\}$ and $\{b_n\}$. Then, the generating function of the sequence $\{b_n\}$ is

$$h(t) = \left(\frac{1}{1-ut}\right) f\left(\frac{st}{1-ut}\right) \tag{1.6}$$

in [7]. The authors have considered special cases of the equation (1.5), which are the $k-$binomial transform, the rising $k-$binomial transform, and the falling $k-$binomial transform in [8]. The binomial transform and its properties are provided in [7, 9, 10].

Many authors have applied binomial transforms to special integer sequences. In addition, they have given numerous properties of them. In [11, 12], the authors defined the binomial transforms of the generalized $k-$Fibonacci and $k-$Fibonacci numbers and provided some identities. The falling binomial, rising binomial, $k-$binomial and binomial transforms of the modified $k-$Fibonacci-like sequence are considered in [13]. In addition, the binomial transform of the balancing polynomials is investigated, and the generating functions and summation formulas are given in [14]. Detailed information on the binomial transforms of special number sequences can be found in [15–17].

Motivated by the above papers, we apply the generalized binomial transform to the bivariate Fibonacci $p-$polynomials and Lucas $p-$polynomials. We obtain a new polynomial sequence from bivariate Fibonacci $p-$polynomials using the generalized binomial transform in Section 2. It is reduced to the falling, rising, $k-$binomial, and binomial transforms of the Jacobsthal, Pell, and Fibonacci numbers. In addition, we provide the generating function and some identities for this new polynomial sequence. In Section 3, we introduce the generalized binomial transform of the bivariate Lucas $p-$polynomials, which are reduced to the falling, rising, $k-$binomial and binomial transforms of Jacobsthal-Lucas, Pell-Lucas, and Lucas numbers. Furthermore, we find the generating function and some identities of them.

2. Main results

We utilize the generalized binomial transform on bivariate Fibonacci $p-$polynomials in this part of the study. Moreover, we present some results for this transform.

Definition 2.1. Let $w_{p,n}(x,y)$ be the generalized binomial transform of the bivariate Fibonacci $p-$polynomial. The definition is as follows:

$$w_{p,n}(x,y)=\sum_{i=0}^{n}\binom{n}{i}u^{n-i}s^{i}F_{p,i}(x,y) \tag{2.1}$$

for $u\neq 0$, $s\neq 0$ or $n\neq 0$ and $w_{p,0}(x,y)=0$.

In the following table, we can see some special cases of the generalized binomial transform of the bivariate Fibonacci $p-$polynomials.

Table 1. Special cases of $w_{p,n}(x,y)$

u	s	p	x	y	Special cases
1	1	1	x	y	The binomial transform of bivariate Fibonacci polynomials
1	1	p	x	1	The binomial transform of Fibonacci $p-$polynomials
1	1	1	x	1	The binomial transform of Fibonacci polynomials
1	1	p	1	1	The binomial transform of Fibonacci $p-$numbers
k	1	1	1	1	The falling binomial transform of Fibonacci numbers
1	k	1	1	1	The rising binomial transform of Fibonacci numbers
1	1	1	1	1	The binomial transform of Fibonacci numbers
1	1	1	$2x$	y	The binomial transform of bivariate Pell polynomials
1	1	p	$2x$	1	The binomial transform of Pell $p-$polynomials
1	1	1	$2x$	1	The binomial transform of Pell polynomials
1	1	p	2	1	The binomial transform of Pell $p-$numbers
k	1	1	2	1	The falling binomial transform of Pell numbers
1	k	1	2	1	The rising binomial transform of Pell numbers
1	1	1	2	1	The binomial transform of Pell numbers
1	1	1	x	$2y$	The binomial transform of bivariate Jacobsthal polynomials
1	1	p	1	$2y$	The binomial transform of Jacobsthal $p-$polynomials
1	1	1	1	$2y$	The binomial transform of Jacobsthal polynomials
1	1	p	1	2	The binomial transform of Jacobsthal $p-$numbers
k	1	1	1	2	The falling binomial transform of Jacobsthal numbers
1	k	1	1	2	The rising binomial transform of Jacobsthal numbers
1	1	1	1	2	The binomial transform of Jacobsthal numbers

Proposition 2.1. *The generating function for the generalized binomial transform of the bivariate Fibonacci $p-$polynomials is derived as follows:*

$$\sum_{n=0}^{\infty}w_{p,n}(x,y)t^{n}=\frac{st}{(ut-1)\left(ut+sxt+syt\left(\frac{st}{1-ut}\right)^{p}-1\right)}.$$

Proof. By using (1.6), we find

$$\sum_{n=0}^{\infty}w_{p,n}(x,y)t^{n}=\left(\frac{1}{1-ut}\right)g_{F}\left(\frac{st}{1-ut}\right).$$

The result is obtained by the equation (1.3). □

Proposition 2.2. *For $n\geq 0$, the generalized binomial transform of the bivariate Fibonacci $p-$polynomials verifies the subsequent relationship:*

$$w_{p,n+1}(x,y)=\sum_{i=0}^{n}\binom{n}{i}u^{n-i}s^{i}\left(uF_{p,i}(x,y)+sF_{p,i+1}(x,y)\right).$$

Proof. Using (2.1), we have

$$w_{p,n+1}(x,y)=\sum_{i=0}^{n+1}\binom{n+1}{i}u^{n+1-i}s^{i}F_{p,i}(x,y).$$

From the following binomial equality

$$\binom{n+1}{i} = \binom{n}{i} + \binom{n}{i-1} \tag{2.2}$$

we find

$$w_{p,n+1}(x,y) = \sum_{i=1}^{n+1} \left(\binom{n}{i} + \binom{n}{i-1} \right) u^{n+1-i} s^i F_{p,i}(x,y).$$

Then

$$w_{p,n+1}(x,y) = u \sum_{i=0}^{n} \binom{n}{i} u^{n-i} s^i F_{p,i}(x,y) + s \sum_{i=0}^{n} \binom{n}{i} u^{n-i} s^i F_{p,i+1}(x,y).$$

The result can be obtained from here. □

Corollary 2.1. *n is a nonnegative integer. The generalized binomial transform of the bivariate Fibonacci $p-$polynomials satisfies the equation below:*

$$w_{p,n+1}(x,y) - u w_{p,n}(x,y) = s \sum_{i=0}^{n} \binom{n}{i} u^{n-i} s^i F_{p,i+1}(x,y). \tag{2.3}$$

Proposition 2.3. *The given equality is satisfied for $n \geq 0$.*

$$w_{p,n+1}(x,y) - (u+sx) w_{p,n}(x,y) - s u^n = sy \sum_{i=1}^{n} \binom{n}{i} u^{n-i} s^i F_{p,i-p}(x,y) \tag{2.4}$$

where $w_{p,n}(x,y)$ denotes the generalized binomial transform of the bivariate Fibonacci $p-$polynomials.

Proof. Using the equality (2.2), we obtain

$$w_{p,n+1}(x,y) = \sum_{i=1}^{n} \binom{n}{i} u^{n-i} s^i \left(u F_{p,i}(x,y) + s F_{p,i+1}(x,y) \right) + s u^n.$$

From (1.1), we determine

$$w_{p,n+1}(x,y) = \sum_{i=1}^{n} \binom{n}{i} u^{n-i} s^i \left((u+sx) F_{p,i}(x,y) + sy F_{p,i-p}(x,y) \right) + s u^n.$$

When the last equation is adjusted, the result is found. □

This part of the study focuses on the generalized binomial transform of the bivariate Lucas $p-$polynomials. We also present some results for this new polynomial sequence.

Definition 2.2. Let $W_{p,n}(x,y)$ represent the generalized binomial transform of the bivariate Lucas $p-$polynomial. The definition is as follows:

$$W_{p,n}(x,y) = \sum_{i=0}^{n} \binom{n}{i} u^{n-i} s^i L_{p,i}(x,y) \tag{2.5}$$

for $u \neq 0$, $s \neq 0$ or $n \neq 0$ and $W_{p,0}(x,y) = p+1$.

In the table below, we provide some particular cases of the generalized binomial transform for the bivariate Lucas $p-$polynomials.

Table 2. Special cases of $W_{p,n}(x,y)$

u	s	p	x	y	Special cases
1	1	1	x	y	The binomial transform of bivariate Lucas polynomials
1	1	p	x	1	The binomial transform of Lucas $p-$polynomials
1	1	1	x	1	The binomial transform of Lucas polynomials
1	1	p	1	1	The binomial transform of Lucas $p-$numbers
k	1	1	1	1	The falling binomial transform of Lucas numbers
1	k	1	1	1	The rising binomial transform of Lucas numbers
1	1	1	1	1	The binomial transform of Lucas numbers
1	1	1	$2x$	y	The binomial transform of bivariate Pell-Lucas polynomials
1	1	p	$2x$	1	The binomial transform of Pell-Lucas $p-$polynomials
1	1	1	$2x$	1	The binomial transform of Pell-Lucas polynomials
1	1	p	2	1	The binomial transform of Pell-Lucas $p-$numbers
k	1	1	2	1	The falling binomial transform of Pell-Lucas numbers
1	k	1	2	1	The rising binomial transform of Pell-Lucas numbers
1	1	1	2	1	The binomial transform of Pell-Lucas numbers
1	1	1	x	$2y$	The binomial transform of bivariate Jacobsthal-Lucas polynomials
1	1	p	1	$2y$	The binomial transform of Jacobsthal-Lucas $p-$polynomials
1	1	1	1	$2y$	The binomial transform of Jacobsthal-Lucas polynomials
1	1	p	1	2	The binomial transform of Jacobsthal-Lucas $p-$numbers
k	1	1	1	2	The falling binomial transform of Jacobsthal-Lucas numbers
1	k	1	1	2	The rising binomial transform of Jacobsthal-Lucas numbers
1	1	1	1	2	The binomial transform of Jacobsthal-Lucas numbers

Proposition 2.4. *The generating function for the generalized binomial transform of $L_{p,n}(x,y)$ is*

$$\sum_{n=0}^{\infty} W_{p,n}(x,y)t^n = \frac{(u+pu+psx)t-p-1}{(1-ut)\left(ut+sxt+syt\left(\frac{st}{1-ut}\right)^p-1\right)}.$$

Proof. Applying (1.6), it follows that

$$\sum_{n=0}^{\infty} W_{p,n}(x,y)t^n = \left(\frac{1}{1-ut}\right) g_L\left(\frac{st}{1-ut}\right).$$

The result is determined using the equation (1.4). □

Proposition 2.5. *The subsequent relation holds for $n \geq 0$.*

$$W_{p,n+1}(x,y) = \sum_{i=0}^{n} \binom{n}{i} u^{n-i}s^i \left(uL_{p,i}(x,y)+sL_{p,i+1}(x,y)\right).$$

Proof. Based on (2.5), we have

$$W_{p,n+1}(x,y) = \sum_{i=0}^{n+1} \binom{n+1}{i} u^{n+1-i}s^i L_{p,i}(x,y).$$

By using (2.2), we find

$$W_{p,n+1}(x,y) = u^{n+1}(p+1) + \sum_{i=1}^{n+1} \left(\binom{n}{i}+\binom{n}{i-1}\right) u^{n+1-i}s^i L_{p,i}(x,y).$$

Thus

$$W_{p,n+1}(x,y) = u\sum_{i=0}^{n} \binom{n}{i} u^{n-i}s^i L_{p,i}(x,y) + s\sum_{i=0}^{n} \binom{n}{i} u^{n-i}s^i L_{p,i+1}(x,y).$$

The result can be derived from the last step. □

Corollary 2.2. *For $n \geq 0$, the following relationship of the generalized binomial transform of the bivariate Lucas $p-$polynomials holds.*

$$W_{p,n+1}(x,y) - uW_{p,n}(x,y) = s\sum_{i=0}^{n}\binom{n}{i}u^{n-i}s^{i}L_{p,i+1}(x,y).$$

Proposition 2.6. *n is a nonnegative integer and $W_{p,n}(x,y)$ denotes the generalized binomial transform of the bivariate Lucas $p-$polynomials. In that case, we obtain*

$$W_{p,n+1}(x,y) - (u+sx)W_{p,n}(x,y) + xpsu^{n} = sy\sum_{i=1}^{n}\binom{n}{i}u^{n-i}s^{i}L_{p,i-p}(x,y).$$

Proof. Considering (2.5), we get

$$W_{p,n+1}(x,y) = \sum_{i=1}^{n}\binom{n}{i}u^{n-i}s^{i}\left(uL_{p,i}(x,y) + sL_{p,i+1}(x,y)\right) + u^{n}(u(p+1)+sx).$$

From (1.2), we have

$$\begin{aligned} W_{p,n+1}(x,y) =& \sum_{i=1}^{n}\binom{n}{i}u^{n-i}s^{i}\left((u+sx)L_{p,i}(x,y) + syL_{p,i-p}(x,y)\right) \\ &+ u^{n}(u(p+1)+sx). \end{aligned}$$

When the last equation is adjusted, the result is found. □

The relation between $w_{p,n}(x,y)$ and $W_{p,n}(x,y)$ is given in the following proposition.

Proposition 2.7. *Let $w_{p,n}(x,y)$ and $W_{p,n}(x,y)$ denote the generalized binomial transforms of the bivariate Fibonacci and Lucas $p-$polynomials. Then, we find*

$$sW_{n}(x,y) = (p+1)w_{n+1}(x,y) - (u+pu+psx)w_{n}(x,y).$$

Proof. If both sides of the equality (2.4) are multiplied by p, we get

$$pw_{p,n+1}(x,y) - p\left((u+sx)w_{p,n}(x,y) + su^{n}\right) = psy\sum_{i=1}^{n}\binom{n}{i}u^{n-i}s^{i}F_{p,i-p}(x,y).$$

From (1), we obtain

$$pw_{p,n+1}(x,y) - p\left((u+sx)w_{p,n}(x,y) + su^{n}\right) = s\sum_{i=1}^{n}\binom{n}{i}u^{n-i}s^{i}(L_{p,i}(x,y) - F_{p,i+1}(x,y)).$$

Hence

$$\begin{aligned} pw_{p,n+1}(x,y) - p(u+sx)w_{p,n}(x,y) =& s\sum_{i=0}^{n}\binom{n}{i}u^{n-i}s^{i}L_{p,i}(x,y) \\ &- s\sum_{i=0}^{n}\binom{n}{i}u^{n-i}s^{i}F_{p,i+1}(x,y). \end{aligned}$$

The result can be derived using the equality (2.3).

□

3. Conclusion

In the paper presented, we focus on applying the generalized binomial transforms to the bivariate Fibonacci and Lucas $p-$polynomials. Moreover, the generating functions and some identities are obtained for these transforms. Note that all results obtained are reduced to the falling, rising, $k-$binomial, and binomial transforms of the Jacobsthal, Jacobsthal-Lucas, Pell, Pell-Lucas, Fibonacci, and Lucas numbers. These particular cases are shown in Table 1 and Table 2. It would be an intriguing study to investigate the Hankel and Catalan transformations of the bivariate Fibonacci and Lucas $p-$polynomials. Additionally, relations among them can be found.

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