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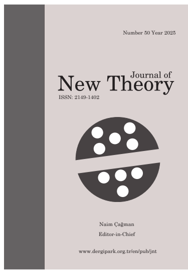

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Bertrand Curves in n-Dimensional Riemann-Otsuki Space

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Abstract— In this paper, we extend the classic properties of Bertrand curves in Euclidean 3-space to an n-dimensional Riemann-Otsuki space. We introduce the concept of infinitesimal deformations of curves within this space, and by applying the Frenet formulas concerning the contravariant component of the covariant derivative, we derive conditions under which a given deformation of a curve corresponds to a Bertrand curve in this n-dimensional space.

Keywords — *General connections, Riemann-Otsuki space, Frenet frame, Bertrand curves*

Mathematics Subject Classification (2020) 53A35, 53B15

1. Introduction

Riemann-Otsuki spaces are defined by a Riemannian metric linked to the concept of a general connection, as introduced by Otsuki [1–5]. When represented in local coordinates, it is established that the components of an affine connection form a geometric object rather than a geometric quantity. This distinction arises because, under coordinate transformations, these components do not transform like the components of a tensor of type (1,2) since they also include terms with second-order partial derivatives of the local coordinates. However, Otsuki showed that the components of an affine connection and those of a tensor of type (1,2) are not entirely separate ideas. Both can be seen as special cases of a more general connection, with the Otsuki connection extending the concept of an affine connection in differentiable manifolds.

By the concept of second-order tangent bundles, denoted as $\mathfrak{T}^2(M)$, Otsuki unified classical connections such as affine, projective, and conformal connections on manifolds within a broader framework. He defined a general connection as a cross-section of the vector bundle $\mathfrak{T}(M)\otimes\mathfrak{D}^2(M)$, where $\mathfrak{D}^2(M)$ is the dual vector bundle of $\mathfrak{T}^2(M)$. Thus, Otsuki introduced the covariant derivative corresponding to this general connection and established its relationship with the basic covariant derivative. Nadj [6] derived Frenet formulas, and in [7], he derived the Gauss, Codazzi, and Kühne equations for Riemann-Otsuki spaces. According to Moor [8, 9], a Riemann-Otsuki space is a special case of a Weyl-Otsuki space. A Riemann-Otsuki space, denoted as $(R - O_n)$, is defined by a general connection that satisfies the equation $D_k g_{ij} = 0$, where D is the covariant derivative concerning the general connection Γ. Recently, Pirinççi [10] investigated the congruences of curves in Weyl-Otsuki spaces.

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Infinitesimal deformations of curves in Riemannian spaces have been explored by Hayden [11], Schouten and van Kampen [12], and Yano et al. [13]. Furthermore, Pears [14] used infinitesimal deformations to study Bertrand curves in n-dimensional Riemannian space $\mathbb{R}^n$. Then, Alo [15] used this method to study generalized helices in n-dimensional Riemann-Otsuki space. Bertrand curves in 3-dimensional Riemann-Otsuki space were studied by Yilmaz and Bektaş [16]. Besides, Li et al. [17] defined a new class of Bertrand curves in Euclidean four-space.

This work uses infinitesimal deformations of curves to examine Bertrand curves in n-dimensional Riemann-Otsuki space. The structure of the paper is as follows: Section 2 presents essential background information on general connections, Frenet formulas for Riemann-Otsuki space, and Bertrand curves in Riemannian space, all of which will be referenced in Section 3. Section 3 provides the main results about Bertrand curves in n-dimensional Riemann-Otsuki space. The last section discusses the need for further research.

2. Preliminaries

In this section, we present some general knowledge of general connections, Frenet formulas for Riemann-Otsuki space, and Bertrand curves in Riemannian space. These results will be used to investigate Bertrand curves in n-dimensional Riemann-Otsuki space.

2.1. General Connections

Otsuki defined a general connection as a cross-section of the vector bundle $T(M) \otimes \mathfrak{D}^2(M)$, where $\mathfrak{D}^2(M)$ is the dual vector bundle of $\mathfrak{T}^2(M)$. In local coordinates u^i, a general connection Γ can be expressed as:

$$\Gamma = \partial u_i \otimes \left(P^i_j d^2 u^k + \Gamma^i_{jk} du^i \otimes du^k \right)$$

and it can be written as $\Gamma = (P^i_j, \Gamma^i_{jk})$. It can be observed that P^i_j represents the components of a (1,1)-tensor, denoted by $P = \lambda(\Gamma)$, which is referred to as the principal endomorphism of $T(M)$. If P is the identity isomorphism of $T(M)$, meaning that $P^i_j = \delta^i_j$, then the connection Γ simplifies to an affine connection. Otsuki also defined the covariant differential concerning this connection as

$$DV^{k_1 \ldots k_p}_{m_1 \ldots m_q} = V^{k_1 \ldots k_p}_{m_1 \ldots m_q;h} du^h$$

where

$$\begin{aligned} V^{k_1 \ldots k_p}_{m_1 \ldots m_q;h} =& P^{k_1}_{i_1} \ldots P^{k_p}_{i_p} \frac{\partial V^{i_1 \ldots i_p}_{j_1 \ldots j_q}}{\partial u^h} P^{j_1}_{m_1} \ldots P^{j_q}_{m_q} + \sum_{s=1}^{p} P^{k_1}_{i_1} \ldots P^{k_{s-1}}_{i_{s-1}} \Gamma^{k_s}_{i_s h} P^{k_{s+1}}_{i_{s+1}} \ldots P^{k_p}_{i_p} V^{i_1 \ldots i_p}_{j_1 \ldots j_q} P^{j_1}_{m_1} \ldots P^{j_q}_{m_q} \\ & - \sum_{t=1}^{q} P^{k_1}_{i_1} \ldots P^{k_p}_{i_p} V^{i_1 \ldots i_p}_{j_1 \ldots j_q} P^{j_1}_{m_1} \ldots P^{j_{t-1}}_{m_{t-1}} \Lambda j_{t m_{t-1}} P^{j_{t+1}}_{m_{t+1}} \ldots P^{j_q}_{m_q} \end{aligned}$$

and $\Lambda^j_{ih} = \Gamma^j_{ih} - \frac{\partial P^j_i}{\partial u^h}$. Moreover, he demonstrated that the product of a tensor Q of type (1,1) and a general connection Γ results in another general connection. These connections, denoted by $'\Gamma = Q\Gamma = (Q^i_k P^k_j, Q^i_k \Gamma^k_{jh})$ and $''\Gamma = \Gamma Q = (P^i_k Q^k_j, \Lambda^i_{kh} Q^k_j)$, are referred to as the contravariant and covariant parts of the connection Γ, respectively. In local coordinates, they can be given by

$$'\Gamma = \partial u_i \otimes \left(Q^i_k P^k_j d^2 u^j + Q^i_k \Gamma^k_{jh} du^j \otimes du^h \right) = \partial u_i Q^i_k \otimes \left(P^k_j d^2 u^j + \Gamma^k_{jh} du^j \otimes du^h \right) = Q\Gamma$$

and

$$''\Gamma = \partial u_i \otimes \left(d\left(P^i_k Q^k_j du^j \right) + \Lambda^i_{kh} Q^k_j du^h \otimes du^j \right) = \Gamma Q$$

If Γ is a regular general connection and $Q = P^{-1}$, then

$$'\Gamma = P^{-1}\Gamma = \left(\delta^i_j, Q^i_k \Gamma^k_{jh}\right) = \left(\delta^i_j, {}'\Gamma^i_{jh}\right)$$

and

$$''\Gamma = \Gamma P^{-1} = \left(\delta^i_j, \Lambda^i_{kh} Q^k_j\right) = \left(\delta^i_j, {}''\Gamma^i_{jh}\right) \tag{2.1}$$

i.e., $'\Gamma$ and $''\Gamma$ are affine connections. Furthermore, Otsuki defined a basic covariant differential by

$$\overline{D}V^{i_1...i_p}_{j_1...j_q} = V^{i_1...i_p}_{j_1...j_q|h} du^h$$

where

$$V^{i_1...i_p}_{j_1...j_q|h} = \frac{\partial V^{i_1...i_p}_{j_1...j_q}}{\partial u^h} + \sum_{s=1}^{p} {}'\Gamma^{i_s}_{kh} V^{i_1...i_{s-1}k i_{s+1}...i_p}_{j_1...j_q} - \sum_{t=1}^{q} {}'\Gamma^{k}_{j_t h} V^{i_1...i_p}_{j_1...j_{t-1}k j_{t+1}...j_q}$$

and showed that

$$V^{i_1...i_p}_{j_1...j_q;m} = P^{i_1}_{k_1}...P^{i_p}_{k_p} V^{k_1...k_p}_{h_1...h_q|m} P^{h_1}_{j_1}...P^{h_q}_{j_q}$$

For a general connection $\gamma = (P^i_j, \Gamma^i_{jh})$ and an identity isomorphism I,

$$\frac{D\delta^i_j}{ds} = \left(\Gamma^i_{hk} P^h_j - P^i_h \Lambda^h_{jk}\right) \frac{dx^k}{ds}$$

and when Γ is regular,

$$\frac{D\delta^i_j}{ds} = \left({}'\Gamma^i_{jk} - {}''\Gamma^i_{jk}\right) \frac{dx^k}{ds} \tag{2.2}$$

which does not necessarily vanish. From (2.1),

$$''\Gamma^i_{kh} P^k_j = \Lambda^i_{lh} Q^l_k P^k_j = \left(-\frac{\partial P^i_l}{\partial u^h} + \Gamma^i_{lh}\right) Q^l_k P^k_j = -\frac{\partial P^i_j}{\partial u^h} + \Gamma^i_{jh} = -\frac{\partial P^i_l}{\partial u^h} + P^i_k {}'\Gamma^k_{jh}$$

and thus the equation obtained called the Otsuki equation,

$$\frac{\partial P^i_l}{\partial u^h} + {}''\Gamma^i_{kh} P^k_j - P^i_k {}'\Gamma^k_{jh} = 0$$

gives the relationship between the covariant and contravariant parts of a general connection Γ.

The components of the curvature tensor with respect to $'\Gamma$ and $''\Gamma$, respectively, are given by

$$'R^i_{jkl} = \frac{\partial' \Gamma^i_{kl}}{\partial x^l} - \frac{\partial' \Gamma^i_{jl}}{\partial x^k} - {}'\Gamma^i_{ak} {}'\Gamma^a_{jl} + {}'\Gamma^i_{al} {}'\Gamma^a_{jk} \tag{2.3}$$

and

$$''R^i_{jkl} = \frac{\partial'' \Gamma^i_{kl}}{\partial x^l} - \frac{\partial'' \Gamma^i_{jl}}{\partial x^k} - {}''\Gamma^i_{ak} {}''\Gamma^a_{jl} + {}''\Gamma^i_{al} {}''\Gamma^a_{jk}$$

The components of the torsion tensor of $'\Gamma$ and $''\Gamma$, respectively, are given by

$$'T^i_{jk} = {}'\Gamma^i_{jk} - {}'\Gamma^i_{kj}$$

and

$$''T^i_{jk} = {}''\Gamma^i_{jk} - {}''\Gamma^i_{kj}$$

2.2. The Frenet Formulas for the Riemann-Otsuki Space

Nadj [6] derived the Frenet formulas for the Riemann-Otsuki space using the contravariant and covariant parts of the general connection Γ. Let C be a curve in n-dimensional Riemann-Otsuki space given by $C : s \to x^i(s)$ with $V_{(1)}, V_{(2)}, ..., V_{(n)}$ the unit tangent, 1-normal, ..., $(n-1)$-normal vector, respectively, and $\kappa_1, \kappa_2, ..., \kappa_{(n-1)}$ its curvatures. Then, *i.* Frenet formulas of the basic covariant differential applied on the contravariant components of the tangent and normal vectors ($'D = \overline{D}$ for contravariant vectors) are as follows:

$$'DV^i_{(\alpha)} = \overline{D}V^i_{(\alpha)} = -\kappa_{\alpha-1}V^i_{(\alpha-1)} + \kappa_\alpha V^i_{(\alpha+1)} + V^q_{(\alpha)}\overline{D}\delta^i_q \tag{2.4}$$

where

$$\kappa_\alpha = \left(g_{ij}\left('DV^i_{(\alpha)} + \kappa_{\alpha-1}V^i_{(\alpha-1)} - V^r_{(\alpha)}{'D}\delta^i_r\right)\left('DV^j_{(\alpha)} + \kappa_{\alpha-1}V^j_{(\alpha-1)} - V^t_{(\alpha)}{'D}\delta^j_t\right)\right)^{\frac{1}{2}}$$

with $\kappa_0 = 0$ and $\kappa_n = 0$ and $\alpha \in \{1, 2, ..., n\}$.

ii. Frenet formulas of the basic covariant differential $''D$ applied on the contravariant components of the tangent and normal vectors are:

$$''DV^j_{(\alpha)} = P^j_i\left(-\kappa^*_{(\alpha-1)}V^i_{(\alpha-1)} + \kappa^*_\alpha V^i_{(\alpha+1)}\right)$$

where $\kappa^*_1 = (g_{ij}{''DV^i_1}{''DV^j_1})^{\frac{1}{2}} > 0$ and $\kappa^*_\alpha = (g_{ij}(''DV^i_\alpha + \kappa^*_{\alpha-1}V^i_{\alpha-1})(''DV^j_\alpha + \kappa^*_{\alpha-1}V^j_{\alpha-1}))^{\frac{1}{2}}$. Here, *, by scalars, denotes that the curvature is expressed with Otsuki's covariant differential $''D$ applied to the contravariant components of the vectors.

iii. Frenet formulas concerning covariant differential $'D$ applied on the covariant components of the tangent and normal vectors are

$$'DV_{(\alpha)i} = P^j_i\left(-\kappa^{**}_{(\alpha-1)}V_{(\alpha-1)j} + \kappa^{**}_\alpha V_{(\alpha+1)j}\right) - V_{(\alpha)r}(D\delta^a_i)Q^r_a$$

where $\alpha \in \{1, 2, ..., n\}$, $\kappa_0 = 0$, $\kappa_n = 0$, and

$$\kappa^*_\alpha = \left(g^{ij}\left('DV_{(\alpha)i} + \kappa^{**}_{\alpha-1}V_{(\alpha-1)i} + V_{(\alpha)r}{'D}\delta^r_i\right)\left('DV_{(\alpha)j} + \kappa^{**}_{\alpha-1}V_{(\alpha-1)j} + V_{(\alpha)r}{'D}\delta^r_j\right)\right)^{\frac{1}{2}}$$

Here, **, by scalars, denotes that the curvature is expressed with Otsuki's covariant differential $'D$ applied to the covariant components of the observed vectors.

iv. Frenet formulas concerning the covariant differential $''D$ applied on covariant components of the tangent and normal vectors ($''D = \overline{D}$ for covariant vectors)

$$''DV_{(\alpha)i} = \bar{D}V_{(\alpha)i} = -\kappa^{***}_{(\alpha-1)}V_{(\alpha-1)i} + \kappa^{***}_\alpha V_{(\alpha+1)i}$$

where

$$\kappa^{***}_\alpha = (g^{ij}(''DV_{(\alpha)i} + \kappa^{***}_{\alpha-1}V_{(\alpha-1)i})(''DV_{(\alpha)j} + \kappa^{***}_{\alpha-1}V_{(\alpha-1)j})^{\frac{1}{2}}$$

Here, ***, by scalars, denotes that the curvature is expressed with Otsuki's covariant differential applied to the covariant components of the observed vectors.

2.3. Bertrand Curves in Riemannian Spaces

Bertrand Curves in 3-dimensional Euclidian Space (E^3) were first introduced by Bertrand in 1850. When a curve $\overline{C}$ can be brought into a point-to-point correspondence with another curve C so that at corresponding points $\overline{P}$ and P, the curves have the same principal normal, then these curves are called Bertrand mates. Pears [14] used the infinitesimal deformations of curves to examine these curves in $\mathbb{R}^n$. He obtained the following results: If the distance between these two curves measured along the normal is small enough for its square to be neglected, then

i. Corresponding points are a fixed distance apart
ii. There are constraints on its curvature
iii. If C is a curve with constant torsion, then the tangent to $\overline{C}$ at $\overline{P}$, when parallel transported back to P, maintains a constant angle with the tangent at P

3. Bertrand Curves in $(R - O_n)$

In this section, we introduce Bertrand curves in n-dimensional Riemann-Otsuki space, defining them as curves that share the same principal normal at corresponding points. We then explore the conditions for two curves to be Bertrand mates. To establish these conditions, we use infinitesimal deformations of the curves since there is no presumption regarding the existence of such pairs of curves in the Riemann-Otsuki space. We prove the following theorem.

Theorem 3.1. Let C be a curve in n-dimensional Riemann-Otsuki space given by $C : s \to x^i(s)$ with unit tangent, 1-normal, ..., $(n-1)$-normal vectors, respectively, $V_{(1)}, V_{(2)}, ..., V_{(n)}$, and curvatures $\kappa_1, \kappa_2, ..., \kappa_{(n-1)}$. Let $\overline{x}^i(\overline{s}) = x^i(s) + \epsilon\lambda V^i_{(2)}$ be an adjacent curve denoted by $\bar{C}$, where ϵ is an infinitesimal constant for its square to be neglected and λ is a function of s. Let P and $\bar{P}$ be corresponding points on C and $\bar{C}$, respectively. Then,

i. The distance λ between corresponding points of these curves satisfies the equation $\frac{d\lambda}{ds} = -\lambda V^q_{(2)} V_{(2)i} \frac{\bar{D}\delta^i_q}{ds}$

ii. If the second curvature of the curve C, and λ are constant functions then the angle between the tangent to C at P and the tangent to $\bar{C}$ at $\bar{P}$, when parallel transported back to P, remains constant.

iii. There are restrictions, called Bertrand equations, on its curvatures.

Proof. Let C be a curve in $(R - O_n)$ space given by

$$x^i = x^i(s)$$

and let $\delta x^i = \epsilon V^i_{(2)}$ be an infinitesimal displacement at each point of this curve, where ϵ is an infinitesimal constant for its square to be neglected. Denote by $\overline{C}$ the deformed curve given by

$$\overline{x}^i(\overline{s}) = x^i(s) + \epsilon\lambda V^i_{(2)} \tag{3.1}$$

where λ is a function of s.

Let P and $\overline{P}$ be corresponding points on C and $\overline{C}$, and $\overline{V}^i_{(2)}$ the principal normal vector at $\overline{P}$. Denote by $\overline{\overline{V}}_{(2)}$ the parallel transported vector of $\overline{V}^i_{(2)}$ from $\overline{P}$ to P and then find conditions which must be satisfied for $\overline{\overline{V}}^i_{(2)} = V^i_{(2)}$.

From (3.1),

$$\frac{d\bar{x}^i}{ds} = \left(\frac{dx^i}{ds} + \epsilon\lambda \frac{dV^i_{(2)}}{ds} + \epsilon \frac{d\lambda}{ds} V^i_{(2)} \right) \frac{ds}{d\bar{s}}$$

and

$$\bar{V}^i_{(1)} = \left(V^i_{(1)} + \epsilon\lambda \frac{dV^i_{(2)}}{ds} + \epsilon \frac{d\lambda}{ds} V^i_{(2)} \right) \frac{ds}{d\bar{s}} \tag{3.2}$$

To find $\frac{ds}{d\bar{s}}$, we substitute (3.2) in

$$\bar{g}_{ij} d\bar{x}^i d\bar{x}^j = (d\bar{s})^2$$

then from (3.1),

$$\bar{g}_{ij} = g_{ij} + \frac{\partial g_{ij}}{\partial x^k} (\bar{x}^k - x^k) = g_{ij} + \epsilon\lambda (g_{i\alpha}{''}\Gamma^\alpha_{jk} + g_{j\alpha}{''}\Gamma^\alpha_{ik}) \tag{3.3}$$

From (2.2), (3.2), and (3.3),

$$\left(\frac{d\bar{s}}{ds} \right)^2 = \left(g_{ij} + \epsilon\lambda V^k_{(2)} \left(g_{ia}{''}\Gamma^\alpha_{jk} + g_{ja}{''}\Gamma^\alpha_{ik} \right) \right) \cdot \left(V^i_{(1)} V^j_{(1)} + V^i_{(1)} \epsilon\lambda \frac{dV^j_{(2)}}{ds} + V^i_{(1)} \epsilon \frac{d\lambda}{ds} V^j_{(2)} + \epsilon\lambda \frac{dV^i_{(2)}}{ds} V^j_{(1)} + \epsilon \frac{d\lambda}{ds} V^i_{(2)} V^j_{(1)} \right)$$

$$\left(\frac{d\bar{s}}{ds}\right)^2 = 1 + 2\epsilon\lambda g_{ij} V^i_{(1)} \left(\frac{dV^j_{(2)}}{ds} + {}''\Gamma^j_{k\alpha} V^k_{(2)} V^\alpha_{(1)}\right)$$

$$\left(\frac{d\bar{s}}{ds}\right)^2 = 1 + 2\epsilon\lambda g_{ij} V^i_{(1)} \left(\frac{dV^j_{(2)}}{ds} + \left({}'\Gamma^j_{ka} - \delta^j_{k|l}\right) V^k_{(2)} V^\alpha_{(1)}\right)$$

and

$$\left(\frac{d\bar{s}}{ds}\right)^2 = 1 + 2\epsilon\lambda g_{ij} V^i_{(1)} \left(\frac{\bar{D}V^j_{(2)}}{ds} - \bar{D}\delta^j_k V^k_{(2)}\right)$$

By using Frenet formulas in (2.4),

$$\left(\frac{d\bar{s}}{ds}\right)^2 = 1 - 2\epsilon\lambda\kappa_1 \quad \text{and} \quad \frac{d\bar{s}}{ds} = 1 - \epsilon\lambda\kappa_1$$

Thus,

$$\frac{ds}{d\bar{s}} = 1 + \epsilon\lambda\kappa_1 \tag{3.4}$$

Substituting (3.4) in (3.2),

$$\begin{aligned} \bar{V}^i_{(1)} &= \left(V^i_{(1)} + \epsilon\lambda\tfrac{dV^i_{(2)}}{ds} + \epsilon\tfrac{d\lambda}{ds}V^i_{(2)}\right)(1 + \epsilon\lambda\kappa_1) \\ &= V^i_{(1)}(1 + \epsilon\lambda\kappa_1) + \epsilon\lambda\tfrac{dV^i_{(2)}}{ds} + \epsilon\tfrac{d\lambda}{ds}V^i_{(2)} \end{aligned} \tag{3.5}$$

Let $\bar{\bar{V}}^i_{(1)}$ be the parallel transported vector of $\bar{V}^i_{(1)}$ from $\bar{P}$ back to P. Since the vector field $\bar{V}^i_{(1)}$ is parallel transported it satisfies the equation

$$\frac{d\bar{V}^i_{(1)}}{ds} + \bar{V}^j_{(1)}\bar{\Gamma}^i_{jk}\frac{d\bar{x}^k}{ds} = 0$$

Then, for infinitesimal deformations,

$$\bar{\bar{V}}^i_{(1)} = \bar{V}^i_{(1)} + {}'\bar{\Gamma}^i_{jk}\bar{V}^j_{(1)}(\bar{x}^k - x^k) = \bar{V}^i_{(1)} + \epsilon\lambda{}'\bar{\Gamma}^i_{jk}\bar{V}^j_{(1)}V^k_{(2)}$$

By using (3.5),

$$\bar{\bar{V}}^i_{(1)} = V^i_{(1)}(1 + \epsilon\lambda\kappa_1) + \epsilon\lambda\frac{dV^i_{(2)}}{ds} + \epsilon\frac{d\lambda}{ds}V^i_{(2)} + \epsilon\lambda\left(V^j_{(1)}(1 + \epsilon\lambda\kappa_1) + \epsilon\lambda\frac{dV^j_{(2)}}{ds} + \epsilon\frac{d\lambda}{ds}V^j_{(2)}\right)V^k_{(2)}\left({}'\Gamma^i_{jk} + \epsilon\lambda\frac{\partial{}'\Gamma^i_{jk}}{\partial x^l}V^l_{(2)}\right)$$

then since ϵ is small enough such that its square is neglected,

$$\bar{\bar{V}}^i_{(1)} = V^i_{(1)}(1 + \epsilon\kappa_1) + \epsilon\lambda\frac{\bar{D}V^i_{(2)}}{ds} + \epsilon\frac{d\lambda}{ds}V^i_{(2)}$$

By using the Frenet formulas in (2.4),

$$\bar{\bar{V}}^i_{(1)} = V^i_{(1)} + \epsilon\lambda\kappa_2 V^i_{(3)} + \epsilon\left(\lambda V^q_{(2)}\frac{\bar{D}\delta^i_q}{ds} + \frac{d\lambda}{ds}V^i_{(2)}\right) \tag{3.6}$$

If we multiply (3.6) by $V_{(2)i}$, since the angles are unchanged during the parallel transport, then

$$\frac{d\lambda}{ds} = -\lambda V^q_{(2)}V_{(2)i}\frac{\bar{D}\delta^i_q}{ds} \tag{3.7}$$

which proves *i*.

If we use (3.7) in (3.6), then

$$\bar{\bar{V}}^i_{(1)} = V^i_{(1)} + \epsilon\lambda\kappa_2 V^i_{(3)}$$

From here, we find $\sin\left(\bar{\bar{V}}_{(1)}, V_{(1)}\right) = \epsilon\lambda\kappa_2$. Hence, the angle between $\bar{\bar{V}}_{(1)}$ and $V_{(1)}$ is constant if and only if λ and κ_2 are constant, which proves the part *ii.*

To determine $\bar{V}_{(2)}$, by using the definition of basic covariant differentiation and the Frenet formulas,

$$\frac{dV^i_{(\alpha)}}{ds} = -\kappa_{\alpha-1}V^i_{(\alpha-1)} + \kappa_\alpha V^i_{(\alpha+1)} + V^q_{(\alpha)}\bar{D}\delta^i_q - {}'\Gamma^i_{jk}V^j_{(\alpha)}V^k_{(1)} \tag{3.8}$$

If we substitute (3.7) into (3.5) and use (3.8), then

$$\bar{V}^i_{(1)} = V^i_{(1)} + \epsilon\lambda\kappa_2 V^i_{(3)} - \epsilon\lambda'\Gamma^i_{jk}V^j_{(2)}V^k_{(1)} \tag{3.9}$$

If we differentiate $\bar{V}^i_{(1)}$ by using the basic covariant differentiation, then

$$\bar{D}\bar{V}^i_{(1)} = \frac{d\bar{V}^i_{(1)}}{ds}\frac{ds}{d\bar{s}} + {}'\bar{\Gamma}^i_{jk}\bar{V}^j_{(1)}\bar{V}^k_{(1)}$$

Then, by using (3.9), (3.4), and Frenet equations,

$$\begin{aligned}\bar{\kappa}_1\bar{V}^i_{(2)} + \bar{V}^q_{(1)}D\delta^i_q = &\left[\frac{d}{ds}\left(V^i_{(1)} + \epsilon\lambda\kappa_2V^i_{(3)} - \epsilon\lambda'\Gamma^i_{jk}V^j_{(2)}V^k_{(1)}\right)\right](1+\epsilon\lambda\kappa_1)\\ &+\left[{}'\Gamma^i_{jk} + \epsilon\lambda\frac{\partial'\Gamma^i_{jk}}{\partial x^l}V^l_{(2)}\right]\left(V^j_{(1)} + \epsilon\lambda\kappa_2V^j_{(3)} - \epsilon\lambda'\Gamma^j_{rs}V^r_{(2)}V^s_{(1)}\right)\left(V^k_{(1)} + \epsilon\lambda\kappa_2V^k_{(3)} - \epsilon\lambda'\Gamma^k_{rs}V^r_{(2)}V^s_{(1)}\right)\end{aligned}$$

Thus,

$$\begin{aligned}\bar{\kappa}_1\bar{V}^i_{(2)} + \left(V^q_{(1)} + \epsilon\lambda\kappa_2V^q_{(3)} - \epsilon\lambda'\Gamma^q_{jk}V^j_{(2)}V^k_{(1)}\right)D\delta^i_q = &\tfrac{dV^i_{(1)}}{ds} + \epsilon\lambda\tfrac{d\kappa_2}{ds}V^i_{(3)} + \epsilon\tfrac{d\lambda}{ds}\kappa_2V^i_{(3)} + \epsilon\lambda\kappa_2\tfrac{dV^i_{(3)}}{ds} - \epsilon\tfrac{d\lambda}{ds}{}'\Gamma^i_{jk}V^j_{(2)}V^k_{(1)}\\ &-\epsilon\lambda\tfrac{\partial'\Gamma^i_{jk}}{\partial x^l}V^l_{(1)}V^j_{(2)}V^k_{(1)} - \epsilon\lambda'\Gamma^i_{jk}\tfrac{dV^j_{(2)}}{ds}V^k_{(1)} - \epsilon\lambda'\Gamma^i_{jk}V^j_{(2)}\tfrac{dV^k_{(1)}}{ds}\\ &+\epsilon\lambda\kappa_1\tfrac{dV^i_{(1)}}{ds} + {}'\Gamma^i_{jk}V^j_{(1)}V^k_{(1)} + \Gamma^i_{jk}\epsilon\lambda\kappa_2V^j_{(1)}V^k_{(3)} - \epsilon\lambda'\Gamma^i_{jk}{}'\Gamma^k_{rs}V^j_{(1)}V^r_{(2)}V^s_{(1)}\\ &+\epsilon\lambda\kappa_2{}'\Gamma^i_{jk}V^j_{(3)}V^k_{(1)} - \epsilon\lambda'\Gamma^i_{jk}{}'\Gamma^j_{rs}V^s_{(1)}V^r_{(2)}V^k_{(1)} + \epsilon\lambda\tfrac{\partial\Gamma^i_{jk}}{\partial x^l}V^{(2)}_lV^j_{(1)}V^k_{(1)}\end{aligned}$$

since ϵ is an infinitesimal constant, which is square to be neglected. By using Frenet equations, after some algebraic operations,

$$\begin{aligned}\bar{\kappa}_1\bar{V}^i_{(2)} - \epsilon\lambda'\Gamma^q_{jk}V^j_{(2)}V^k_{(1)}D\delta^i_q =&\kappa_1V^i_{(2)} + \epsilon\lambda\left((\kappa_1^2-\kappa_2^2)V^i_{(2)} + \frac{d\kappa_2}{ds}V^i_{(3)} + \kappa_2\kappa_3V^i_{(4)} - \kappa'\Gamma^i_{jk}V^j_{(2)}V^k_{(2)}\right)\\ &-\epsilon\lambda\left(-\frac{\partial'\Gamma^i_{kl}}{\partial x^j} + \frac{\partial'\Gamma^i_{jl}}{\partial x^k} + {}'\Gamma^i_{ak}{}'\Gamma^a_{jl} - {}'\Gamma^i_{al}{}'\Gamma^a_{jk}\right)V^j_{(1)}V^k_{(2)}V^l_{(1)}\\ &+\epsilon\frac{d\lambda}{ds}\kappa_2V^i_{(3)} - \epsilon\kappa_1V^i_{(1)}\frac{d\lambda}{ds}\end{aligned}$$

Then, by substituting (2.3),

$$\begin{aligned}\bar{\kappa}_1\bar{V}^i_{(2)} =&\kappa_1V^i_{(2)} + \epsilon\lambda\left((\kappa_1^2-\kappa_2^2)V^i_{(2)} + \frac{d\kappa_2}{ds}V^i_{(3)} + \kappa_2\kappa_3V^i_{(4)} - \kappa_1{}'\Gamma^i_{jk}V^j_{(2)}V^k_{(2)} + R^i_{jkl}V^j_{(1)}V^k_{(2)}V^l_{(1)}\right)\\ &+\epsilon\frac{d\lambda}{ds}\kappa_2V^i_{(3)} - \epsilon\kappa_1V^i_{(1)}\frac{d\lambda}{ds}\end{aligned} \tag{3.10}$$

Since $\bar{\bar{V}}_{(2)}$ is the parallel transport of the normal vector $\bar{V}_{(2)}$ of $\bar{C}$ at the point $\bar{P}$ to the point P of C,

$$\bar{\bar{V}}^i_{(2)} = \bar{V}^i_{(2)} + {}'\bar{\Gamma}^i_{jk}V^j_{(2)}(\bar{x}^k - x^k) = \bar{V}^i_{(2)} + \epsilon\lambda'\bar{\Gamma}^i_{jk}V^j_{(2)}V^k_{(2)} \tag{3.11}$$

From (3.11),

$$\bar{V}^i_{(2)} = \bar{\bar{V}}^i_{(2)} - \epsilon\lambda'\bar{\Gamma}^i_{jk}V^j_{(2)}V^k_{(2)}$$

and

$$\begin{aligned}\bar{\kappa}_1\bar{V}^i_{(2)} &= \bar{\kappa}_1\left(\bar{\bar{V}}^i_{(2)} - \epsilon\lambda{}'\Gamma^i_{jk}V^j_{(2)}V^k_{(2)}\right) = \bar{\kappa}_1\bar{V}^i_{(2)} - \epsilon\lambda\bar{\kappa}_1{}'\Gamma^i_{jk}\bar{V}^j_{(2)}V^k_{(2)}\\ &= \bar{\kappa}_1\bar{V}^i_{(2)} - \epsilon\lambda\left(\kappa_1 + \epsilon\lambda\tfrac{\partial\kappa_1}{\partial x^l}V^l_{(1)}V^k_{(1)}\right){}'\Gamma^i_{jk}\bar{V}^j_{(2)}V^k_{(2)}\\ &= \bar{\kappa}_1\bar{V}^i_{(2)} - \epsilon\lambda\kappa_1{}'\Gamma^i_{jk}\bar{V}^j_{(2)}V^k_{(2)}\end{aligned}$$

Subtituting this equation in (3.10),

$$\begin{aligned}\bar{\kappa}_1\bar{\bar{V}}^i_{(2)} - \epsilon\lambda\kappa_1{}'\Gamma^i_{jk}V^j_{(2)}V^k_{(2)} &= \kappa_1 V^i_{(2)}\\ &+\epsilon\lambda\left(\left(\kappa_1^2-\kappa_2^2\right)V^i_{(2)} + \tfrac{d\kappa_2}{ds}V^i_{(3)} + \kappa_2\kappa_3 V^i_{(4)} + {}'R^i_{jkl}V^j_{(1)}V^k_{(2)}V^l_{(1)} - \kappa_1\Gamma^i_{jk}V^j_{(2)}V^k_{(2)}\right)\\ &+\epsilon\tfrac{d\lambda}{ds}\kappa_2 V^i_{(3)} - \epsilon\kappa_1 V^i_{(1)}\tfrac{d\lambda}{ds}\end{aligned}$$

or

$$\begin{aligned}\bar{\kappa}_1\bar{\bar{V}}^i_{(2)} &= \kappa_1 V^i_{(2)} + \epsilon\lambda\left(\left(\kappa_1^2-\kappa_2^2\right)V^i_{(2)} + \tfrac{d\kappa_2}{ds}V^i_{(3)} + \kappa_2\kappa_3V^i_{(4)} + {}'R^i_{jkl}V^j_{(1)}V^k_{(2)}V^l_{(1)}\right)\\ &+\epsilon\tfrac{d\lambda}{ds}\kappa_2V^i_{(3)} - \epsilon\kappa_1V^i_{(1)}\tfrac{d\lambda}{ds}\end{aligned} \tag{3.12}$$

Multiplying (3.12) by $V_{(3)i}$,

$$\begin{aligned}\bar{\kappa}_1\bar{\bar{V}}^i_{(2)}V_{(3)i} &= \kappa_1 V^i_{(2)}V_{(3)i}\\ &+\epsilon\lambda\left(\left(\kappa_1^2-\kappa_2^2\right)V^i_{(2)}V_{(3)i} + \tfrac{d\kappa_2}{ds}V^i_{(3)}V_{(3)i} + \kappa_2\kappa_3V^i_{(4)}V_{(3)i} + {}'R^i_{jkl}V^j_{(1)}V^k_{(2)}V^l_{(1)}V_{(3)i}\right)\\ &+\epsilon\tfrac{d\lambda}{ds}\kappa_2V^i_{(3)}V_{(3)i} - \epsilon\kappa_1V^i_{(1)}V_{(3)i}\tfrac{d\lambda}{ds}\end{aligned}$$

and

$$0 = \epsilon\lambda\left(\frac{d\kappa_2}{ds} + {}'R^i_{jkl}V^j_{(1)}V^k_{(2)}V^l_{(1)}V_{(3)i}\right) + \epsilon\frac{d\lambda}{ds}\kappa_2$$

Then, by using (3.7) and the notation ${}'R^i_{jkl}V^j_{(1)}V^k_{(2)}V^l_{(1)}V_{(3)i} = {}'\gamma_{3121}$,

$${}'\gamma_{3121} = -\frac{d\kappa_2}{ds} + V^q_{(2)}V_{(2)i}\frac{\bar{D}\delta^i_q}{ds}\kappa_2 \tag{3.13}$$

Multiplying (3.12) by $V_{(4)i}$,

$$\kappa_2\kappa_3 + {}'\gamma_{4121} = 0$$

or

$${}'\gamma_{4121} = -\kappa_2\kappa_3 \tag{3.14}$$

Finally, multiplying (3.12) by $V_{(p)i}$, for $p > 4$,

$${}'\gamma_{p121} = 0 \quad p > 4 \tag{3.15}$$

(3.13)-(3.15) represent the $n-2$ Bertrand equations, which establish the relationship between $n-1$ curvatures of the curve C in n-dimensional Riemann-Otsuki space, thereby proving part *iii* of the theorem. □

4. Conclusion

In this study, we use the infinitesimal deformations of curves to examine the properties of Bertrand curves in n-dimensional Riemann-Otsuki space, as there is no assumption regarding the existence of these curves within this space. Theorem 3.1 provides the conditions for such curves' existence. From part i of Theorem 3.1, we conclude that the distance λ between corresponding points of the Bertrand mates satisfies the differential equation (3.7). This condition corresponds to λ being constant in $\mathbb{R}^n$.

From part *ii* of Theorem 3.1, we deduce that the angle between tangents at corresponding points remains constant if both the second curvature κ_2 and λ are constant functions. In part *iii* of Theorem 3.1, we derive the $n-2$ Bertrand relations, as given by (3.13)-(3.15), which relate the $n-1$ curvatures of the given curve C in n-dimensional Riemann-Otsuki space. These relations are consistent with the corresponding relations in Riemannian space, with $\frac{\bar{D}\delta^i_q}{ds}=0$, obtained by Pears [14]. Furthermore, applying these results to Euclidean space yields the well-known result in $\mathbb{E}^3$, κ_2 being constant. In $\mathbb{E}^n$ for $n \geq 4$, we have $\kappa_2 = 0$ or $\kappa_3 = 0$, implying that the Bertrand curves in $\mathbb{E}^n$ for $n \geq 4$ are degenerate.

Future research may focus on $(1,3)-V$ Bertrand curves in the Riemann-Otsuki space. These curves, as introduced in [17], are characterized by the property that the plane spanned by the principal normal and second binormal of the curve coincides with the plane spanned by the principal normal and second binormal of its Bertrand mate. Moreover, the focus of interest is other unique curves in the Riemann-Otsuki space.

Author Contributions

All the authors contributed equally to this work. The first author's doctoral dissertation, supervised by the second author (late), is the basis for this paper. They all read and approved the final version of the paper.

Conflicts of Interest

All the authors declare no conflict of interest.

Ethical Review and Approval

No approval from the Board of Ethics is required.

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