Sakarya University Journal of Science, 22 (6), 1628-1630, 2018.

SAKARYA ÜNİVERSİTESİ FEN BİLİMLERİ ENSTİTÜSÜ DERGİSİ
SAKARYA UNIVERSITY JOURNAL OF SCIENCE

e-ISSN: 2147-835X
Dergi sayfası: http://www.saujs.sakarya.edu.tr

Received
16-02-2018
Accepted
26-03-2018

Doi
10.16984/saufenbilder.395615

A Lyapunov Function For Logistic Equation On Time Scales

Veysel Fuat Hatipoğlu*[1]

ABSTRACT

In this study we focus on the stability of dynamic logistic equation which is used in single species population dynamics. Here we have introduced a quadratic Lyapunov function for generalized dynamic logistic equation on time scales. By using proposed Lyapunov function, asymptotic stability conditions for the equilibrium solution of dynamic logistic equation have been investigated.

Keywords: Dynamic logistic equation, Lyapunov Function, Stability, Time Scales

1. INTRODUCTION

Logistic equation (or Verhulst equation) is often used to provide a model for single species population model. In the last century logistic equation has been applied to various branches of science such as ecology, medicine, physics and chemistry. For continuous case logistic equation is the differential equation

$$\frac{dx(t)}{dt} = rx(t)\left(1 - \frac{x(t)}{N}\right), t \geq 0, r > 0, \qquad (1)$$

where $r > 0$ is the intrinsic growth rate, $x(t)$ is the population size at time *t* and $N > 0$ is the carrying capacity of the environment. For discrete case logistic equation is described by the difference equation

$$X_{t+1} = rX_t\left(1 - \frac{X_t}{N}\right), r > 0, \qquad (2)$$

where $N > 0$ is the carrying capacity of the environment.

Unification and extension of continuous and discrete times is possible with time scales calculus. Therefore implementing equations (1) and (2) in a single equation is possible. In this manner logistic equation is considered on time scales by many authors [1] – [5]. Qualitative behavior of both continuous and discrete versions of logistic equation has been studied by several authors [6], [7]. However to the best of our knowledge, few articles are published on the asymptotic behavior of dynamic logistic equation on time scales. In [3] authors give the sufficient conditions for exponential asymptotic stability of a critical point of an almost linear dynamic equation and apply the findings to dynamic logistic equation. Their results show that the zero solution of dynamic logistic equation is unstable on any time scales and the solution of other equilibrium point $x = 1$ is exponentially stable depending on graininess function μ.

* Corresponding Author
[1] Muğla Sıtkı Koçman University, veyselfuat.hatipoglu@mu.edu.tr

2. PRELIMINARIES

A time scale $\mathbb{T}$ is an arbitrary nonempty closed subset of $\mathbb{R}$. For $t \in \mathbb{T}$, forward jump operator $\sigma: \mathbb{T} \to \mathbb{T}$ is defined by

$$\sigma(t) := \inf\{s \in \mathbb{T}: s > t\}$$

The graininess function $\mu: \mathbb{T} \to [0, \infty)$ is defined by

$$\mu(t) = \sigma(t) - t.$$

The function $p: \mathbb{T} \to \mathbb{R}$ is said to be regressive provided $1 + \mu(t)p(t) \neq 0$ for all $t \in \mathbb{T}$. The set of all regressive rd-continuous functions $f: \mathbb{T} \to \mathbb{R}$ is denoted by $\mathcal{R}$. We refer to [1], [2] for detailed literature with the calculus of time scales. [2] gives the generalized logistic equation that is used in population dynamics as

$$y^{\Delta} = \frac{py\left(1-\frac{y}{N}\right)}{1+\frac{\mu p}{N}y} \qquad (3)$$

with $\frac{py}{N} \in \mathcal{R}$, where $N \neq 0$ is the carrying capacity of the population and $p \in \mathcal{R}$ is the growth rate. The solution of (3) is

$$y(t) = \frac{N}{\left(\frac{N}{y_0}-1\right)e_{\ominus p}(t,t_0)+1} \qquad (4)$$

satisfying $y(t_0) = y_0$ where the function $\ominus p$ defined by $(\ominus p)(t) := \frac{p(t)}{1+\mu(t)p(t)}$ for all $t \in \mathbb{T}$. Note that $y = 0$ and $y = N$ are the equilibrium solutions of (3). If we consider time scales as $\mathbb{T} = \mathbb{R}$ and $\mathbb{T} = \mathbb{Z}$ we see that (3) is identical with (1) and (2) respectively.

Definition of Lyapunov functions on time scales is as follows,

Definition 1. [8] A function $V(x): \mathbb{R}^n \to \mathbb{R}$, is called a time scale Lyapunov function for system (3) if

1. $V(x) \geq 0$ with equality if and only if $x = 0$, and
2. $V^{\Delta}(x(t)) \leq 0$.

In the next section we state the Lyapunov stability theorem on time scales and show the asymptotic stability of dynamic logistic equation on time scales.

3. STABILITY RESULTS

Following theorem gives the asymptotic stability conditions by using Lyapunov functions on time scales.

Theorem 2. [4] Given system (3) with equilibrium $x = 0$, if there exists an associated Lyapunov function $V(x)$, then $x = 0$ is Lyapunov stable. Furthermore, if $V^{\Delta}(x(t)) < 0$, then $x = 0$ is asymptotically stable.

Stability of the equilibria $y_1 = 0$ and $y_2 = N$ of equation (3) have been studied by Gard and Hoffacker [3] with $N = 1$. Authors showed that the equilibrium y = 0 of equation (3) is unstable on any time scale. In this section to show the Lyapunov stability of the equilibrium $y = N$ of logistic equation (3), we construct a Lyapunov function candidate in the form

$$V(t, y(t)) = (y - N)^2 \qquad (5)$$

where N is the carrying capacity of the population. Letting $z = 1 - \frac{y}{N}$ we transform equation (3) to

$$z^{\Delta} = \frac{-pz(1-z)}{1+\mu p(1-z)} \qquad (6)$$

where p is the growth rate with $p(1 - z) \in \mathcal{R}$. Also Lyapunov function (5) transforms to

$$V(t, z(t)) = (Nz)^2\ . \qquad (7)$$

We observe the stability of the zero solution $z_0 = 0$ of (6) (the $y_1 = N$ solution of logistic equation (3)). It is obvious that $V(t, z(t)) = (Nz)^2 \geq 0$. To verify $V(t, z(t))$ is a Lyapunov function for (6) we need to show $V^{\Delta}(t, z(t)) \leq 0$ for all $t \in \mathbb{T}$. Taking the $\Delta -$ derivative of (7),

$$V^{\Delta}(t, z(t)) = N^2(z + z^{\sigma})z^{\Delta}.$$

Then we have,

$$V^{\Delta}(t, z(t)) = N^2(2z + \mu(t))z^{\Delta}. \qquad (8)$$

Here we consider (8) in cases as $z < 0$, $0 < z < 1$, and $z = 0$.

Case 1. If $z < 0$ ($y > N$) then we have

$$p(t)z(t)(1 - z(t)) < 0 \qquad (9)$$

and we have,

$$1 + \mu(t)p(t)(1 - z(t)) > 0. \qquad (10)$$

Using the relations (9) and (10) in the equation (6), one can see that $z^{\Delta} > 0$ with $z < 0$ for all $t \in [t_0, \infty)_{\mathbb{T}}$. Therefore we can conclude that

$$V^{\Delta}(t,z(t)) = N^2(2z+\mu(t))z^{\Delta} < 0.$$

for any time scales satisfying $|z| > \frac{\mu(t)}{2}$.

Case 2. If $0 < z < 1$ $(0 < y < N)$ we see that

$$p(t)z(t)(1-z(t)) > 0 \quad (11)$$

and

$$1+\mu(t)p(t)(1-z(t)) > 0. \quad (10)$$

Writing (11) and (12) into (6) we obtain that $z^{\Delta} < 0$ for all $t \in [t_0,\infty)_{\mathbb{T}}$. Therefore since $\mu(t) \geq 0$ we have

$$V^{\Delta}(t,z(t)) = N^2(2z+\mu(t))z^{\Delta} < 0.$$

Hence (7) is a Lyapunov function for the equation (6) for the case $0 < z < 1$.

Case 3. When $z = 0$ $(y = N)$, both $V(0,0) = 0$ and $V^{\Delta}(0,0) = 0$.

From the results of all cases equation (7) is a Lyapunov function for the dynamic logistic equation (6). Therefore by Theorem 1, $z = 0$ $(y = N)$ solution of (6) is asymptotically Lyapunov stable for any time scale satisfying $|z| > \frac{\mu(t)}{2}$. Asymptotically Lyapunov stability of the zero solution of dynamic logistic equation (6) indicates the asymptotically stability of the equilibrium solution $y = N$ of equation (3).

4. CONCLUSION

Population dynamics has been an important topic in ecology. Modeling and investigating the behavior of ecological phenomena have always been attractive for mathematicians. So this study is built on analying the Lyapunov stability of dynamic logistic equation on time scales. A Lyapunov function for dynamic logistic equation has been constructed. The study states that the equilibrium point $y = N$ namely carrying capacity of the population is asymptotically Lyapunov stable on time scales satisfying, $|z| > \frac{\mu(t)}{2}$. For future studies researchers might focus on the qualitative behavior of impulsive model of dynamic logistic equation.

REFERENCES

[1] M. Bohner and A. Peterson, *Dynamic Equations on Time Scales: An Introduction with Applications*. Springer Science & Business Media, 2001.

[2] M. Bohner and A. Peterson, *Advances in dynamic equations on time scales*. Springer Science & Business Media, 2002.

[3] T. Gard and J. Hoffacker, “Asymptotic behavior of natural growth on time scales,” *Dynamic Systems and Applications*, vol. 12, no. 1/2, pp. 131–148, 2003.

[4] V. Lakshmikantham, S. Sivasundaram, and B. Kaymakcalan, *Dynamic systems on measure chains (Vol. 370).* Springer Science & Business Media, 2013.

[5] J. Zhang, M. Fan, and H. Zhu, “Periodic solution of single population models on time scales,” *Mathematical and Computer Modelling,* vol. 52, no. 3, pp. 515–521, 2010.

[6] P. Cull, “Stability of discrete one-dimensional population models,” *Bulletin of Mathematical Biology*, vol. 50, no. 1, pp. 67-75, 1988.

[7] M. W. Hirsch, S. Smale, and R. L. Devaney, *Differential equations, dynamical systems, and an introduction to chaos*. Academic press, 2012.

[8] J. M. Davis, I. A. Gravagne, R. J. Marks, and A.A. Ramos, “Algebraic and dynamic Lyapunov equations on time scales.” In Proc. IEEE 42nd Southeastern Symposium on System Theory (SSST), 2010, pp. 329-334.