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EULER-LAGRANGE EQUATIONS ON THREE-DIMENSIONAL SPACE

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Abstract: In this article, it is aimed to introduce the Euler-Lagrange equations using a three-dimensional space for mechanical systems. In addition to, the geometrical-physical results related to three-dimensional space for mechanical systems are also given.

Keywords: *Euler-Lagrange, Mechanic, System, Space, Formalism*.

MSC 2010: 70H03, 35Q70, 82C21.

ÜÇ BOYUTLU UZAYDA EULER-LAGRANGE DENKLEMLERİ

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Özet: Bu makale ile üç boyutlu uzay kullanılarak mekanik sistemler için Euler-Lagrange denklemlerini tanıtmak amaçlanmıştır. Ek olarak, üç boyutlu uzaydaki mekanik sistemler için geometrik ve fiziksel sonuçlar da verilmiştir.

Anahtar Kelimeler: *Euler-Lagrange, Mekanik, Sistem, Uzay, Formalizm*.

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1. INTRODUCTION

Today, there is helpful use in differential geometry for applied sciences. Differential geometry has a lots of different applications in the branches of science. These applications, came into our lives, are used in many areas and the popular science. We can say that differential geometry provides a good working area for studying Lagrangians of classical mechanics and field theory. That dynamic equation for moving bodies is obtained for Lagrangian mechanics. This dynamic equation is illustrated as follows:

2. LAGRANGE DYNAMICS EQUATION [1,2,3].

Let M be an n-dimensional manifold and TM its tangent bundle with canonical projection $\tau_M : TM \to TM$. TM is called the phase space of velocities of the base manifold M. Let $L : TM \to \mathbb{R}$ be a differentiable function on TM called the Lagrangian function. We consider the closed 2-form on TM given by $\Phi_L = -dd_J L.$ (1)

Consider the equation

$$i_x \Phi_L = dE_L. \tag{2}$$

Then X is a vector field and i_x is reduction function that it is $i_x \Phi_L = \Phi_L(X)$. We shall see that (2) under a certain condition on X is the intrinsical expression of the Euler-Lagrange equations of motion. This equation is named as **Lagrange dynamical equation**. We shall see that for motion in a potential,

$$E_L = V(L) - L \tag{3}$$

is an energy function and $V = J(X)$ a Liouville vector field. Here dE_L denotes the differential of E. The triple (TM, Φ_L, X) is known as Euler-Lagrangian system on the tangent bundle TM. If it is continued the operations on (2) for any coordinate system $(q^i(t), p_i(t))$, infinite dimension Lagrange's equation is obtained the form below:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}^i}\right) = \frac{\partial L}{\partial q^i}, \quad \frac{\partial q^i}{\partial t} = \dot{q}^i, \; i = 1, ..., n. \tag{4}$$

There are many studies about Euler-Lagrangian dynamics, mechanics, formalisms, systems and equations. There are real, complex, paracomplex and other analogues for these studies. It is well-known that Euler-Lagrangian analogues are very important tools. They have a simple method to describe the model for mechanical systems. The models about mechanical systems are given as follows. Some examples of the Euler-Lagrangian is applied to model the problems include harmonic oscillator, charge Q in electromagnetic fields, Kepler problem of the earth in orbit around the sun, pendulum, molecular and fluid dynamics, LC networks, Atwood's machine, symmetric top etc. Previous works done on that subject can be given as follows.

Vries examined that the Lagrangian motion equations have a very simple interpretation in relativistic quantum mechanics [4]. *Tekkoyun* showed that paracomplex analogue of the Euler-Lagrange equations was obtained in the framework of para-Kählerian manifold and the geometric results on a paracomplex mechanical systems were found [5]. *Liu* studied that electronic origins, molecular dynamics simulations, computational nanomechanics, multiscale modelling of materials fields were contributed [6]. Bi-paracomplex analogue of Lagrangian systems was shown on Lagrangian distributions by *Tekkoyun* and *Sari* [7]. *Tekkoyun* and *Yayli* presented generalized-quaternionic Kählerian analogue of Lagrangian and Hamiltonian mechanical systems. Eventually, the geometric-physical results related to generalized-quaternionic Kählerian mechanical systems are provided [8]. *Kasap* and *Tekkoyun* obtained Lagrangian and Hamiltonian formalism for mechanical systems using para/pseudo-Kähler manifolds, representing an interesting multidisciplinary field of research. Also, the geometrical, relativistically, mechanical and physical results related to para/pseudo-Kähler mechanical systems were given, too [9]. *Enge* and *Maiber* proposed a method for modelling electromechanical systems (EMS) with variable structure in the electrical subsystem [10].

In the present paper, we will present equations related to Euler-Lagrangian mechanical systems on three-dimensional space.

3. PRELIMINARIES

In this study, all the manifolds and geometric objects are of $C^{\infty}(TM)$. The Einstein summation convention $\sum_{j=1}^{n} a_j x^j = a_j x^j$ will in use. Also, TM is tangent manifold, T^*M is cotangent manifold and M will always denote an n-dimensional smooth manifold. In addition to, $\{X,Y\}$, A, $F(TM)$, $\chi(TM)$ and $\wedge^1(TM)$ denote the set of vector fields, paracomplex numbers, the set of (para)-complex functions on TM, the set of (para)-complex vector fields on TM and the set of (para)-complex 1-forms on TM, respectively.

4. J-HOLOMORPHIC CURVES

A pseudoholomorphic curve (or J-holomorphic curve) is a smooth map from a Riemann surface into an almost complex manifold that satisfies the Cauchy-Riemann equation. Introduced in 1985 by Gromov, pseudoholomorphic curves have since revolutionized the study of symplectic manifolds. The theory of J holomorphic curves is one of the new techniques which have recently revolutionized the study of symplectic geometry, making it possible to study the global structure of symplectic manifolds. The methods are also of interest in the study of Kähler manifolds, since often when one studies properties of holomorphic curves in such manifolds it is necessary to perturb the complex structure to be generic. The effect of this is to ensure that one is looking at persistent rather than accidental features of these curves. However, the perturbed structure may no longer be integrable, and so again one is led to the study of curves which are holomorphic with respect to some non-integrable almost complex structure J [11]. A complex-valued function f of a complex variable z is said to be holomorphic at a point a if it is differentiable at every point within some open disk centered at a Negative curvature (pseudosphere).

5. THE CAUCHY-RIEMANN EQUATION

The Cauchy-Riemann differential equations in complex analysis consist of a system of two partial differential equations which must be satisfied if it is know that a complex function is complex differentiable. Moreover, the equations are necessary and sufficient conditions for complex differentiation once it seen that its real and
imaginary parts are differentiable real functions of two variables. The Cauchy-Riemann equations on a pair of real-valued functions of two real variables $u(x,y)$ and $v(x,y)$ are the two equations:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \ \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} \quad (5)$$

Typically u and v are taken to be the real and imaginary parts respectively of a complex-valued function of a single complex variable

$$z = x + iy; \quad f(x+iy) = u(x,y) + i\,v(x,y) \quad (6)$$

6. SYMPLECTIC GEOMETRY

A symplectic manifold is a smooth manifold (M) equipped with a closed nondegenerate differential 2-form ω called the symplectic form. The study of symplectic manifolds is called symplectic geometry or symplectic topology. Symplectic manifolds arise naturally in abstract formulations of classical mechanics and analytical mechanics as the cotangent bundles of manifolds which provides one of the major motivations for the field. The set of all possible configurations of a system is modelled as a manifold, and cotangent bundle of this manifold describes the phase space of the system. The basic example of an almost complex symplectic manifold is standard Euclidean space $\left(R^{2n},\omega_0\right)$ with its standard almost complex structure J_0 obtained from the usual identification with C^n. Thus, one sets

$$z_{\mathrm{j}} = x_{2j-1} + i\,x_{2j} \tag{7}$$

for $j = 1,...,n$ and defines J_0 by

$$J_0\left(\partial_{2j-1}\right) = \partial_{2j},\quad J_0\left(\partial_{2j}\right) = -\partial_{2j-1} \tag{8}$$

where $\partial_j = \partial / \partial x_j$ is the standard basis of $T_x R^{2n}$ [11].

7. ALMOST (PARA) COMPLEX STRUCTURE

Let V be a vector space over R. Recall that a *complex structure* on V is a linear operator J on V such that $J^2 = -I$, where $J^2 = J \circ J$, and I is the identity operator on V. A prototypical example of a complex structure is given by the map $J : V \to V$ defined by $J(v,w) = (-w,v)$ where $V = R^n \oplus R^n$. An *almost complex structure* on a manifold M is a differentiable map $J : TM \to TM$ on the tangent bundle TM of M such that J preserves each fiber. If $J^2 = I$, J is a *paracomplex structure*, $Tr(J) = 0$.

A celebrated theorem of Newlander and Nirenberg [12] says that an almost complex structure is a complex structure if and only if its Nijenhuis tensor or torsion N vanishes, where, for vector fields X and Y on M.

Theorem : The almost complex structure J on M is integrable if and only if the tensor N_J vanishes identically, where N_J is defined on two vector fields X and Y by

$$N_J(X,Y) = [JX, JY] - J[JX,Y] - J[X,JY] - [X,Y]. \tag{9}$$

The tensor (2,1) is called the Nijenhuis tensor (9). We say that J is torsion free if $N_J = 0$. Paracomplex Nijenhuis tensor of an almost (para)-complex manifold (M,J) is given by (9). Let $(x_1,...,x_{2n})$ be a local coordinate system. The torsion tensor is bilinear, for if $X = \dfrac{\partial}{\partial x_j}$ and $Y = \dfrac{\partial}{\partial x_k}$ are vector fields and J_j^i are the components of J, then by direct calculation the i^{th} component of the torsion tensor is given by

$$\begin{aligned} N\left(\frac{\partial}{\partial x_j},\frac{\partial}{\partial x_k}\right)^i &= N_{jk}^i \\ &= \sum_{h=0}^{2n}\left(J_j^h\partial_h J_k^i - J_k^h\partial_h J_j^i - J_h^i\partial_j J_k^h - J_h^i\partial_k J_j^h\right) \end{aligned} \tag{10}$$

where ∂_h denotes partial differentiation ∂_{x_h}. It disappears if and only if J is an integrable almost (para)-complex structure, i.e. given any point $P \in N$, there are local coordinates which are centered at P, so

$$\begin{aligned} &I.\ J\left(\frac{\partial}{\partial x}\right) = \frac{\partial}{\partial y} + \frac{\partial}{\partial z}, \\ &II.\ J\left(\frac{\partial}{\partial y}\right) = \frac{\partial}{\partial x} + \frac{\partial}{\partial z}, \\ &III.\ J\left(\frac{\partial}{\partial z}\right) = -\frac{\partial}{\partial z}. \end{aligned} \tag{11}$$

Holomorphic properties of these structures are

$$
\begin{aligned}
I.\ J^2\left(\frac{\partial}{\partial x}\right) &= J\circ J\left(\frac{\partial}{\partial x}\right) = J\left(\frac{\partial}{\partial y}\right) + J\left(\frac{\partial}{\partial z}\right) \\
&= \frac{\partial}{\partial x} + \frac{\partial}{\partial z} - \frac{\partial}{\partial z} = \frac{\partial}{\partial x} \\
II.\ J^2\left(\frac{\partial}{\partial y}\right) &= J\circ J\left(\frac{\partial}{\partial y}\right) = J\left(\frac{\partial}{\partial x}\right) + J\left(\frac{\partial}{\partial z}\right) \\
&= \frac{\partial}{\partial y} + \frac{\partial}{\partial z} - \frac{\partial}{\partial z} = \frac{\partial}{\partial y} \\
III\ J^2\left(\frac{\partial}{\partial z}\right) &= J\circ J\left(\frac{\partial}{\partial z}\right) = -J\left(\frac{\partial}{\partial z}\right) = \frac{\partial}{\partial z}.
\end{aligned}
\tag{12}
$$

As it is seen from above, they are paracomplex structures. The system is based on three variables and three-dimensional for (x, y, z). In this study, above holomorpfic structures will be used.

8. LAGRANGIAN EQUATIONS

In this section, we get Euler-Lagrange equations for quantum and classical mechanics on three-dimensional space. Firstly, take J as the local basis element on three-dimensional space and (x, y, z) be its coordinate functions on three-dimensional space. Let ξ be the vector field decided by

$$\xi = X\frac{\partial}{\partial x} + Y\frac{\partial}{\partial y} + Z\frac{\partial}{\partial z}. \tag{13}$$

The vector field described by

$$
\begin{aligned}
V = J(\xi) &= J\left(X\frac{\partial}{\partial x} + Y\frac{\partial}{\partial y} + Z\frac{\partial}{\partial z}\right) \\
&= X\left(\frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right) + Y\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial z}\right) - Z\frac{\partial}{\partial z}
\end{aligned}
\tag{14}
$$

is said to be Liouville vector field on three-dimensional space. Three-dimensional space form is the closed 2-form which is given by $\Phi_L = -dd_J L$ so that

$$d_J = \left(\frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)dx + \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial z}\right)dy - \frac{\partial}{\partial z}dz \tag{15}$$

If we use (2), we obtain the equations given by

$$
\begin{aligned}
\Phi_L &= \left(\frac{\partial^2 L}{\partial x\partial y} + \frac{\partial^2 L}{\partial x\partial z}\right)dx\wedge dx \\
&+ \left(\frac{\partial^2 L}{\partial x\partial x} + \frac{\partial^2 L}{\partial x\partial z}\right)dy\wedge dx - \frac{\partial^2 L}{\partial x\partial z}dz\wedge dx \\
&+ \left(\frac{\partial^2 L}{\partial y\partial y} + \frac{\partial^2 L}{\partial y\partial z}\right)dx\wedge dy \\
&+ \left(\frac{\partial^2 L}{\partial y\partial x} + \frac{\partial^2 L}{\partial y\partial z}\right)dy\wedge dy - \frac{\partial^2 L}{\partial y\partial z}dz\wedge dy \\
&+ \left(\frac{\partial^2 L}{\partial z\partial y} + \frac{\partial^2 L}{\partial z\partial z}\right)dx\wedge dz \\
&+ \left(\frac{\partial^2 L}{\partial z\partial x} + \frac{\partial^2 L}{\partial z\partial z}\right)dy\wedge dz - \frac{\partial^2 L}{\partial z\partial z}dz\wedge dz.
\end{aligned}
\tag{16}
$$

The following processes

$$
\begin{aligned}
&1.\ f\wedge g = -g\wedge f \\
&2.\ f\wedge g(v) = f(v)g - g(v)f \\
&3.\ \frac{\partial x}{\partial x} = \delta_x^x = 1,\ \frac{\partial x}{\partial y} = \delta_y^x = 0
\end{aligned}
\tag{17}
$$

considering the external product and kronecker delta features that then we calculated

$$
\begin{aligned}
i_\xi\Phi_L = \Phi_L(\xi) = {}& \\
+X&\left[\begin{array}{l}\left(\frac{\partial^2 L}{\partial x\partial y} + \frac{\partial^2 L}{\partial x\partial z}\right)dx - \left(\frac{\partial^2 L}{\partial x\partial y} + \frac{\partial^2 L}{\partial x\partial z}\right)dx \\ -\left(\frac{\partial^2 L}{\partial x\partial x} + \frac{\partial^2 L}{\partial x\partial z}\right)dy + \frac{\partial^2 L}{\partial x\partial z}dz + \left(\frac{\partial^2 L}{\partial z\partial y} + \frac{\partial^2 L}{\partial z\partial z}\right)dz\end{array}\right] \\
+Y&\left[\begin{array}{l}-\left(\frac{\partial^2 L}{\partial y\partial y} + \frac{\partial^2 L}{\partial y\partial z}\right)dx + \left(\frac{\partial^2 L}{\partial y\partial x} + \frac{\partial^2 L}{\partial y\partial z}\right)dy \\ -\left(\frac{\partial^2 L}{\partial y\partial x} + \frac{\partial^2 L}{\partial y\partial z}\right)dy + \frac{\partial^2 L}{\partial y\partial z}dz + \left(\frac{\partial^2 L}{\partial z\partial x} + \frac{\partial^2 L}{\partial z\partial z}\right)dz\end{array}\right] \\
+Z&\left[\begin{array}{l}-\frac{\partial^2 L}{\partial x\partial z}dx - \frac{\partial^2 L}{\partial y\partial z}dy - \left(\frac{\partial^2 L}{\partial z\partial y} + \frac{\partial^2 L}{\partial z\partial z}\right)dx \\ -\left(\frac{\partial^2 L}{\partial z\partial x} + \frac{\partial^2 L}{\partial z\partial z}\right)dy - \frac{\partial^2 L}{\partial z\partial z}dz + \frac{\partial^2 L}{\partial z\partial z}dz\end{array}\right].
\end{aligned}
\tag{18}
$$

Energy function and its differential can be written as in the following:

$$
\begin{aligned}
E_L &= J(\xi) - L \\
&= X\left(\frac{\partial L}{\partial y} + \frac{\partial L}{\partial z}\right) + Y\left(\frac{\partial L}{\partial x} + \frac{\partial L}{\partial z}\right) - Z\frac{\partial L}{\partial z} - L
\end{aligned}
\tag{19}
$$

Its differential form is as follows:

$$
\begin{aligned}
dE_L = {} & X\left(\frac{\partial^2 L}{\partial x \partial y} + \frac{\partial^2 L}{\partial x \partial z}\right)dx + Y\left(\frac{\partial^2 L}{\partial x \partial x} + \frac{\partial^2 L}{\partial x \partial z}\right)dx \\
& - Z\frac{\partial^2 L}{\partial x \partial z}dx - \frac{\partial L}{\partial x}dx + X\left(\frac{\partial^2 L}{\partial y \partial y} + \frac{\partial^2 L}{\partial y \partial z}\right)dy \\
& + Y\left(\frac{\partial^2 L}{\partial y \partial x} + \frac{\partial^2 L}{\partial y \partial z}\right)dy - Z\frac{\partial^2 L}{\partial y \partial z}dy - \frac{\partial L}{\partial y}dy \\
& + X\left(\frac{\partial^2 L}{\partial z \partial y} + \frac{\partial^2 L}{\partial z \partial z}\right)dz + Y\left(\frac{\partial^2 L}{\partial z \partial x} + \frac{\partial^2 L}{\partial z \partial z}\right)dz \\
& - Z\frac{\partial^2 L}{\partial z \partial z}dz - \frac{\partial L}{\partial z}dz.
\end{aligned} \tag{20}
$$

If we use (2) we obtain the equations given by

$$
\begin{aligned}
I.\; & -X\frac{\partial^2 L}{\partial x \partial y} - X\frac{\partial^2 L}{\partial x \partial z} - Y\frac{\partial^2 L}{\partial y \partial y} - Y\frac{\partial^2 L}{\partial y \partial z} \\
& - Z\frac{\partial^2 L}{\partial z \partial y} - Z\frac{\partial^2 L}{\partial z \partial z} = -\frac{\partial L}{\partial x}dx, \\
II.\; & -X\frac{\partial^2 L}{\partial x \partial x} - X\frac{\partial^2 L}{\partial x \partial z} - Y\frac{\partial^2 L}{\partial y \partial x} - Y\frac{\partial^2 L}{\partial y \partial z} \\
& - Z\frac{\partial^2 L}{\partial z \partial x} - Z\frac{\partial^2 L}{\partial z \partial z} = -\frac{\partial L}{\partial y}dy \\
III.\; & X\frac{\partial^2 L}{\partial x \partial z} + Y\frac{\partial^2 L}{\partial y \partial z} + Z\frac{\partial^2 L}{\partial z \partial z} = -\frac{\partial L}{\partial z}dz.
\end{aligned} \tag{21}
$$

Considering the curve α , an integral curve of ξ i.e. $\xi(\alpha(t)) = \dfrac{\partial \alpha}{\partial t}$ we can find the equations as follows:

$$
\begin{aligned}
I.\; & -\xi\left(\frac{\partial L}{\partial y}\right) - \xi\left(\frac{\partial L}{\partial z}\right) + \frac{\partial L}{\partial x} = 0, \\
II.\; & -\xi\left(\frac{\partial L}{\partial x}\right) - \xi\left(\frac{\partial L}{\partial z}\right) + \frac{\partial L}{\partial y} = 0, \\
III.\; & \xi\left(\frac{\partial L}{\partial z}\right) + \frac{\partial L}{\partial z} = 0,
\end{aligned} \tag{22}
$$

or using the definition of the integral curve

$$
\begin{aligned}
I.\; & -\frac{\partial}{\partial t}\left(\frac{\partial L}{\partial y}\right) - \frac{\partial}{\partial t}\left(\frac{\partial L}{\partial z}\right) + \frac{\partial L}{\partial x} = 0, \\
II.\; & -\frac{\partial}{\partial t}\left(\frac{\partial L}{\partial x}\right) - \frac{\partial}{\partial t}\left(\frac{\partial L}{\partial z}\right) + \frac{\partial L}{\partial y} = 0, \\
III.\; & \frac{\partial}{\partial t}\left(\frac{\partial L}{\partial z}\right) + \frac{\partial L}{\partial z} = 0,
\end{aligned} \tag{23}
$$

such that these equations are called ***Euler–Lagrange equations*** on three-dimensional space. Thus the triple (M, Φ_L, ξ) is named as a ***Euler–Lagrange mechanical system*** on three-dimensional space.

9. EQUATIONS CLOSED SOLUTION

These partial differential equations (23) are depending on time. We can solve these equations using symbolic computation program. The software codes and solutions of these equations as follows:

I. Codes:
-diff(diff(L1(x,y,z,t),y),t)-diff(diff(L1(x,y,z,t),z),t)+diff(L1(x,y,z,t),x) =0;
Solutions:
L1(x,y,z,t)=F1(x)*F2(y)*F3(z)*F4(t);
diff(F1(x),x)=c1*F1(x), diff(F2(y),y)=c2*F2(y),
diff(F3(z),z)=c3*F3(z),
diff(F4(t),t)=-c1*F4(t)/(-c2-c3).

II. Codes:
-diff(diff(L2(x,y,z,t),x),t)-diff(diff(L2(x,y,z,t),z),t)+diff(L2(x,y,z,t),y) =0;
Solutions:
L2(x,y,z,t)=F1(x)*F2(y)*F3(z)*F4(t);
diff(F1(x),x)=c1*F1(x),
diff(F2(y),y)=c2*F2(y),
diff(F3(z),z)=c3*F3(z),
diff(F4(t),t)=-c2*F4(t)/(-c1-c3)

III. Codes:
diff(diff(L3(x,y,z,t),z),t)+ diff(L3(x,y,z,t),z)=0;
Solutions:
L3(x,y,z,t)=F1(t,y,x)+exp(-t)*F2(z,y,x) (24)

10. CONCLUSION

The equations found by (23) easily seen extremely useful in applications from Euler-Lagrangian mechanics, quantum physics, optimal control, biology and fluid dynamics. The obtained equations are very important to explain the rotational spatial mechanical-physical problems. For this reason, the obtained equations are only considered to be a first step to realize how a generalized on three-dimensional space geometry. They have been used in solving problems in different physical areas. In addition in the equations, using the symbolic computation program, closed solutions (24) were found.

Our proposal for future research, the Lagrange mechanical equations derived on a generalized on three-dimensional space are suggested to deal with problems in electrical, magnetical and gravitational fields of quantum and classical mechanics of physics [13,14,15].

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