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RESEARCH ARTICLE

Catenaries, Cycloids and Warped Products

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ABSTRACT

We study warped products derived from catenaries and cycloids. We give an example of non-homogeneous semi-symmetric 3-space closely related to cycloids.

Keywords: *Warped product; surface of revolution; catenary; catenoid; cycloid.*
AMS Subject Classification (2020): *Primary: 53A04; Secondary: 53A10, 53C25*

1. Introduction

The notion of warped product was introduced by Bishop and O'Neill [4] to construct complete Riemannian manifolds of negative curvature. Let (B, g_B) and (F, g_F) be Riemannian manifolds. Take a positive smooth function f on B, then one obtains a Riemannian manifold

$$B \times_f F = (B \times F, g_B + f^2 g_F).$$

The resulting Riemannian manifold $B \times_f F$ is called the *warped product* with *base* B, *standard fiber* F and the *warping function* f. Before the publication of [4], the same object was introduced by Kručkovič [27] by the name *semireducible Riemannian space*.

Warped products have been used to construct explicit examples of many differential geometric problems. For instance, Ejiri [19] proved the existence of warped products of the form $M = \mathbb{S}^1 \times_f F$, where F is a compact Riemannian manifold of constant positive scalar curvature, which provide counterexamples to the following conjecture:

Conjecture 1. *Let M, g be a Riemannian manifold of dimension $n > 2$ and has constant scalar curvature. Assume that the identity component $\mathrm{Conf}_\circ(M)$ of the full conformal transformation group $\mathrm{Conf}(M)$ is strictly larger than the identity component $\mathrm{Iso}_\circ(M)$ of the full isometry group $\mathrm{Iso}(M)$, then M is isometric to spheres.*

Let us turn our attention to catenary and cycloid. The catenoid in the Euclidean 3-space $\mathbb{E}^3$ has two fundamental characterizations. First, it is characterized as the only non-planar minimal surfaces of revolution. The other one is that the catenoid is the surface of revolution whose profile curve is the catenary. It is known that catenary is characterized as a geodesic in the upper half plane equipped with a certain Riemannian metric (see Section 4).

Analogously, the cycloid is characterized as a geodesic in the upper half plane equipped with a certain Riemannian metric (see Section 5). Obviously, the surface of revolution with profile curve is cycloid (we call it "*cycnoid*" in this paper) is *non-minimal* in $\mathbb{E}^3$.

On the other hand, Ejiri [20] and Kokubu [26] studied minimal surfaces in warped products. By using their results, one can see that the cycnoids can be minimally immersed in some warped products.

This paper discusses some topics concerning on warped products derived from the catenoid and the cycnoid.

This article is dedicated to professor Bang-yen Chen. Professor Chen has made substantial progress of submanifold geometry via warped products (see *e.g.*, [8]–[12] and the book [13]).

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2. Warped products

Let (B, g_B) and (F, g_F) be Riemannian manifolds and f a positive smooth function on B. Then the *warped product* $B \times_f F$ with *base* B, *standard fiber* F and the *warping function* f is the product manifold $B \times F$ equipped with the *warped metric* $g_B + f^2 g_F$. For a point $p \in B$, the submanifold

$$F_p := \{(p, q) \mid q \in F\}$$

of $B \times_f F$ is called the *fiber* over $p \in B$. On the other hand, the submanifold

$$B_q := \{(p, q) \mid p \in B\}$$

is called the *leaf* over $q \in F$. Every leaf is totally geodesic in $B \times_f F$. On the other hand, every fiber is totally umbilical in $B \times_f F$.

Remark 2.1. Bishop [3] and Chen [7] proposed a generalization of the notion of warped product in the following manner. Take a positive smooth function h on the product manifold $B \times F$. Then the Riemannian manifold $(B \times F, g_B + h^2 g_F)$ is called the *umbilic product* in the sense of Bishop [3] and *twisted product* in the sense of Chen [7].

For fundamental properties of warped product, we refer to [2, Section 9.J], [13, Chapter 3], [36, Chapter 7].

3. Surfaces of revolution in Euclidean 3-space

3.1.

Let us denote by $\mathbb{R}^3(x, y, z)$ be the Cartesian 3-space with linear coordinates (x, y, z). We equip the Euclidean metric $g_0 = dx^2 + dy^2 + dz^2$. The resulting Riemannian 3-manifold $\mathbb{E}^3(x, y, z) = (\mathbb{R}^3(x, y, z), g_0)$ is referred to as the *Euclidean 3-space*.

Take a regular curve

$$\gamma(t) = (x(t), z(t))$$

in the xz-plane defined on an interval $\mathcal{I}$ and satisfying $x(t) > 0$. Then the *surface of revolution*

$$M_\gamma = \{(x(t)\cos\theta, x(t)\sin\theta, z(t)) \mid t \in \mathcal{I},\ 0 \leq \theta < 2\pi\}, \tag{3.1}$$

with *profile curve* γ in $\mathbb{E}^3(x, y, z)$ is the surface defined as the image of γ in $\mathbb{E}^3$ under the rotation around z-axis. The first fundamental form I of M_γ is given by

$$\mathrm{I} = (\dot{x}(t)^2 + \dot{z}(t)^2)\, dt^2 + x(t)^2\, d\theta^2. \tag{3.2}$$

Choose a unit normal vector field

$$\frac{1}{\sqrt{\dot{x}(t)^2 + \dot{z}(t)^2}} \left(-\dot{z}(t)\cos\theta, -\dot{z}(t)\sin\theta, \dot{x}(t)\right),$$

then the second fundamental form $\mathrm{I\!I}$ derived from this unit normal vector field is given by

$$\mathrm{I\!I} = \frac{\dot{x}(t)\ddot{z}(t) - \ddot{x}(t)\dot{z}(t)}{\sqrt{\dot{x}(t)^2 + \dot{z}(t)^2}}\, dt^2 + \frac{x(t)\dot{z}(t)}{\sqrt{\dot{x}(t)^2 + \dot{z}(t)^2}}\, d\theta^2.$$

The Gauss curvature K_{E} and the mean curvature H_{E} are given by

$$K_{\mathsf{E}} = \frac{\dot{z}(t)\,(\dot{x}(t)\ddot{z}(t) - \ddot{x}(t))\dot{z}(t)}{x(t)\,(\dot{x}(t)^2 + \dot{z}(t)^2)^2}, \quad H_{\mathsf{E}} = \frac{1}{2}\left\{\frac{\dot{z}(t)}{x(t)\sqrt{\dot{x}(t)^2 + \dot{z}(t)^2}} + \frac{\dot{x}(t)\ddot{z}(t) - \ddot{x}(t)\dot{z}(t)}{(\dot{x}(t)^2 + \dot{z}(t)^2)^{3/2}}\right\}.$$

It should be remarked that the signed curvature κ_{E} of the profile curve with respect to the Euclidean metric $dx^2 + dz^2$ is given by

$$\kappa_{\mathsf{E}}(t) = -\frac{\dot{x}(t)\ddot{z}(t) - \ddot{x}(t)\dot{z}(t)}{(\dot{x}(t)^2 + \dot{z}(t)^2)^{3/2}}$$

under the orientation determined by $dz \wedge dx$.

3.2.

Let us reparametrize the profile curve by the arc length parameter s with respect to the Euclidean metric $dx^2 + dz^2$, then (3.2) is rewritten as

$$\mathrm{I} = ds^2 + x(s)^2\, d\theta^2, \quad s \in \mathcal{I}(s), \tag{3.3}$$

where $\mathcal{I}(s)$ is an interval. Then the second fundamental form $\mathrm{I\!I}$, Gauss curvature K_{E} and the mean curvature H_{E} are simplified as

$$\mathrm{I\!I} = (\dot{x}(s)\ddot{z}(s) - \ddot{x}(s)\dot{z}(s)\,)\, ds^2 + x(s)\dot{z}(s)\, d\theta^2 = -\kappa_{\mathsf{E}}(s)\, ds^2 + x(s)\dot{z}(s)\, d\theta^2,$$

$$K_{\mathsf{E}} = -\frac{\kappa_{\mathsf{E}}(s)\dot{z}(s)}{x(s)} = -\frac{\dot{x}(s)}{x(s)},$$

$$H(s) = \frac{1}{2}\left(-\kappa_{\mathsf{E}}(s) + \frac{\dot{z}(s)}{x(s)}\right) = \frac{1}{2}\left(\frac{\dot{z}(s)}{x(s)} - \frac{\ddot{x}(s)}{\dot{z}(s)}\right). \tag{3.4}$$

The surface M_γ is the image of isometric immersion of the warped product $\mathcal{I}(s) \times_x \mathbb{S}^1 = (\mathcal{I}(s) \times \mathbb{S}^1, ds^2 + x(s)^2\, d\theta^2)$ into $\mathbb{E}^3$. Thus surfaces of revolution are typical examples of warped product submanifold.

We may regard the profile curve as a curve in the *right half plane*:

$$\mathbb{R}^2_+(x,z) = \{(x,z) \in \mathbb{R}^2(x,z) \mid x > 0\}$$

equipped with the Poincaré metric

$$\frac{dx^2 + dz^2}{x^2}.$$

Then we can reparametrize the profile curve γ by the arclength parameter

$$u = \int_{\mathcal{I}} \frac{1}{x(t)} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt = \int_0^s \frac{1}{x(s)}\, ds$$

with respect to the Poincaré metric. Then the first and second fundamental forms of M_γ are rewritten as

$$\mathrm{I} = x(u)^2\,(du^2 + d\theta^2), \quad \mathrm{I\!I} = -\kappa_{\mathsf{E}}(u)x(u)du^2 + \frac{dz}{du}\, d\theta^2.$$

Thus (u,θ) is an isothermal and curvature-line coordinate system for M_γ. An isothermal curvature-line coordinate system is traditionally called an *isothermic coordinate system*.

Proposition 3.1. *Every surface of revolution in $\mathbb{E}^3$ is reparametrized by isothermic coordinate systems.*

The Gauss curvature and the mean curvature are rewritten as

$$K = -\frac{1}{x(u)^2}\frac{dz}{du}(u)\kappa_{\mathsf{E}}(u), \quad H = \frac{1}{2}\left(-\kappa_{\mathsf{E}}(u) + \frac{1}{x(u)^2}\frac{dz}{du}(u)\right).$$

Note that the signed geodesic curvature κ_{H} of $(z(s), x(s))$ with respect to the Poincaré metric is given by

$$\kappa_{\mathsf{H}} = \frac{1}{2}\left(\frac{1}{x(u)^2}\left(\frac{dz}{du}(u)\frac{d^2x}{du^2}(u) - \frac{d^2z}{du^2}(u)\frac{dx}{du}(u)\right) + \frac{1}{x(u)}\frac{dz}{du}(u)\right)$$

with respect to the orientation determined by $dz \wedge dx$.

3.3.

Hereafter we use the arc length parametrization for γ with respect to Euclidean metric. The form of the first fundamental form (3.3) motivates us to consider the product manifold

$$\mathbb{R}^2_+(x,z)\times\mathbb{S}^1=\{(x,z,e^{i\theta})\mid x>0,\ z\in\mathbb{R},\ e^{i\theta}\in\mathbb{S}^1\}.$$

On the product manifold $\mathbb{R}^2_+(x,z)\times\mathbb{S}^1$, we equip the warped metric $dx^2+dz^2+f(x,z)^2\,d\theta^2$ with warping function $f(x,z)=x$. Then the warped product $\mathbb{R}^2_+(x,z)\times_x\mathbb{S}^1$ is isometric to an open subset of Euclidean 3-space $\mathbb{E}^3$. Indeed, if we set $r:=x$ and

$$\xi:=r\cos\theta,\quad \eta:=r\sin\theta,\quad \zeta:=z,$$

then the metric $dx^2+dz^2+x^2\,d\theta^2$ is rewritten as

$$d\xi^2+d\eta^2+d\zeta^2.$$

One can see that the warped product $\mathbb{R}^2_+(x,z)\times_x\mathbb{S}^1$ is nothing but the *cylindrical coordinates representation* of $\mathbb{E}^3(\xi,\eta,\zeta)\smallsetminus\{\zeta\text{-axis}\}$.

3.4.

For an arc length parametrized curve $\gamma(s)=(x(s),z(s))$ in the Euclidean right half plane $(\mathbb{R}^2_+(x,z),dx^2+dz^2)$, the surface M_γ of revolution is regarded a surface

$$\hat{M}_\gamma:=\{(x(s),z(s),e^{i\theta})\mid s\in\mathcal{I}(s),\ e^{i\theta}\in\mathbb{S}^1\}\tag{3.5}$$

in the warped product $\mathbb{R}^2_+(x,z)\times\mathbb{S}^1$. Indeed, if we equip a warped product metric g on $\mathbb{R}^2_+(x,z)\times\mathbb{S}^1$ by

$$g=dx^2+dz^2+x^2\,d\theta^2,$$

then the metric on $\hat{M}_\gamma$ induced from the warped product metric g coincides with the first fundamental form I. Thus the Riemannian 2-manifold $(\mathcal{I}(s)\times\mathbb{S}^1,\mathrm{I})$ is isometrically immersed in the warped product $\mathbb{R}^2_+(x,z)\times_x\mathbb{S}^1$. The following result is a special case of Ejiri's theorem [20].

Theorem 3.1. *Let $\gamma(s)=(x(s),z(s))$ be a unit speed curve in the right half plane $\mathbb{R}^2_+(x,z)$ equipped with a Riemannian metric $f(x,z)^2(dx^2+dz^2)$. Here $f(x,z)$ is a smooth positive function on $\mathbb{R}^2_+(x,z)$. Let M_γ be the surface of revolution in $\mathbb{E}^3$ with profile curve γ. Then the surface $\hat{M}_\gamma$ defined by the parametrization (3.5) is a minimal surface in the warped product manifold*

$$\mathbb{R}^2_+(x,z)\times_f\mathbb{S}^1=(\mathbb{R}^2_+(x,z)\times\mathbb{S}^1,dx^2+dz^2+f^2\,d\theta^2)$$

if and only if γ is a geodesic in $(\mathbb{R}^2_+(x,z),f(x,z)^2(dx^2+dz^2))$.

Related to Ejiri's theorem, Kokubu [26] obtained the following theorem.

Theorem 3.2. *Let $\gamma(s)=(x(s),z(s))$ be a unit speed curve in the right half plane $\mathbb{R}^2_+(x,z)$ equipped with a Riemannian metric $f(x,z)^2(dx^2+dz^2)$. Here $f(x,z)$ is a smooth positive function on $\mathbb{R}^2_+(x,z)$. Let M_γ be the surface of revolution in $\mathbb{E}^3$ with profile curve γ. Then the surface $\hat{M}_\gamma$ defined by the parametrization (3.5) is a minimal surface in the warped product manifold*

$$\mathbb{R}^2_+(x,z)\times_f\mathbb{S}^1=(\mathbb{R}^2_+(x,z)\times\mathbb{S}^1,dx^2+dz^2+f^2\,d\theta^2)$$

if and only if it is minimal in the Riemannian 3-manifold

$$(\mathbb{R}^2_+(x,z)\times\mathbb{S}^1,(dx^2+dz^2)/h^2+h^2f^2\,d\theta^2)$$

for any positive function $h(x,z)$ on $\mathbb{R}^2_+(x,z)$.

3.5.

For later use, here we collect some fundamental formulas for the Riemannian 2-manifold $(\mathbb{R}^2_+(x,z),f(x,z)^2\,(dx^2+dz^2))$. Take an orthonormal frame field

$$\bar{e}_1=\frac{1}{f}\frac{\partial}{\partial x},\quad \bar{e}_2=\frac{1}{f}\frac{\partial}{\partial z}.$$

Then its metrically dual coframe field is given by

$$\bar{\vartheta}^1 = f\,dx, \quad \bar{\vartheta}^2 = f\,dz.$$

The connection 1-form ${\bar{\omega}_2}^1 = -{\bar{\omega}_1}^2$ defined by the *first structure equations*:

$$d\bar{\vartheta}^1 + {\bar{\omega}_2}^1 \wedge \bar{\vartheta}^2 = 0, \quad d\bar{\vartheta}^2 + {\bar{\omega}_1}^2 \wedge \bar{\vartheta}^1 = 0$$

is computed as

$${\bar{\omega}_2}^1 = \frac{1}{f^2}(f_z \bar{\vartheta}^1 - f_x \bar{\vartheta}^2) = \frac{1}{f}(f_z\,dx^f_z\,dz).$$

The Christoffel symbols are given by

$$\Gamma_{xx}^{\ x} = \frac{f_x}{f}, \quad \Gamma_{xx}^{\ z} = -\frac{f_z}{f}, \quad \Gamma_{xz}^{\ x} = \frac{f_z}{f}, \quad \Gamma_{xz}^{\ z} = \frac{f_x}{f}, \quad \Gamma_{zz}^{\ x} = -\frac{f_x}{f}, \quad \Gamma_{zz}^{\ z} = \frac{f_z}{f}. \tag{3.6}$$

The geodesic equation is the system:

$$\frac{dx^2}{d\tau^2} + \frac{f_x}{f}\left(\frac{dx}{d\tau}\right)^2 + \frac{2f_z}{f}\frac{dx}{d\tau}\frac{dz}{d\tau} - \frac{f_x}{f}\left(\frac{dz}{d\tau}\right)^2 = 0, \quad \frac{dz^2}{d\tau^2} - \frac{f_z}{f}\left(\frac{dx}{d\tau}\right)^2 + \frac{2f_x}{f}\frac{dx}{d\tau}\frac{dz}{d\tau} + \frac{f_z}{f}\left(\frac{dz}{d\tau}\right)^2 = 0$$

under the arc length condition:

$$\left(\frac{dx}{d\tau}\right)^2 + \left(\frac{dz}{d\tau}\right)^2 = \frac{1}{f(x(\tau), z(\tau))^2}.$$

The Gauss curvature is

$$-\frac{1}{f^2}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2}\right)\log f.$$

3.6.

Let us choose $f(x) = x^{k/2}$ for some fixed constant k. Then the geodesic equation for an arc length parametrized curve $(x(\tau), z(\tau))$ of $(\mathbb{R}^2_+(x,z), x^k(dx^2 + dz^2))$ is the system:

$$x'' + \frac{k}{2x}\left((x')^2 - (z')^2\right) = 0, \quad z'' + \frac{k}{x}x'z' = 0 \tag{3.7}$$

under the arc length condition:

$$(x')^2 + (z')^2 = \frac{1}{x^k}.$$

Here the prime is the differentiation by τ.

The Gauss curvature is given by

$$\frac{k}{2x^{k+2}}.$$

Note that $\mathbb{E}^3(x,y,z) \smallsetminus \{(0,0,0)\}$ has another warped product representation. Introduce the *polar coordinates* (also called the *spherical coordinates*) (r, θ, φ) by

$$(x,y,z) = (r\sin\theta\cos\varphi, r\sin\theta\sin\varphi, r\cos\theta), \quad r > 0, \quad 0 \leq \theta \leq \pi, \quad 0 \leq \varphi < 2\pi.$$

Then the Euclidean metric $g_0 = dx^2 + dy^2 + dz^2$ is rewritten as

$$g_0 = dr^2 + r^2(d\theta^2 + \sin^2\theta).$$

Here the induced metric of the unit 2-sphere $\mathbb{S}^2$ coincides with $d\theta^2 + \sin^2\theta$. Hence $\mathbb{E}^3 \smallsetminus \{(0,0,0)\}$ is represented as the warped product $\mathbb{R}^+(r) \times_r \mathbb{S}^2$.

4. The catenary and the catenoid

4.1.

Let us consider a non-planar minimal surfaces M_γ of revolution in Euclidean 3-space $\mathbb{E}^3(x,y,z)$. By solving the ordinary differential equation $H=0$ by using (3.4), non-planar minimal surfaces are congruent to the surface of revolution with profile curve

$$\gamma(s) = (x(s), z(s)) = \left(\sqrt{s^2+a^2}, \pm a \sinh^{-1}\frac{s}{a}\right), \quad a>0. \tag{4.1}$$

This profile curve is catenary $x = a\cosh(z/a)$ lies in the xz-plane $(\mathbb{R}^2(x,z), dx^2+dz^2)$. The minimal surface of revolution with profile curve (4.1) is called a *catenoid*. The catenoid

$$M_\gamma = \left\{ \left(\sqrt{s^2+a^2}\,\cos\theta, \sqrt{s^2+a^2}\,\sin\theta, a\sinh^{-1}\frac{s}{a}\right) \,\middle|\, s\in\mathbb{R},\ e^{i\theta}\in\mathbb{S}^1 \right\}$$

has the first and second fundamental forms:

$$\mathrm{I} = ds^2 + (s^2+a^2)\,d\theta^2, \quad \mathrm{II} = -\frac{a}{s^2+a^2}\,ds^2 + a\,d\theta^2.$$

The Gauss curvature is given by

$$K = -\frac{a^2}{(s^2+a^2)^2}.$$

Let us take the arc length parameter u for γ with respect to the Poincaré metric $(dx^2+dz^2)/x^2$. The parameter u is given by

$$u = \int_0^s \frac{ds}{\sqrt{s^2+a^2}} = \sinh^{-1}\frac{s}{a}.$$

The catenary γ is reparametrized as $\gamma(u) = (a\cosh u, au)$. Hence we obtain the following *isothermic parametrization* of the catenoid:

$$M_\gamma = \left\{ (a\cosh u\,\cos\theta, a\cosh u\,\sin\theta, au) \mid u\in\mathbb{R},\ e^{i\theta}\in\mathbb{S}^1 \right\}$$

with first and second fundamental forms:

$$\mathrm{I} = a^2\cosh^2 u(du^2+d\theta^2), \quad \mathrm{II} = a(-\,du^2 + \,d\theta^2),$$

The Gauss curvature is expressed as

$$K = -\frac{1}{a^2\cosh^4 u}.$$

Remark 4.1. Blair [5] proposed a higher dimensional generalization of catenoid.

Remark 4.2. Surfaces of revolution with nonzero constant mean curvature are described by Delaunay [17]. Those are nodoids or unduloids. The profile curve of a nodoid is called the *nodary* which is the roulette of foci of a hyperbola by rolling along a fixed line. On the other hand, the profile curve of an unduloid is called the *undurary* which is the roulette of foci of an ellipse by rolling along a fixed line.

4.2.

Let us consider the geodesic equation for the Riemannian 2-manifold $(\mathbb{R}^2_+(x,z), x^2(dx^2+dz^2))$. From (3.7), an arc length parametrized curve $(x(\tau), z(\tau))$ is a geodesic in $(\mathbb{R}^2_+(x,z), x^2(dx^2+dz^2))$ if and only if

$$x'' = \frac{1}{x}(\,(z')^2 - (x')^2), \quad z'' = \frac{2}{x}\,x'z'. \tag{4.2}$$

Here τ is the arc length parameter with respect to the metric $x^2(dx^2+dz^2)$ and the prime denotes the differentiation by τ.

Proposition 4.1. *The catenary $x = a\cosh(z/a)$ is a geodesic of the Riemannian 2-manifold $(\mathbb{R}^2_+(x,z), x^2(dx^2+dy^2))$.*

Proof. The parameter τ of the catenary (4.1) is related to the Euclidean arc length parameter s by

$$\tau = \int_0^s \sqrt{s^2 + a^2}\, ds = \frac{s}{2} + \frac{s^2}{2} \log\left(s + \sqrt{s^2 + a^2}\right).$$

Hence we obtain

$$x' = \frac{s}{s^2 + a^2}, \quad z' = \frac{a}{s^2 + a^2}, \quad x'' = \frac{a^2 - s^2}{(s^2 + a^2)^{5/2}}, \quad z'' = \frac{-2as}{(s^2 + a^2)^{5/2}}.$$

By using these, one can confirm that $(x(\tau), z(\tau))$ satisfies the geodesic equation (4.2). □

Note that τ is rewritten as

$$\tau = \frac{a^2}{4}(2u + \sinh(2u))$$

in terms of the arc length parameter u with respect to the Poincaré metric.

By Theorem 3.1, the catenoid M_γ is regarded as a minimal surface

$$\{(\sqrt{s^2 + a^2}, \pm a \sinh^{-1}(s/a), e^{i\theta}) \mid s \in \mathbb{R},\ 0 \leq \theta < 2\pi\}$$

in the Euclidean 3-space $(\mathbb{R}^2_+(x, z) \times \mathbb{S}^1, dx^2 + dz^2 + x^2\, d\theta^2)$. Moreover, the catenoid is minimal in the warped product $\mathbb{H}^2(-1) \times_{x^2} \mathbb{S}^1$. Applying Kokubu's theorem, we obtain

Proposition 4.2. *Let $(x(s), z(s))$ be a curve in the Riemannian 2-manifold $(\mathbb{R}^2_+(x, z), x^2(dx^2 + dz^2))$. Then*

- *$(x(s), z(s))$ is a geodesic in $(\mathbb{R}^2_+(x, z), x^2(dx^2 + dz^2))$.*
- *$(x(s), z(s))$ is a catenary in the Euclidean half plane $(\mathbb{R}^2_+(x, z), dx^2 + dz^2)$.*
- *$(x(s), z(s), \theta)$ is a minimal surface in the Euclidean 3-space*

$$\mathbb{E}^3 = \mathbb{R}^2_+(x, z) \times_x \mathbb{S}^1 = (\mathbb{R}^2_+(x, z) \times \mathbb{S}^1, dx^2 + dz^2 + x^2\, d\theta^2).$$

- *$(x(s), z(s), \theta)$ is a minimal surface in the warped product*

$$\mathbb{H}^2(x, z) \times_{x^2} \mathbb{S}^1 = (\mathbb{R}^2_+(x, z) \times \mathbb{S}^1, (dx^2 + dz^2)/x^2 + x^4\, d\theta^2).$$

Remark 4.3. Parker [37] proposed the following problem:

> *Are there any functions which have the property that the ratio of area under the curve to the curve's arc length is independent of the interval over which they are measured ?*

Take a closed interval $[a, b]$ and draw the graph of a function $x = f(z) : [a, b] \to \mathbb{R}$ in the zx-plane. Then Parker's problem is interpreted as a problem to determine the function f satisfying the relation:

$$\int_a^b x(z)\, dz = k \int_a^b \sqrt{1 + \left(\frac{dx}{dz}\right)^2}\, dz,$$

where $k > 0$ is a constant. One can conform that the solutions to this equation are $x = k$ (constant function) or the catenary $x = k \cosh\{(z - z_0)/k\}$. Parker also showed that iff a surface of revolution has the property that the ratio of the volume it encloses to its surface area is independent of the interval on which it is defined, then the surface is a catenoid.

As is well known, the catenary is derived from the *hanging chain problem* in the Euclidean plane (*cf.* [30]). López [29] studied hanging chain problems in the 2-sphere $\mathbb{S}^2$ as well as the hyperbolic plane $\mathbb{H}^2$. López proposed the hanging chain problem with respect to a circle in Euclidean plane [31]. da Silva and Lopez [15] introduce the notion of catenary in arbitrary Riemannian 2-manifolds. Minimal surfaces derived from catenaries in hyperbolic geometry are studied in [16].

5. The cycloid

5.1.

As is well known, cycloids are nothing but the *Brachistochrones*. Take a point (x_0, z_0) of $\{(x,z) \in \mathbb{R}^2 \mid x \geq 0\}$. Consider the space Ω of all C^1-paths from the origin $(0,0)$ to $(x_0, z_0) \in \mathbb{R}^2_+(x,z)$. We parametrize a path $\gamma \in \Omega$ as $\gamma(z) = (x(z), z)$. The Brachistochrones are critical points of the functional:

$$T(\gamma) = \int_0^{z_0} \sqrt{\frac{1}{2\mathrm{g}\, x}\left(1 + \left(\frac{dx}{dz}\right)^2\right)}\, dz, \quad \gamma \in \Omega.$$

The positive constant g is understood as the gravitational acceleration constant. For more information on Brachistochrones, we refer to [1, 14]. Cycloid-shaped crystals of $\mathrm{TaSe_3}$ were investigated in [32].

5.2.

On the right half plane $\mathbb{R}^2(x,z)_+$, we equip a Riemannian metric

$$\frac{1}{2\mathrm{g}\, x}\,(dx^2 + dz^2).$$

Then the Euler-Lagrange equation of the functional T coincides with the geodesic equation of $(\mathbb{R}^2(x,z)_+, (dx^2 + dz^2)/(2\mathrm{g}\, x)\,)$.

Proposition 5.1. *Cycloids are geodesics of the Riemannian* 2*-manifold* $(\mathbb{R}^2(x,z)_+, (dx^2 + dz^2)/(2\mathrm{g}\, x)\,)$.

Hereafter, for simplicity we use the unit system so that $\mathrm{g} = 1/2$. Note that $(\mathbb{R}^2(x,z), x(dx^2 + dz^2))$ has non-constant negative Gauss curvature $-1/(2x)$.

The arc length parameter s of the cycloid

$$\gamma(t) = (a(1 - \cos t), a(t - \sin t)\,)\,, \quad 0 < t < 2\pi, \;\; a > 0$$

is given by

$$s(t) = \int_0^t \sqrt{\dot{x}(t)^2 + \dot{z}(t)^2}\, dt = \int_0^t 2a \sin\frac{t}{2}\, dt = 4a\left(1 - \cos\frac{t}{2}\right).$$

The arc length parameter u with respect to the Poincaré metric is given by

$$u(t) = \int_0^t \sqrt{\frac{\dot{x}(t)^2 + \dot{z}(t)^2}{x(t)^2}}\, dt = \int_0^t \frac{dt}{\sin\frac{t}{2}} = \log\tan\frac{t}{4}, \quad 0 < t < 2\pi.$$

The arc length parameter τ with respect to the metric $(dx^2 + dz^2)/x$ is

$$\tau(t) = \sqrt{\frac{\dot{x}(t)^2 + \dot{z}(t)^2}{x(t)}}\, dt = \sqrt{2a}\, t.$$

Thus when $a = 1/2$, we have $\tau = t$. Note that the metric $(dx^2 + dz^2)/x$ is incomplete.

5.3.

Let us consider the *cycnoid*, that is the surface of revolution whose profile curve is the cycloid. Cycnoid is non-minimal in $\mathbb{E}^3$. Morita [33] confirmed that the first fundamental form of the cycnoid does not satisfy the *Ricci condition* (see *e.g.*, [25]).

The cycnoid is parametrized as

$$\{(a(1 - \cos t)\cos\theta, a(1 - \cos t)\sin\theta, t - \sin t) \mid 0 < t < 2\pi,\; e^{i\theta} \in \mathbb{S}^1\}.$$

5.4.

By Ejiri-Kokubu's theorem (Theorem 3.1 and Theorem 3.2), we obtain the following fact which motivates the present study.

Proposition 5.2. *Let $(x(s), z(s))$ be a curve in the Riemannian 2-manifold $(\mathbb{R}^2_+(x,z), x(dx^2+dz^2))$. Then*

- *$(x(s), z(s))$ is a geodesic in $(\mathbb{R}^2_+(x,z), (dx^2+dz^2)/x)$.*
- *$(x(s), z(s))$ is a cycloid in the Euclidean half plane $(\mathbb{R}^2_+(x,z), dx^2+dz^2)$.*
- *$(x(s), z(s), e^{i\theta})$ is a minimal surface in the warped product*

$$\mathbb{E}^2_+(x,z)\times_{1/\sqrt{x}}\mathbb{S}^1=(\mathbb{R}^2_+(x,z)\times\mathbb{S}^1, dx^2+dz^2+d\theta^2/x).$$

- *$(x(s), z(s), e^{i\theta})$ is a minimal surface in the warped product*

$$\mathbb{H}^2(x,z)\times_{\sqrt{x}}\mathbb{S}^1=(\mathbb{R}^2_+(x,z)\times\mathbb{S}^1, (dx^2+dz^2)/x^2+x\,d\theta^2).$$

5.5.

Several attempts have been done to generalize the notion of catenary (see *e.g.*, [28, 39]). For example, in [39], Euclidean planar curves derived from the variational problem for the functional

$$c\int_{z_1}^{z_2} x^{\alpha}\sqrt{1+\left(\frac{dx}{dz}\right)^2}\,dz,$$

for curves $(x(z), z)$ in $\mathbb{R}^2_+(x,z)$, where c is a positive constant and α is a constant. Curves in $(\mathbb{R}^2_+(x,z), dx^2+dz^2)$ which satisfy the Euler-Lagrange equation of the above functional are called *α-catenary* in [39].

From Riemannian geometric point of view, α-catenary in the sense of [39] is a geodesic of $(\mathbb{R}^2_+(x,z), x^k(dx^2+dz^2))$ with $k=2\alpha$. In the next section we study curvature property of the warped products of the form $\mathbb{R}^2_+(x,z)\times_{x^{k/2}}\mathbb{S}^1$.

6. The warped product $\mathbb{R}^2_+(x,z)\times_f\mathbb{S}^1$

6.1.

Motivated by Proposition 5.2, we study curvature properties of warped products of the form $\mathbb{R}^2_+(x,z)\times_f\mathbb{S}^1$ with warping function $f(x)=x^{k/2}$, where k is a fixed real number. Take an orthonormal frame field

$$e_1=\frac{\partial}{\partial x},\quad e_2=\frac{\partial}{\partial z},\quad e_3=x^{-k/2}\frac{\partial}{\partial\theta}.$$

Then its metrically dual coframe field is

$$\vartheta^1=dx,\quad \vartheta^2=dz,\quad \vartheta^3=x^{k/2}\,d\theta.$$

The connections form $\{\omega_j{}^i\}$ is defined by the *first structure equation*:

$$d\vartheta^i+\sum_{j=1}^{3}\omega_j{}^i\wedge\vartheta^j=0,\quad i=1,2,3.$$

The connection forms are computed as

$$\omega_2{}^1=\omega_3{}^2=0,\quad \omega_3{}^1=\frac{k}{2x}\,\vartheta^3.$$

The Levi-Civita connection ∇ is described as

$$\nabla_{e_3}e_1=-\frac{k}{2x}e_3,\quad \nabla_{e_3}e_3=\frac{k}{2x}e_1,\quad \nabla_{e_i}e_j=0\ \text{for}\ (i,j)\neq(3,1),\,(3,3).$$

The curvature forms $\{\Omega_j{}^i\}$ are defined by the *second structure equation*:

$$\Omega_j{}^i = d\omega_j{}^i + \sum_{k=1}^{3} \omega_k{}^i \wedge \omega_j{}^k, \quad i, j = 1, 2, 3.$$

The curvature forms are given by

$$\Omega_2{}^1 = \Omega_3{}^2 = 0, \quad \Omega_3{}^1 = \frac{k(k-2)}{4x^2}\,\vartheta^1 \wedge \vartheta^3.$$

Hence the sectional curvatures are given by

$$K(e_1 \wedge e_2) = K(e_2 \wedge e_3) = 0, \quad K(e_1 \wedge e_3) = \frac{k(k-2)}{4x^2}.$$

The Ricci operator is given by

$$\frac{k(k-2)}{4x^2}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

This formula implies the following fact.

Proposition 6.1. *The warped product $\mathbb{R}^2_+(x,z) \times_{x^{k/2}} \mathbb{S}^1$ is semi-symmetric, that is, its Riemannian curvature R satisfies $R \cdot R = 0$. In particular, $\mathbb{R}^2_+(x,z) \times_{x^{k/2}} \mathbb{S}^1$ is locally symmetric when and only when $k = 0$ or $k = 2$.*

Remark 6.1. A Riemannian manifold (M, g) is said to be *semi-symmetric* if its Riemannian curvature satisfies $R \cdot R = 0$. Here $R \cdot R$ is the derivative of R by R itself. When $\dim M = 3$, M is semi-symmetric if and only if locally symmetric or principal Ricci curvatures ρ_1, ρ_2 and ρ_3 satisfies $\rho_1 = \rho_2 \neq \rho_3$ and $\rho_3 = 0$ up to numeration. Nomizu [35] conjectured that complete irreducible semi-symmetric Riemannian manifolds of dimension $n \geq 3$ are locally symmetric. Hitoshi Takagi [40] gave a counter-example to Nomizu's conjecture. More precisely, Takagi constructed a hypersurface of Euclidean 4-space of type number 2 such that it is complete irreducible semi-symmetric but not locally symmetric. Dušek and Kowalski proved the local rigidity of Takagi's hypersurface [18].

Since the scalar curvature of $\mathbb{R}^2_+(x,z) \times_{x^{\frac{k}{2}}} \mathbb{S}^1$ is non-constant unless $k = 0$, the warped products $\mathbb{R}^2_+(x,z) \times_{x^{\frac{k}{2}}} \mathbb{S}^1$ with $k \neq 0, 2$ provide examples of *non-homogeneous* semi-symmetric spaces.

Remark 6.2. In [38], Peñafiel, Quaglia and Trejos studied Weingarten surfaces in the warped product

$$(\mathbb{R}^+(r) \times \mathbb{S}^1) \times_f \mathbb{R}(z) = (\{(r, e^{i\theta}, z) \mid r > 0,\ e^{i\theta} \in \mathbb{S}^1,\ z \in \mathbb{R}\}, dr^2 + r^2 d\theta^2 + f(r)^2 dz^2).$$

Set $x := r$, then this warped product is rewritten as

$$(\mathbb{R}^+(x) \times_f \mathbb{R}(z)) \times_x \mathbb{S}^1 = (\{(x, z, e^{i\theta}) \in \mathbb{R}^2_+(x,z) \times \mathbb{S}^1\}, dx^2 + f(x)^2 dz^2 + x^2\, d\theta^2).$$

More precisely, they studied surfaces in $(\mathbb{R}^+(r) \times \mathbb{S}^1) \times_f \mathbb{R}(z)$ satisfying $H = \lambda(H^2 - K)$ for some continuously differentiable function $\lambda : (-\varepsilon, \infty) \to \mathbb{R}$ such that $4z\dot{\lambda}(z)^2 < 1$ and $\lambda(0) = 0$. They call Weingarten surfaces of this type by the name elliptic Weingarten surfaces of minimal type.

7. Solvable Lie group models of $\mathbb{H}^2_+ \times_f \mathbb{S}^1$

7.1.

We saw that catenoid induces a minimal surface in $\mathbb{H}^2_+ \times_{x^2} \mathbb{S}^1$. On the other hand, cycnoid induces a minimal surface in $\mathbb{H}^2_+ \times_{\sqrt{x}} \mathbb{S}^1$. These observations motivate us to study warped products

$$\mathcal{M}_k := \mathbb{H}^2_+ \times_{x^{k/2+1}} \mathbb{S}^1 = (\mathbb{H}^2(x,z) \times \mathbb{S}^2, g_k), \quad g_k = \frac{dx^2 + dz^2}{x^2} + x^{k+2}\, d\theta^2 \quad k \in \mathbb{R}.$$

Next, we set

$$\widetilde{\mathcal{M}}_k := \mathbb{H}^2_+ \times_{x^{k/2+1}} \mathbb{R} = (\mathbb{H}^2_+ \times \mathbb{R}(y), \tilde{g}_k), \quad \tilde{g}_k = \frac{dx^2 + dz^2}{x^2} + x^{k+2}\, dy^2, \quad k \in \mathbb{R}.$$

The warped product $\widetilde{\mathcal{M}}_k$ is the universal covering of $\mathcal{M}_k$. Each member of this one-parameter family $\{\widetilde{\mathcal{M}}_k\}_{k\in\mathbb{R}}$ of Riemannian 3-manifolds is homogeneous. Indeed $\widetilde{\mathcal{M}}_k$ and $\mathcal{M}_k$ are identified with the solvable Lie groups

$$\widetilde{\mathcal{M}}_k = \left\{ \left(\begin{array}{ccc} x & 0 & z \\ 0 & x^{-(k+2)} & y \\ 0 & 0 & 1 \end{array} \right) \middle| \; x, y, z \in \mathbb{R},\; x > 0 \right\},$$

$$\mathcal{M}_k = \left\{ \left(\begin{array}{ccc} x & 0 & z \\ 0 & x^{-(k+2)} & \theta \\ 0 & 0 & 1 \end{array} \right) \middle| \; x, z \in \mathbb{R},\; x > 0,\; \theta \in \mathbb{R}/2\pi\mathbb{Z} \right\},$$

respectively. Ejiri-Kokubu theorem is rephrased as follows:

Corollary 7.1. *Let $\gamma(s) = (x(s), z(s))$ be a unit speed curve in the right half plane $\mathbb{R}^2_+(x, z)$ equipped with a Riemannian metric $x^k(dx^2 + dz^2)$. Then the following properties are mutually equivalent:*

1. *$\gamma(s)$ is a geodesic in $(\mathbb{R}^2_+(x, z), x^k(dx^2 + dz^2))$.*
2. *The surface $(x(s), z(s), e^{i\theta})$ is a minimal surface in the warped product manifold*
$$\mathbb{R}^2_+(x, z) \times_{x^{k/2}} \mathbb{S}^1 = (\mathbb{R}^2_+(x, z) \times \mathbb{S}^1, dx^2 + dz^2 + x^k\, d\theta^2).$$
3. *The surface $(x(s), z(s), e^{i\theta})$ is a minimal surface in the warped product manifold*
$$\mathcal{M}_k = (\mathbb{H}^2_+(x, z) \times_{x^{(k+2)/2}} \mathbb{S}^1, (dx^2 + dz^2)/x^2 + x^{k+2}\, d\theta^2),$$

Example 7.1. Here we consider the case $k = 0$. Every geodesics in $(\mathbb{R}^2_+(x, z), dx^2 + dz^2)$ induces minimal sufaces in the Riemannian product $\mathbb{E}^2_+ \times \mathbb{S}^1 = (\mathbb{R}^2_+ \times \mathbb{S}^1, dx^2 + dz^2 + d\theta^2)$ and in $\mathcal{M}_0 = \mathbb{H}^2(x, z) \times_x \mathbb{S}^1$.

Geodesics are (half) lines and classified as follows:

1. The *horizontal line* defined by the equation $z = z_0$, where z_0 is a constant:

   The surface of revolution $(s\cos\theta, s\sin\theta, z_0)$ in $\mathbb{E}^3$ with profile curve γ is a totally geodesic plane $z = z_0$ excluding the z-axis. Obviously this surface is smoothly extended to the whole plane $z = z_0$. The totally geodesic plane $z = z_0$ corresponds to the minimal surface $(s, z_0, e^{i\theta})$ in the Riemannian product $\mathbb{E}^2(x, z) \times \mathbb{S}^1$ as well as the minimal surface $(s, z_0, e^{i\theta})$ in the warped product $\mathbb{H}^2(x, z) \times_x \mathbb{S}^1$.

2. The *vertical line* defined by the equation $x = x_0$, where x_0 is a positive constant:

   The surface of revolution $(x_0\cos\theta, x_0\sin\theta, s)$ in $\mathbb{E}^3$ with profile curve γ is a circular cylinder which is flat and of non-zero constant mean curvature. The circular cylinder corresponds to the minimal surface $(x_0, s, e^{i\theta})$ in the Riemannian product $\mathbb{E}^2(x, z) \times \mathbb{S}^1$ as well as the minimal surface $(x_0, s, e^{i\theta})$ in the warped product $\mathbb{H}^2(x, z) \times_x \mathbb{S}^1$.

3. The *oblique line* $(x(s), z(s)) = (as + x_0, bs + z_0)$, where $x_0 > 0$ and $a^2 + b^2 = 1$:

   The surface of revolution in $\mathbb{E}^3$ with profile curve γ is a circular cone without the vertex which is flat.

   The circular cone corresponds to the minimal surface $(as + x_0, bs + z_0, e^{i\theta})$ in the Riemannian product $\mathbb{E}^2(x, z) \times \mathbb{S}^1$ as well as the minimal surface $(as + x_0, bs + z_0, e^{i\theta})$ in the warped product $\mathbb{H}^2(x, z) \times_x \mathbb{S}^1$.

As we saw before, the surface of revolution whose profile curve is a catenary is minimal in $\mathbb{E}^3$ well as $\mathbb{H}^2(x, z) \times_{x^2} \mathbb{S}^1$. Next, the surface of revolution whose profile curve is a cycloid is minimal in $\mathbb{R}^2_+(x, z) \times_{1/\sqrt{x}} \mathbb{S}^1$ as well as $\mathbb{H}^2(x, z) \times_{\sqrt{x}} \mathbb{S}^1$. Now we know that every flat surfaces of revolution in $\mathbb{E}^3$ is minimally immersed into the Riemannian product $\mathbb{E}^2_+(x, z) \times \mathbb{S}^1$ as well as the warped product $\mathbb{H}^2(x, z) \times_x \mathbb{S}^1$.

Example 7.2. In case $k = -2$, geodesics in $(\mathbb{R}^2_+(x, z), (dx^2 + dz^2)/x^2) = \mathbb{H}^2(x, z)$ are classified as

1. The half line $x = x_0 > 0$ with arc length parametrization $\gamma(\tau) = (x_0, x_0\,\tau + z_0)$: The surface of revolution $(x_0\cos\theta, x_0\sin\theta, x_0\,\tau + z_0)$ in $\mathbb{E}^3$ with profile curve γ is a circular cylinder which is flat and of non-zero constant mean curvature. The first fundamental form is given by $x_0^2\, d\tau^2 + d\theta^2$.

   The circular cylinder corresponds to the minimal surface $(x_0, x_0\,\tau + z_0, e^{i\theta})$ in the warped product $\mathbb{R}^2(x, z) \times_{1/x} \mathbb{S}^1$ as well as to the minimal surface $(x_0, x_0\,\tau + z_0, e^{i\theta})$ in the Riemannian product $\mathbb{H}^2(x, z) \times \mathbb{S}^1$.

2. The geodesic $\{(x,z) \in \mathbb{H}^2(x,z) \mid x^2+(z-z_0)^2 = R^2,\ z_0 - R < z < z_0 + R\}$ for some $R > 0$ and $z_0 \in \mathbb{R}$:

   The geodesic γ is parametrized as $(R\cos\phi, R\sin\phi + z_0)$. Note that ϕ is not the arc length parameter with respect to the Poincaré metric $(dx^2+dz^2)/z^2$. The surface of revolution in $\mathbb{E}^3$ with this profile curve is the round sphere $\{(x,y,z) \in \mathbb{E}^3 \mid x^2+y^2+(z-z_0)^2 = R^2\}$ without north pole $(0,0,z_0+R)$ and the south pole $(0,0,z_0-R)$. Obviously the surface of revolution is smoothly extended to the whole round sphere.

   The round sphere corresponds to the minimal surface $(R\cos\phi, R\sin\phi + z_0, , e^{i\theta})$ in the warped product $\mathbb{R}^2(x,z) \times_{1/x} \mathbb{S}^1$ as well as the minimal surface $(R\cos\phi, R\sin\phi + z_0, e^{i\theta})$ in the Riemannian product $\mathbb{H}^2(x,z) \times \mathbb{S}^1$.

7.2. Another model of $\widetilde{\mathcal{M}}_k$

Let us perform the coordinate transformation:

$$z := u, \quad x = e^w, \quad y := v,$$

then $\tilde{g}_k$ is written as

$$\tilde{g}_k = e^{-2w}du^2 + e^{(k+2)w}dv^2 + dw^2.$$

Thus $\widetilde{\mathcal{M}}_k$ is isometric to

$$(\mathbb{R}^3(u,v,w), e^{-2w}du^2 + e^{(k+2)w}dv^2 + dw^2).$$

The solvable Lie group model of $\widetilde{\mathcal{M}}_k$ is rewritten as

$$\widetilde{G}_k = \left\{ \left(\begin{array}{ccc} e^w & 0 & u \\ 0 & e^{-(k+2)w/2} & v \\ 0 & 0 & 1 \end{array} \right) \middle|\ u,v,w \in \mathbb{R} \right\}.$$

We introduce a representation

$$\rho_k : (\mathbb{R}(w), +) \to \mathrm{GL}_2\mathbb{R}; \quad \rho_k(w) = \left(\begin{array}{cc} e^w & 0 \\ 0 & e^{-(k/2+1)w} \end{array} \right).$$

The solvable Lie group $\widetilde{G}_k$ is realized as the semi-direct product $\mathbb{R}^2 \times_{\rho_k} \mathbb{R}$ with respect to the representation ρ_k. Thus $\widetilde{G}_k$ is identified with $\mathbb{R}^3(u,v,w)$ with multiplication:

$$(u_1,v_1,w_1) * (u_1,v_2,w_2) = (u_1 + e^{w_1}u_2, v_1 + e^{-(k+2)w_1/2}v_2, w_1+w_2).$$

The Riemannian metric $\tilde{g}_k = e^{-2w}du^2 + e^{(k+2)w}dv^2 + dw^2$ is left invariant with respect to this Lie group structure. Take a discrete subgroup

$$\varGamma = \left\{ \left(\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 2\pi m \\ 0 & 0 & 1 \end{array} \right) \middle|\ m \in \mathbb{Z} \right\} \cong 2\pi\mathbb{Z}$$

of $\widetilde{G}_k$, then $G_k := \widetilde{G}_k/\varGamma$ is diffeomorphic to $\mathcal{M}_k$. Moreover the left invariant Riemannian metric $\tilde{g}_k$ descends to a metric g_k on G_k. As a result, $\mathcal{M}_k$ is isometric to G_k.

Take a left invariant orthonormal frame field:

$$e_1 = e^w\frac{\partial}{\partial u}, \quad e_2 = e^{-(k+2)w/2}\frac{\partial}{\partial v}, \quad e_3 = \frac{\partial}{\partial w}.$$

The connection forms relative to the coframe field $\{\vartheta^1, \vartheta^2, \vartheta^3\}$ metrically dual to $\{e_1,e_2,e_3\}$ are given by

$$\omega_2{}^1 = 0, \quad \omega_3{}^1 = -\vartheta^1, \quad \omega_3{}^2 = \frac{k+2}{2}\,\vartheta^2.$$

The curvature forms are given by

$$\Omega_2{}^1 = \frac{k+2}{2}(\vartheta^1 \wedge \vartheta^2), \quad \Omega_3{}^1 = -(\vartheta^2 \wedge \vartheta^3), \quad \Omega_3{}^2 = -\frac{(k+2)^2}{4}(\vartheta^2 \wedge \vartheta^3).$$

The sectional curvatures are given by

$$K(e_1 \wedge e_2) = \frac{k+2}{2}, \quad K(e_1 \wedge e_3) = -1, \quad K(e_2 \wedge e_3) = -\frac{(k+2)^2}{4}.$$

The Ricci operator is given by

$$\begin{pmatrix} -k/2 & 0 & 0 \\ 0 & k(k+2)/4 & 0 \\ 0 & 0 & -1-(k+2)^2/4 \end{pmatrix}.$$

Example 7.3 ($k = 2$)**.** $\mathcal{M}_2 \cong G_2$ is the universal covering

$$(\mathbb{R}^3(u,v,w), e^{-2w}du^2 + e^{4w}dv^2 + dw^2)$$

of the warped product which contains catenoid as a minimal surface.

Example 7.4 ($k = -1$)**.** $\widetilde{\mathcal{M}}_{-1} \cong \tilde{G}_{-1}$ is the universal covering

$$(\mathbb{R}^3(u,v,w), e^{-2w}du^2 + e^{w}dv^2 + dw^2)$$

of the warped product which contains cycnoid as a minimal surface.

Example 7.5 ($k = 0$)**.** $\widetilde{\mathcal{M}}_0 \cong \tilde{G}_0$ is the model space

$$\mathrm{Sol}_3 = (\mathbb{R}^3(u,v,w), e^{-2w}du^2 + e^{2w}dv^2 + dw^2)$$

of the solvegeometry.

Example 7.6 ($k = -4$)**.** $\widetilde{\mathcal{M}}_{-4}$ is the warped product model

$$\mathbb{R}(w) \times_{e^{-w}} \mathbb{R}^2(u,v) = (\mathbb{R}^3(u,v,w), e^{-2w}(du^2+dv^2) + dw^2)$$

of the hyperbolic 3-space $\mathbb{H}^3$ of constant curvature -1.

Example 7.7 ($k = -2$)**.** $\widetilde{\mathcal{M}}_{-2}$ is the Riemannian product

$$\mathbb{H}^2 \times \mathbb{R} = (\mathbb{R}(w) \times_{e^{-w}} \mathbb{R}(u)) \times \mathbb{R}(v) = (\mathbb{R}^3(u,v,w), e^{-2w}du^2 + dv^2 + dw^2).$$

7.3.

An integral representation formula for minimal surfaces in $\widetilde{G}_k$ was obtained in [21, 22, 24]. Here we recall it. Let $(u,v,w) : \Sigma \to \widetilde{G}_k$ be a conformal immersion of a Riemann surface Σ into the solvable Lie group $\widetilde{G}_k$ with unit normal vector field N. Let us identify the Lie algebra $\mathfrak{g}_k$ of $\widetilde{G}_k$ with Euclidean 3-space via the orthonormal basis $\{e_1, e_2, e_3\}$. Then the mapping $(u,v,w)^{-1}N$ takes value in the unit 2-sphere $\mathbb{S}^2 \subset \mathfrak{g}_k \cong \mathbb{E}^3$. Next let us consider the stereographic projection $\mathfrak{g}_k \supset \mathbb{S}^2 \setminus \{e_3\} \to \mathbb{C} = \mathbb{R}e_1 \oplus \mathbb{R}e_2$ with pole e_3. Then the image G of $(u,v,w)^{-1}N$ under the stereographic projection is called the *normal Gauss map* of the surface. By using G, $(u,v,w)^{-1}N$ is expressed as

$$\frac{1}{1+|\mathsf{G}|^2}\left(2\mathrm{Re}\,(\mathsf{G})e_1 + 2\mathrm{Im}\,(\mathsf{G})e_2 + (|\mathsf{G}|^2-1)e_3\right).$$

Under the stereographic projection $\mathbb{S}^2 \setminus \{\infty\} \to \mathbb{C} = \mathbb{R}e_1 \oplus \mathbb{R}e_2$,

Theorem 7.1. *Let* F *and* G *be* $(\mathbb{C} \cup \{\infty\})$*-valued functions defined on a simply connected region* $\mathbb{D} \subset \mathbb{C}$ *satisfying*:

$$\frac{\partial \mathsf{F}}{\partial \bar{\zeta}} = \frac{|\mathsf{F}|^2\,\mathsf{G}}{2}\left((1-\mathsf{G}^2) + \frac{k+2}{2}(1+\bar{\mathsf{G}}^2)\right),$$

$$\frac{\partial \mathsf{G}}{\partial \bar{\zeta}} = -\frac{\mathsf{F}}{4}\left((1+\mathsf{G}^2)(1-\bar{\mathsf{G}}^2) - \frac{k+2}{2}(1-\mathsf{G}^2)(1+\bar{\mathsf{G}}^2)\right).$$

Then $(u(\zeta,\bar{\zeta}), v(\zeta,\bar{\zeta}), w(\zeta,\bar{\zeta})\,) : \mathbb{D} \to \tilde{G}_k$ *defined by*

$$\begin{aligned} w =& 2\int_{\zeta_0}^{\zeta} \mathrm{Re}\, \mathsf{FG}\, d\zeta, \\ u =& 2\int_{\zeta_0}^{\zeta} \mathrm{Re}\ \exp w \left(\frac{1}{2}\,\mathsf{F}(1-\mathsf{G}^2)\right) d\zeta, \\ v =& 2\int_{\zeta_0}^{\zeta} \mathrm{Re}\ \exp\left(-\frac{(k+2)w}{2}\right)\left(\frac{i}{2}\mathsf{F}(1+\mathsf{G}^2)\right) d\zeta \end{aligned}$$

is a weakly conformal harmonic map. If it is conformal, then it defines a minimal surface in $\tilde{G}_k$*.*

7.4.

Here we exhibit some typical submanifolds in $\widetilde{G}_k$.

Example 7.8. For any constants u_0 and v_0, the surfaces

$$\mathcal{L}_{u=u_0} = \{(u_0, v, w) \in \widetilde{G}_k\}, \quad \mathcal{L}_{v=v_0} = \{(u, v_0, w) \in \widetilde{G}_k\}$$

are totally geodesic in $\widetilde{G}_k$ and of constant curvature $-(k+2)^2/4$ and -1, respectively. On the other hand

$$\mathcal{L}_{w=w_0} = \{(u, v, w_0) \in \widetilde{G}_k\}$$

is flat and of constant mean curvature $-k/2$.

Nistor [34] studied constant angle surfaces in $\widetilde{G}_k$.

7.5.

Ejiri-Kokubu theorem is rephrased as follows:

Corollary 7.2. *Let* $\gamma(s) = (u(s), w(s))$ *be a unit speed curve in the Cartesian plane* $\mathbb{R}^2(u, w)$ *equipped with a Riemannian metric* $e^{kw}du^2 + e^{(k+2)w}dw^2$*. Then the following properties are mutually equivalent:*

1. $\gamma(s)$ *is a geodesic in* $(\mathbb{R}^2(u, w), e^{kw}du^2 + e^{(k+2)w}dw^2\,)$*.*
2. *The surface* $(u(s), v, w(s))$ *is a minimal surface in the warped product manifold*

$$\mathcal{M}_k = (\mathbb{R}^3(u, v, w), e^{-2w}du^2 + e^{(k+2)w}dv^2 + dw\,).$$

From Example 7.1, we obtain the following fact.

Corollary 7.3. *Every flat surface of revolution in* $\mathbb{E}^3$ *can be minimally immersed in* Sol_3/Γ*.*

Next, from Example 7.2, we obtain the following fact.

Corollary 7.4. *Circular cylinder and round spheres can be minimally immersed in* $\mathbb{H}^2 \times \mathbb{S}^1$*.*

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Competing interests

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