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Research Article

New Analytical Wave Structures for the (2+1)-Dimensional Chaffee-Infante Equation

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Abstract

The focus of this paper is the (2+1)-dimensional Chaffee-Infante equation (CIE). The model describes the diffusion of a gas in a homogeneous medium, which makes it an important tool in the research of mathematics and physics. The modified extended Tanh expansion method is employed. Many soliton solutions have been obtained by rigorous analysis and calculation. This method can generate various types of solutions including trigonometric, trigonometric-hyperbolic, rational, kink, singular, and periodic singular solitons. We also present some of the obtained solutions' 3D, contour, and 2D plots. In order to tackle complex nonlinear issues, the solutions are dependable, efficient, and manageable, and the generated results provide a basis for further research. The study's method used in this paper is characterised by its ability to generate simple, reliable and original solutions to nonlinear partial differential equations (NLPDEs) in mathematical physics. To the best of our knowledge, no such work has been done before for this problem. The Maple software has been used to check the correctness of each solution found.

1. Introduction

NLPDEs are frequently used to describe complex physical events in the disciplines of chemistry, biology, mechanics, and physics [1–5] . It's an exciting attempt to find accurate solutions to NLPDEs, and academics have made great strides in this direction [6] . Many techniques have been developed over time to obtain analytical solutions for these kinds of issues. Because of their innately unpredictable behaviors, NLPDEs continue to provide substantial management and control issues despite these developments [7, 8] . A system can change quickly even with tiny modifications to some of the influencing variables. Therefore, scientists from different disciplines are investigating analytical form solutions of nonlinear equations to understand and investigate complex processes [9–16] . These solutions aid in our understanding of the behaviors of many nonlinear occurrences by illuminating their conceptual and visual connections. Therefore, in order to obtain deeper understanding, scholars who are interested in nonlinear phenomena, whether in engineering and other scientific fields, have been examining these analytic-form solutions [17, 18] .

The (2+1)-dimensional CIE is a NLEE that was first developed for use in combustion chemistry and combustion physics research. Characterizing the kinetics of chemical interactions during combustion processes is essential, especially when premixed flames are involved. Since its development, the CIE has been extensively studied and applied in the field of science. In many other disciplines, including electrical science, nuclear physics, ecology, fluid dynamics, and others, it has been widely used to explain the physical processes of mass movement and particle dispersion. In mathematics and physics, the (2+1)-dimensional CIE is primarily studied because it offers a valuable model for examining the diffusion phenomenon that occurs a gas in a homogeneous medium.

In this paper, motivated by other studies, we used the modified extended tanh expansion method (METEM) approach to study the (2+1)-dimensional CIE and get soliton solutions. Our ability to represent different wave patterns of complex physical events in scientific disciplines is made possible by these innovative discoveries. The physical processes of mass transport and particle diffusion can be described using the well-known reaction diffusion equation known as the CI equation. Numerous scientific and technological domains, including as fluid dynamics, plasma physics, ion-acoustic waves in the plasma, sound waves, electromagnetic waves, and signal processing through optical cables, use this equation. It is now known as the Chafee–Infante equation, and it was initially proposed by Nathaniel Chafee and Ettore Infante.

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The study and identification of different types of optical solitons is crucial to the investigation and application of this equation. In this context, researchers have carried out several studies. Sulaiman et al. obtained new lump solutions for the (2+1)-dimensional CIE using the Hirota bilinear form (HBF) [19] . Zhu et al. obtained analytical solutions to the main equation by the application of two methods involving the solutions of the trigonometric and hyperbolic functions [20] . To find closed-form solitary wave solutions for (2+1)-dimensional CIE, Akbar et al. used the first integral method [21] . Ay and Yaşar Painlevé constructed Bäcklund transformations and the other symmetries in non-local structures by using the shortened expansion approach. They put these symmetries in place and constructed the corresponding single variable Lie conversion group using an extended system. Also they proposed novel correct solution profiles in this transformation group by combining other simple accurate solution structures [22] .
In this paper, we consider the (2+1)-dimensional CIE in the form [19–22] :

$$V_{xt}+\left(-V_{xx}-\sigma V^{3}-\sigma V\right)_{x}+\eta V_{yy}=0. \tag{1.1}$$

Here $V(x,y,t)$ is the function which describes the intricate wave profile σ, η are the coefficient for diffusion and degradation, respectively. The primary motivation for studying the (2+1)-dimensional CIE in mathematics and physics is that it offers a practical model for examining the diffusion happening of a gas in a homogeneous medium. Obtaining several families of analytical soliton solutions and illustrating the dynamics of solitonic structures of the (2+1)-dimensional CIE will be the main goals of this study. These new discoveries enable us to depict different wave patterns of complex physical occurrences in a variety of scientific fields.
The rest of the paper is structured as follows: Description of the METEM is given in §2. The implementation of the proposed method are provided in §3. The dynamic behaviors for the various solutions is shown in 3D, contour, and 2D graphs in §4, and the findings are explained. Lastly, §5 some conclusions are presented.

2. Description of Applied Method

Assume that the presence of a NLPDE in the following form [23] :

$$N(V,V_x,V_y,V_t,V_{xx},V_{yy},V_{xy},V_{xt},\dots)=0, \tag{2.1}$$

where N stands for both the polynomial in space and time and its partial derivatives of $V(x,y,t)$. In order to solve Eq. (2.1), the subsequent wave transforms are used:

$$V(x,y,t)=U(\xi),\quad \xi=ax+by-ct, \tag{2.2}$$

where $U(\xi)$ represents the pulse's form and a, b, c are the non-zero real constants.
The following ordinary differential equation (ODE) is derived from Eq. (2.1):

$$O(U,U',U'',U''',\dots)=0. \tag{2.3}$$

The main components of METEM are summarized below.
Step 1. To solve the ODE, take Eq. (2.3) in the series form as:

$$U(\xi)=A_0+\sum_{i=1}^{n}\left(A_i\Phi^{i}(\xi)+B_i\Phi^{-i}(\xi)\right), \tag{2.4}$$

where A_i,B_i are the ordinary constant parameters to be determined later. The function $\Phi(\xi)$, which will be determined later, satisfies the auxiliary equation as follows:

$$\frac{d\Phi(\xi)}{d\xi}=\varphi+(\Phi(\xi))^{2}. \tag{2.5}$$

In Eq. (2.5), are constants discovered be later. In subsequent, the use of Eq. (2.5) is shown.
Family 1. For $\varphi<0$, the general solutions for the ansatz Eq. (2.5), given the following hyperbolic solutions:

$$\Phi_1(\xi)=-\sqrt{-\varphi}\tanh\left(\sqrt{-\varphi}\,(\xi+\xi_0)\right), \tag{2.6}$$

$$\Phi_2(\xi)=-\sqrt{-\varphi}\coth\left(\sqrt{-\varphi}\,(\xi+\xi_0)\right), \tag{2.7}$$

$$\Phi_3(\xi)=-\sqrt{-\varphi}\left(\tanh\left(2\sqrt{-\varphi}\,(\xi+\xi_0)\right)+i\varepsilon\,\mathrm{sech}\left(2\sqrt{-\varphi}\,(\xi+\xi_0)\right)\right), \tag{2.8}$$

$$\Phi_4(\xi)=\frac{-\sqrt{-\varphi}\tanh\left(\sqrt{-\varphi}\,(\xi+\xi_0)\right)+\varphi}{\sqrt{-\varphi}\tanh\left(\sqrt{-\varphi}\,(\xi+\xi_0)\right)+1}, \tag{2.9}$$

$$\Phi_5(\xi)=\frac{\sqrt{-\varphi}\left(-4\cosh\left(2\sqrt{-\varphi}\,(\xi+\xi_0)\right)+5\right)}{\left(4\sinh\left(2\sqrt{-\varphi}\,(\xi+\xi_0)\right)+3\right)}, \tag{2.10}$$

$$\Phi_6(\xi)=\frac{\varepsilon\sqrt{-w\left(a^2+b^2\right)}-a\sqrt{-\varphi}\cosh\left(2\sqrt{-\varphi}\left(\xi+\xi_0\right)\right)}{a\sinh\left(2\sqrt{-\varphi}\left(\xi+\xi_0\right)\right)+b}, \tag{2.11}$$

$$\Phi_7(\xi)=\varepsilon\sqrt{-\varphi}\left[1-\frac{2a}{a+\cosh\left(2\sqrt{-\varphi}\left(\xi+\xi_0\right)\right)-\varepsilon\sinh\left(2\sqrt{-\varphi}\left(\xi+\xi_0\right)\right)}\right]. \tag{2.12}$$

Family 2. For $\varphi>0$, the general solutions for the ansatz Eq. (2.5), given the following trigonometric solutions:

$$\Phi_8(\xi)=\sqrt{\varphi}\tan\left(\sqrt{\varphi}\left(\xi+\xi_0\right)\right), \tag{2.13}$$

$$\Phi_9(\xi)=-\sqrt{\varphi}\cot\left(\sqrt{\varphi}\left(\xi+\xi_0\right)\right), \tag{2.14}$$

$$\Phi_{10}(\xi)=\sqrt{\varphi}\left(\tan\left(2\sqrt{\varphi}\left(\xi+\xi_0\right)\right)+\varepsilon\sec\left(2\sqrt{\varphi}\left(\xi+\xi_0\right)\right)\right), \tag{2.15}$$

$$\Phi_{11}(\xi)=-\frac{\sqrt{\varphi}\left(1-\tan\left(\sqrt{\varphi}\left(\xi+\xi_0\right)\right)\right)}{\left(1+\tan\left(\sqrt{\varphi}\left(\xi+\xi_0\right)\right)\right)}, \tag{2.16}$$

$$\Phi_{12}(\xi)=\frac{\sqrt{\varphi}\left(-5\cos\left(2\sqrt{\varphi}\left(\xi+\xi_0\right)\right)+4\right)}{\left(5\sin\left(2\sqrt{\varpi}\left(\xi+\xi_0\right)\right)+3\right)}, \tag{2.17}$$

$$\Phi_{13}(\xi)=\frac{\varepsilon\sqrt{\varphi\left(a^2+b^2\right)}-a\sqrt{\varphi}\cos\left(2\sqrt{\varphi}\left(\xi+\xi_0\right)\right)}{a\sin\left(2\sqrt{\varphi}\left(\xi+\xi_0\right)\right)+b}, \tag{2.18}$$

$$\Phi_{14}(\xi)=i\varepsilon\sqrt{\varphi}\left[1-\frac{2a}{a+\cos\left(2\sqrt{\varphi}\left(\xi+\xi_0\right)\right)-i\varepsilon\sin\left(2\sqrt{\varphi}\left(\xi+\xi_0\right)\right)}\right]. \tag{2.19}$$

Family 3. For $\varphi=0$, the general solutions for the Eq. (2.5), the following rational solution is given:

$$\Phi_{15}(\xi)=-\frac{1}{\xi+\xi_0}. \tag{2.20}$$

Here, the real arbitrary parameters are $\varepsilon=\pm1,\ a,\ b,\ \varphi,\ \xi_0$.
Step 2. In order to balance the nonlinear terms in Eq. (2.3) with the highest order derivative, we determine n for Eq. (2.4).
Step 3. Inserting Eq. (2.4) and its derivatives in Eq. (2.3) with regard to Eq. (2.5), we get a polynomial for $U(\xi)$. By taking the coefficients of each power to zero, we yield a system of equations with unknown parameters $\varphi,\ A_i,\ B_i\ (i=1,2,3,\ldots,n)$ and solving this system we obtain the analytic solutions of Eq. (2.3).
Step 4. Finally, we get several analytical solutions to Eq. (2.1) by applying the transformation to Eq. (2.2) and using the solutions to Eq. (2.3). Through the consideration of the above three families, we have acquired the analytical solutions for Eq. (1.1).

3. Analysis of Solitons for the (2+1)-Dimensional CIE

In this section of the study, we build different soliton solutions for Eq. (1.1) while taking the METEM into consideration.
The (2+1)-dimensional CIE is consider as:

$$V(x,y,t)=U(\xi),\quad \xi=(ax+by-ct). \tag{3.1}$$

When we apply in Eq. (3.1) to Eq. (1.1), we get

$$-a^3U'''(\xi)+b^2\eta U''(\xi)-caU''(\xi)-a\sigma U'(\xi)+3a\sigma U'(\xi)U^2(\xi)=0. \tag{3.2}$$

After we integrate once with respect to ξ, Eq. (3.2) changes to as follows:

$$-a^3U''(\xi)+\left(b^2\eta-ca\right)U'(\xi)-a\sigma U(\xi)+a\sigma U^3(\xi)=0, \tag{3.3}$$

in which we obtain $n=1$ by balancing $U^3(\xi)$ with $U''(\xi)$. The general solution to Eq. (3.3) is as follows:

$$U(\xi)=A_0+A_1\Phi(\xi)+B_1\frac{1}{\Phi(\xi)}. \tag{3.4}$$

By inserting Eq. (3.4) together with Eq. (3.3) into Eq. (2.5) and setting the coefficients to zero for various powers of $\Phi(\xi)$, we have a system of equations. And we solve using the Maple software program to get the subsequent equation system:

$$\begin{cases} \Phi(\xi)^3: a\sigma A_1^3 - 2a^3 A_1 = 0, \\ \Phi(\xi)^2: -\left((-3\sigma A_0 A_1 + c)a - b^2\eta\right) A_1 = 0, \\ \Phi(\xi)^1: 3A_1 a\sigma\left(B_1 A_1 + A_0^2 - \frac{1}{3}\right) - 2a^3 A_1 \varphi = 0, \\ \Phi(\xi)^0: \left((6B_1\sigma A_0 - c\varphi)A_1 + \sigma A_0^3 + cB_1 - \sigma A_0\right) a - b^2\eta\left(-A_1\varphi + B_1\right) = 0, \\ \Phi(\xi)^{-1}: 3B_1 a\left(B_1 A_1 + A_0^2 - \frac{1}{3}\right)\sigma - 2a^3 B_1 \varphi = 0, \\ \Phi(\xi)^{-2}: \left((3B_1\sigma A_0 + c\varphi)a - b^2\eta\varphi\right) B_1 = 0, \\ \Phi(\xi)^{-3}: -2a^3 B_1 \varphi^2 + a\sigma B_1^3 = 0. \end{cases}$$

By solving above system for B_1, A_1, A_0, c, and φ, we get these set solutions:
Set 1:

$$B_1 = 0, \quad A_1 = \sqrt{\frac{2}{\sigma}}a, \quad A_0 = 0, \quad c = \frac{b^2\eta}{a}, \quad \varphi = -\frac{\sigma}{2a^2}.$$

Set 2:

$$B_1 = 0, \quad A_1 = \sqrt{\frac{2}{\sigma}}a, \quad A_0 = \frac{1}{2}, \quad c = \frac{3a^2\sigma\sqrt{\frac{2}{\sigma}} + 2b^2\eta}{2a}, \quad \varphi = -\frac{\sigma}{8a^2}.$$

Set 3:

$$B_1 = \frac{\sqrt{2\sigma}}{2a}, \quad A_1 = 0, \quad A_0 = 0, \quad c = \frac{b^2\eta}{a}, \quad \varphi = -\frac{\sigma}{2a^2}.$$

Set 4:

$$B_1 = \frac{3\sqrt{2\sigma} + \sqrt{\sigma}}{16a}, \quad A_1 = \sqrt{\frac{2}{\sigma}}a, \quad A_0 = 0, \quad c = \frac{b^2\eta}{a}, \quad \varphi = \frac{4\sigma}{16a^2}.$$

Set 5:

$$B_1 = \frac{\sqrt{2\sigma}}{32a}, \quad A_1 = \sqrt{\frac{2}{\sigma}}a, \quad A_0 = \frac{1}{2}, \quad c = \frac{3a^2\sqrt{2\sigma} + 2b^2\eta}{2a}, \quad \varphi = -\frac{\sigma}{32a^2}.$$

Set 6:

$$B_1 = \frac{\sqrt{2\sigma}}{8a}, \quad A_1 = 0, \quad A_0 = -\frac{1}{2}, \quad c = \frac{-3a^2\sigma\sqrt{2} + 2\sqrt{\sigma}b^2\eta}{2a\sqrt{\sigma}}, \quad \varphi = -\frac{\sigma}{8a^2}.$$

For **Set 1**, substituting the values of constants into Eq. (3.4) and Eq. (3.1) along with Eq. (2.6)-Eq. (2.20) and simplifying, yields next solutions:
Family 1: When $\varphi < 0$ and the hyperbolic solutions of Eq. (1.1) are given as follows:

$$V_{1,1}(x,y,t) = -\tanh\left(\sqrt{\frac{\sigma}{2a^2}}\left(ax + by - \frac{b^2\eta t}{a}\right)\right), \tag{3.5}$$

$$V_{1,2}(x,y,t) = -\coth\left(\sqrt{\frac{\sigma}{2a^2}}\left(ax + by - \frac{b^2\eta t}{a}\right)\right),$$

$$V_{1,3}(x,y,t) = -\left(\tanh\left(\sqrt{\frac{2\sigma}{a^2}}\left(ax + by - \frac{b^2\eta t}{a}\right)\right) + i\,\mathrm{sech}\left(\sqrt{\frac{2\sigma}{a^2}}\left(ax + by - \frac{b^2\eta t}{a}\right)\right)\right),$$

$$V_{1,4}(x,y,t) = \frac{\sqrt{\frac{2}{\sigma}}a\left(-\frac{\sigma}{2a^2} - \sqrt{\frac{\sigma}{2a^2}}\tanh\left(\sqrt{\frac{\sigma}{2a^2}}\left(ax + by - \frac{b^2\eta t}{a}\right)\right)\right)}{\left(1 + \sqrt{\frac{\sigma}{2a^2}}\tanh\left(\sqrt{\frac{\sigma}{2a^2}}\left(ax + by - \frac{b^2\eta t}{a}\right)\right)\right)}, \tag{3.6}$$

$$V_{1,5}(x,y,t) = \frac{5 - 4\cosh\left(\sqrt{\frac{2\sigma}{a^2}}\left(ax + by - \frac{b^2\eta t}{a}\right)\right)}{\left(3 + 4\sinh\left(\sqrt{\frac{2\sigma}{a^2}}\left(ax + by - \frac{b^2\eta t}{a}\right)\right)\right)},$$

$$V_{1,6}(x,y,t)=\frac{\sqrt{\frac{2a^2}{\sigma}}\left(\sqrt{\frac{2\sigma(a^2+b^2)}{a^2}}-\sqrt{2\sigma}\cosh\left(\sqrt{\frac{2\sigma}{a^2}}\left(ax+by-\frac{b^2\eta t}{a}\right)\right)\right)}{\left(2a\sinh\left(\sqrt{\frac{2\sigma}{a^2}}\left(ax+by-\frac{b^2\eta t}{a}\right)\right)+b\right)},$$

$$V_{1,7}(x,y,t)=1-\frac{2a}{\left(a+\cosh\left(\sqrt{\frac{2\sigma}{a^2}}\left(ax+by-\frac{b^2\eta t}{a}\right)\right)-\sinh\left(\sqrt{\frac{2\sigma}{a^2}}\left(ax+by-\frac{b^2\eta t}{a}\right)\right)\right)}.$$

Family 2: Given $\varphi>0$, using the offered method obtained the following trigonometric solutions for Eq. (1.1):

$$V_{1,8}(x,y,t)=i\tan\left(\sqrt{-\frac{\sigma}{2a^2}}\left(ax+by-\frac{b^2\eta t}{a}\right)\right),$$

$$V_{1,9}(x,y,t)=-i\cot\left(\sqrt{-\frac{\sigma}{2a^2}}\left(ax+by-\frac{b^2\eta t}{a}\right)\right),$$

$$V_{1,10}(x,y,t)=i\left(\tan\left(\sqrt{-\frac{2\sigma}{a^2}}\left(ax+by-\frac{b^2\eta t}{a}\right)\right)+\sec\left(\sqrt{-\frac{2\sigma}{a^2}}\left(ax+by-\frac{b^2\eta t}{a}\right)\right)\right),$$

$$V_{1,11}(x,y,t)=-\frac{i\left(1-\tan\left(\sqrt{-\frac{\sigma}{2a^2}}\left(ax+by-\frac{b^2\eta t}{a}\right)\right)\right)}{\left(1+\tan\left(\sqrt{-\frac{\sigma}{2a^2}}\left(ax+by-\frac{b^2\eta t}{a}\right)\right)\right)},$$

$$V_{1,12}(x,y,t)=\frac{i\left(4-5\cos\left(\sqrt{-\frac{2\sigma}{a^2}}\left(ax+by-\frac{b^2\eta t}{a}\right)\right)\right)}{\left(3+5\sin\left(\sqrt{-\frac{2\sigma}{a^2}}\left(ax+by-\frac{b^2\eta t}{a}\right)\right)\right)},$$

$$V_{1,13}(x,y,t)=\frac{\sqrt{\frac{2a^2}{\sigma}}\left(\sqrt{-\frac{\sigma(a^2-b^2)}{2a^2}}-\sqrt{-\frac{\sigma}{2}}\cos\left(\sqrt{-\frac{2\sigma}{a^2}}\left(ax+by-\frac{b^2\eta t}{a}\right)\right)\right)}{\left(a\sin\left(\sqrt{-\frac{2\sigma}{a^2}}\left(ax+by-\frac{b^2\eta t}{a}\right)\right)+b\right)},$$

$$V_{1,14}(x,y,t)=1-\frac{2a}{\left(a+\cos\left(\sqrt{-\frac{2\sigma}{a^2}}\left(ax+by-\frac{b^2\eta t}{a}\right)\right)-i\sin\left(\sqrt{-\frac{2\sigma}{a^2}}\left(ax+by-\frac{b^2\eta t}{a}\right)\right)\right)}.$$

Family 3: When $\varphi=0$, then Eq. (1.1) has rational solution:

$$V_{1,15}(x,y,t)=-\frac{\sqrt{\frac{2a^2}{\sigma}}}{ax+by-\frac{b^2\eta t}{a}}.$$

For **Set 2**, substituting the values of constants into Eq. (3.4) and Eq. (3.1) along with Eq. (2.6)-Eq. (2.20) and simplifying, yields following solutions, respectively:

Family 1: When $\varphi<0$ and the hyperbolic solutions of Eq. (1.1) are given as follows:

$$V_{2,1}(x,y,t)=\frac{1}{2}\left(1-\tanh\left(\sqrt{\frac{\sigma}{8a^2}}\left(ax+by-\frac{3a^2\sqrt{2\sigma}+2b^2\eta t}{2a}\right)\right)\right),$$

$$V_{2,2}(x,y,t)=\frac{1}{2}\left(1-\coth\left(\sqrt{\frac{\sigma}{8a^2}}\left(ax+by-\frac{3a^2\sqrt{2\sigma}+2b^2\eta t}{2a}\right)\right)\right),$$

$$V_{2,3}(x,y,t)=\frac{1}{2}\left(1-\left(\tanh\left(\sqrt{\frac{\sigma}{2a^2}}\left(ax+by-\frac{3a^2\sqrt{2\sigma}+2b^2\eta t}{2a}\right)\right)+i\,\mathrm{sech}\left(\sqrt{\frac{\sigma}{2a^2}}\left(ax+by-\frac{3a^2\sqrt{2\sigma}+2b^2\eta t}{2a}\right)\right)\right)\right),$$

$$V_{2,4}(x,y,t)=\frac{1}{2}+\frac{\sqrt{\frac{2}{\sigma}}a\left(-\frac{\sigma}{8a^2}-\sqrt{\frac{\sigma}{8a^2}}\tanh\left(\sqrt{\frac{\sigma}{8a^2}}\left(ax+by-\frac{3a^2\sqrt{2\sigma}+2b^2\eta t}{2a}\right)\right)\right)}{\left(1+\sqrt{\frac{\sigma}{8a^2}}\tanh\left(\sqrt{\frac{\sigma}{8a^2}}\left(ax+by-\frac{3a^2\sqrt{2\sigma}+2b^2\eta t}{2a}\right)\right)\right)},$$

$$V_{2,5}(x,y,t)=\frac{1}{2}\left(1+\frac{5-4\cosh\left(\sqrt{\frac{\sigma}{2a^2}}\left(ax+by-\frac{3a^2\sqrt{2\sigma}+2b^2\eta t}{2a}\right)\right)}{3+4\sinh\left(\sqrt{\frac{\sigma}{2a^2}}\left(ax+by-\frac{3a^2\sqrt{2\sigma}+2b^2\eta t}{2a}\right)\right)}\right),$$

$$V_{2,6}(x,y,t)=\frac{1}{2}+\frac{\sqrt{\frac{2a^2}{\sigma}}\left(\sqrt{\frac{\sigma(a^2+b^2)}{8a^2}}-\sqrt{\frac{\sigma}{8}}\cosh\left(\sqrt{\frac{\sigma}{2a^2}}\left(ax+by-\frac{3a^2\sqrt{2\sigma}+2b^2\eta t}{2a}\right)\right)\right)}{\left(a\sinh\left(\sqrt{\frac{\sigma}{2a^2}}\left(ax+by-\frac{3a^2\sqrt{2\sigma}+2b^2\eta t}{2a}\right)\right)+b\right)},$$

$$V_{2,7}(x,y,t)=1-\frac{a}{\left(\begin{array}{l}a+\cosh\left(\sqrt{\frac{\sigma}{2a^2}}\left(ax+by-\frac{3a^2\sqrt{2\sigma}+2b^2\eta t}{2a}\right)\right)\\ -\sinh\left(\sqrt{\frac{\sigma}{2a^2}}\left(ax+by-\frac{3a^2\sqrt{2\sigma}+2b^2\eta t}{2a}\right)\right)\end{array}\right)}.$$

Family 2: Given $\varphi>0$, using the method obtained the following trigonometric solutions for Eq. (1.1):

$$V_{2,8}(x,y,t)=\frac{1}{2}\left(1+i\tan\left(\sqrt{-\frac{\sigma}{8a^2}}\left(ax+by-\frac{3a^2\sqrt{2\sigma}+2b^2\eta t}{2a}\right)\right)\right),$$

$$V_{2,9}(x,y,t)=\frac{1}{2}\left(1-i\cot\left(\sqrt{\frac{\sigma}{8a^2}}\left(ax+by-\frac{3a^2\sqrt{2\sigma}+2b^2\eta t}{2a}\right)\right)\right),$$

$$V_{2,10}(x,y,t)=\frac{1}{2}\left(1+i\left(\begin{array}{l}\tan\left(\sqrt{-\frac{\sigma}{2a^2}}\left(ax+by-\frac{3a^2\sqrt{2\sigma}+2b^2\eta t}{2a}\right)\right)\\ +\sec\left(\sqrt{-\frac{\sigma}{2a^2}}\left(ax+by-\frac{3a^2\sqrt{2\sigma}+2b^2\eta t}{2a}\right)\right)\end{array}\right)\right),$$

$$V_{2,11}(x,y,t)=\frac{1}{2}\left(1-\frac{i\left(1-\tan\left(\sqrt{-\frac{\sigma}{8a^2}}\left(ax+by-\frac{3a^2\sqrt{2\sigma}+2b^2\eta t}{2a}\right)\right)\right)}{1+\tan\left(\sqrt{-\frac{\sigma}{8a^2}}\left(ax+by-\frac{3a^2\sqrt{2\sigma}+2b^2\eta t}{2a}\right)\right)}\right),$$

$$V_{2,12}(x,y,t)=\frac{1}{2}\left(1+\frac{i\left(4-5\cos\left(\sqrt{-\frac{\sigma}{2a^2}}\left(ax+by-\frac{3a^2\sqrt{2\sigma}+2b^2\eta t}{2a}\right)\right)\right)}{3+5\sin\left(\sqrt{-\frac{\sigma}{2a^2}}\left(ax+by-\frac{3a^2\sqrt{2\sigma}+2b^2\eta t}{2a}\right)\right)}\right),$$

$$V_{2,13}(x,y,t)=\frac{1}{2}+\frac{\sqrt{\frac{2a^2}{\sigma}}\left(\sqrt{-\frac{\sigma(a^2-b^2)}{8a^2}}-\sqrt{-\frac{\sigma}{8}}\cos\left(\sqrt{-\frac{\sigma}{2a^2}}\left(ax+by-\frac{3a^2\sqrt{2\sigma}+2b^2\eta t}{2a}\right)\right)\right)}{\left(a\sin\left(\sqrt{-\frac{\sigma}{2a^2}}\left(ax+by-\frac{3a^2\sqrt{2\sigma}+2b^2\eta t}{2a}\right)\right)+b\right)},$$

$$V_{2,14}(x,y,t)=1-\frac{a}{\left(\begin{array}{l}a+\cos\left(\sqrt{-\frac{\sigma}{2a^2}}\left(ax+by-\frac{3a^2\sqrt{2\sigma}+2b^2\eta t}{2a}\right)\right)\\ -i\sin\left(\sqrt{-\frac{\sigma}{2a^2}}\left(ax+by-\frac{3a^2\sqrt{2\sigma}+2b^2\eta t}{2a}\right)\right)\end{array}\right)}.$$

Family 3: When $\varphi=0$, then Eq. (1.1) has rational solution:

$$V_{2,15}(x,y,t)=\frac{1}{2}-\frac{\sqrt{\frac{2a^2}{\sigma}}}{ax+by-\frac{3a^2\sigma\sqrt{\frac{2}{\sigma}}+2b^2\eta t}{2a}}.$$

For **Set 3**, substituting the values of constants into Eq. (3.4) and Eq. (3.1) along with Eq. (2.6)-Eq. (2.20) and simplifying, yields following solutions, respectively:
Family 1: When $\varphi<0$ and the hyperbolic solutions of Eq. (1.1) are given as follows:

$$V_{3,1}(x,y,t)=-\frac{1}{\tanh\left(\sqrt{\frac{\sigma}{2a^2}}\left(ax+by-\frac{b^2\eta t}{a}\right)\right)},$$

$$V_{3,2}(x,y,t)=-\frac{1}{\coth\left(\sqrt{\frac{\sigma}{2a^2}}\left(ax+by-\frac{b^2\eta t}{a}\right)\right)},$$

$$
\begin{aligned}
V_{3,3}(x,y,t) &= -\frac{1}{\tanh\left(\sqrt{\frac{2\sigma}{a^2}}\left(ax+by-\frac{b^2\eta t}{a}\right)\right)+i\,\mathrm{sech}\left(\sqrt{\frac{2\sigma}{a^2}}\left(ax+by-\frac{b^2\eta t}{a}\right)\right)},\\
V_{3,4}(x,y,t) &= \frac{\sqrt{2\sigma}\left(1+\sqrt{\frac{\sigma}{2a^2}}\tanh\left(\sqrt{\frac{\sigma}{2a^2}}\left(ax+by-\frac{b^2\eta t}{a}\right)\right)\right)}{2a\left(-\frac{\sigma}{2a^2}-\sqrt{\frac{\sigma}{2a^2}}\tanh\left(\sqrt{\frac{\sigma}{2a^2}}\left(ax+by-\frac{b^2\eta t}{a}\right)\right)\right)},\\
V_{3,5}(x,y,t) &= \frac{3+4\sinh\left(\sqrt{\frac{2\sigma}{a^2}}\left(ax+by-\frac{b^2\eta t}{a}\right)\right)}{5-4\cosh\left(\sqrt{\frac{2\sigma}{a^2}}\left(ax+by-\frac{b^2\eta t}{a}\right)\right)},\\
V_{3,6}(x,y,t) &= \frac{\sqrt{2\sigma}\left(a\sinh\left(\sqrt{\frac{2\sigma}{a^2}}\left(ax+by-\frac{b^2\eta t}{a}\right)\right)+b\right)}{a\left(\sqrt{\frac{2\sigma(a^2+b^2)}{a^2}}-\sqrt{2\sigma}\cosh\left(\sqrt{\frac{2\sigma}{a^2}}\left(ax+by-\frac{b^2\eta t}{a}\right)\right)\right)},\\
V_{3,7}(x,y,t) &= 1-\frac{2a}{\left(a+\cosh\left(\sqrt{\frac{2\sigma}{a^2}}\left(ax+by-\frac{b^2\eta t}{a}\right)\right)-\sinh\left(\sqrt{\frac{2\sigma}{a^2}}\left(ax+by-\frac{b^2\eta t}{a}\right)\right)\right)}.
\end{aligned}
\tag{3.7}
$$

Family 2: Given $\varphi > 0$, using the method obtained the following trigonometric solutions for Eq. (1.1):

$$
\begin{aligned}
V_{3,8}(x,y,t) &= \frac{1}{i\tan\left(\sqrt{-\frac{\sigma}{2a^2}}\left(ax+by-\frac{b^2\eta t}{a}\right)\right)},\\
V_{3,9}(x,y,t) &= -\frac{1}{i\cot\left(\sqrt{-\frac{\sigma}{2a^2}}\left(ax+by-\frac{b^2\eta t}{a}\right)\right)},\\
V_{3,10}(x,y,t) &= \frac{1}{i\left(\tan\left(\sqrt{-\frac{2\sigma}{a^2}}\left(ax+by-\frac{b^2\eta t}{a}\right)\right)+\sec\left(\sqrt{-\frac{2\sigma}{a^2}}\left(ax+by-\frac{b^2\eta t}{a}\right)\right)\right)},\\
V_{3,11}(x,y,t) &= -\frac{\left(1+\tan\left(\sqrt{-\frac{\sigma}{2a^2}}\left(ax+by-\frac{b^2\eta t}{a}\right)\right)\right)}{i\left(1-\tan\left(\sqrt{-\frac{\sigma}{2a^2}}\left(ax+by-\frac{b^2\eta t}{a}\right)\right)\right)},\\
V_{3,12}(x,y,t) &= \frac{3+5\sin\left(\sqrt{-\frac{2\sigma}{a^2}}\left(ax+by-\frac{b^2\eta t}{a}\right)\right)}{i\left(4-5\cos\left(\sqrt{-\frac{2\sigma}{a^2}}\left(ax+by-\frac{b^2\eta t}{a}\right)\right)\right)},\\
V_{3,13}(x,y,) &= \frac{\sqrt{2\sigma}\left(a\sin\left(\sqrt{-\frac{2\sigma}{a^2}}\left(ax+by-\frac{b^2\eta t}{a}\right)\right)+b\right)}{a\left(\sqrt{-\frac{2\sigma(a^2-b^2)}{a^2}}-\sqrt{-2\sigma}\cos\left(\sqrt{-\frac{2\sigma}{a^2}}\left(ax+by-\frac{b^2\eta t}{a}\right)\right)\right)},\\
V_{3,14}(x,y,t) &= -1+\frac{2a}{\left(a+\cos\left(\sqrt{-\frac{2\sigma}{a^2}}\left(ax+by-\frac{b^2\eta t}{a}\right)\right)-i\sin\left(\sqrt{-\frac{2\sigma}{a^2}}\left(ax+by-\frac{b^2\eta t}{a}\right)\right)\right)}.
\end{aligned}
$$

Family 3: When $\varphi = 0$, then Eq. (1.1) has rational solution:

$$
V_{3,15}(x,y,t) = -\frac{\sqrt{2\sigma}}{2a}\left(ax+by-\frac{b^2\eta t}{a}\right).
$$

For **Set 4**, substituting the values of constants into Eq. (3.4) and Eq. (3.1) along with Eq. (2.6)-Eq. (2.20) and simplifying, yields following solutions, respectively:

Family 1: When $\varphi < 0$ and the hyperbolic solutions of Eq. (1.1) are given as follows:

$$
\begin{aligned}
V_{4,1}(x,y,t) = &-\frac{\sqrt{-2}}{2}\tanh\left(\sqrt{-\frac{\sigma}{4a^2}}\left(ax+by-\frac{b^2\eta t}{a}\right)\right)\\
&-\frac{1}{\sqrt{-2}\tanh\left(\sqrt{-\frac{\sigma}{4a^2}}\left(ax+by-\frac{b^2\eta t}{a}\right)\right)},
\end{aligned}
$$

$$V_{4,2}(x,y,t)=-\frac{\sqrt{-2}}{2}\coth\left(\sqrt{-\frac{\sigma}{4a^2}}\left(ax+by-\frac{b^2\eta t}{a}\right)\right)-\frac{1}{\sqrt{-2}\coth\left(\sqrt{-\frac{\sigma}{4a^2}}\left(ax+by-\frac{b^2\eta t}{a}\right)\right)},$$

$$V_{4,3}(x,y,t)=-\frac{\sqrt{-2}}{2}\begin{pmatrix}\tanh\left(\sqrt{-\frac{\sigma}{4a^2}}\left(ax+by-\frac{b^2\eta t}{a}\right)\right)\\ +i\,\mathrm{sech}\left(\sqrt{-\frac{\sigma}{4a^2}}\left(ax+by-\frac{b^2\eta t}{a}\right)\right)\end{pmatrix}-\frac{1}{\sqrt{-2}\begin{pmatrix}\tanh\left(\sqrt{-\frac{\sigma}{4a^2}}\left(ax+by-\frac{b^2\eta t}{a}\right)\right)\\ +i\,\mathrm{sech}\left(\sqrt{-\frac{\sigma}{4a^2}}\left(ax+by-\frac{b^2\eta t}{a}\right)\right)\end{pmatrix}},$$

$$V_{4,4}(x,y,t)=-\frac{\sqrt{\frac{2a^2}{\sigma}}\left(\frac{\sigma}{4a^2}-\sqrt{-\frac{\sigma}{4a^2}}\tanh\left(\sqrt{-\frac{\sigma}{4a^2}}\left(ax+by-\frac{b^2\eta t}{a}\right)\right)\right)}{1+\sqrt{-\frac{\sigma}{4a^2}}\tanh\left(\sqrt{-\frac{\sigma}{4a^2}}\left(ax+by-\frac{b^2\eta t}{a}\right)\right)}+\frac{\sqrt{2\sigma}\left(1+\sqrt{-\frac{\sigma}{4a^2}}\tanh\left(\sqrt{-\frac{\sigma}{4a^2}}\left(ax+by-\frac{b^2\eta t}{a}\right)\right)\right)}{4a\left(\frac{\sigma}{4a^2}-\sqrt{-\frac{\sigma}{4a^2}}\tanh\left(\sqrt{-\frac{\sigma}{4a^2}}\left(ax+by-\frac{b^2\eta t}{a}\right)\right)\right)},$$

$$V_{4,5}(x,y,t)=\frac{\sqrt{-2}\left(5-4\cosh\left(\sqrt{-\frac{\sigma}{4a^2}}\left(ax+by-\frac{b^2\eta t}{a}\right)\right)\right)}{2\left(3+4\sinh\left(\sqrt{-\frac{\sigma}{4a^2}}\left(ax+by-\frac{b^2\eta t}{a}\right)\right)\right)}+\frac{\left(3+4\sinh\left(\sqrt{-\frac{\sigma}{4a^2}}\left(ax+by-\frac{b^2\eta t}{a}\right)\right)\right)}{\sqrt{-2}\left(5-4\cosh\left(\sqrt{-\frac{\sigma}{4a^2}}\left(ax+by-\frac{b^2\eta t}{a}\right)\right)\right)},$$

$$V_{4,6}(x,y,t)=\frac{\sqrt{\frac{2a^2}{\sigma}}\left(\sqrt{-\frac{\sigma(a^2+b^2)}{4a^2}}-\sqrt{-\frac{\sigma}{4}}\cosh\left(\sqrt{-\frac{\sigma}{a^2}}\left(ax+by-\frac{b^2\eta t}{a}\right)\right)\right)}{a\sinh\left(\sqrt{-\frac{\sigma}{a^2}}\left(ax+by-\frac{b^2\eta t}{a}\right)\right)+b}+\frac{\sqrt{2\sigma}\left(a\sinh\left(\sqrt{-\frac{\sigma}{a^2}}\left(ax+by-\frac{b^2\eta t}{a}\right)\right)+b\right)}{4a\left(\sqrt{-\frac{\sigma(a^2+b^2)}{4a^2}}-\sqrt{-\frac{\sigma}{4}}\cosh\left(\sqrt{-\frac{\sigma}{a^2}}\left(ax+by-\frac{b^2\eta t}{a}\right)\right)\right)},$$

$$V_{4,7}(x,y,t)=\sqrt{-\frac{1}{2}}-\frac{\sqrt{-\frac{1}{2}}2a}{\begin{pmatrix}a+\cosh\left(\sqrt{-\frac{\sigma}{a^2}}\left(ax+by-\frac{b^2\eta t}{a}\right)\right)\\ -\sinh\left(\sqrt{-\frac{\sigma}{a^2}}\left(ax+by-\frac{b^2\eta t}{a}\right)\right)\end{pmatrix}}+\frac{1}{\sqrt{-2}}-\frac{\frac{1}{\sqrt{-2}}2a}{\begin{pmatrix}a+\cosh\left(\sqrt{-\frac{\sigma}{a^2}}\left(ax+by-\frac{b^2\eta t}{a}\right)\right)\\ -\sinh\left(\sqrt{-\frac{\sigma}{a^2}}\left(ax+by-\frac{b^2\eta t}{a}\right)\right)\end{pmatrix}}.$$

Family 2: Given $\varphi>0$, using the method obtained the following trigonometric solutions for Eq. (1.1):

$$V_{4,8}(x,y,t)=\frac{1}{\sqrt{2}}\tan\left(\sqrt{\frac{\sigma}{4a^2}}\left(ax+by-\frac{b^2\eta t}{a}\right)\right)+\frac{1}{\sqrt{2}\tan\left(\sqrt{\frac{\sigma}{4a^2}}\left(ax+by-\frac{b^2\eta t}{a}\right)\right)},$$

$$V_{4,9}(x,y,t)=-\frac{1}{\sqrt{2}}\cot\left(\sqrt{\frac{\sigma}{4a^2}}\left(ax+by-\frac{b^2\eta t}{a}\right)\right)-\frac{1}{\sqrt{2}\cot\left(\sqrt{\frac{\sigma}{4a^2}}\left(ax+by-\frac{b^2\eta t}{a}\right)\right)},$$

$$V_{4,10}(x,y,t)=\frac{1}{\sqrt{2}}\begin{pmatrix}\tan\left(\sqrt{\frac{\sigma}{a^2}}\left(ax+by-\frac{b^2\eta t}{a}\right)\right)\\+\sec\left(\sqrt{\frac{\sigma}{a^2}}\left(ax+by-\frac{b^2\eta t}{a}\right)\right)\end{pmatrix}+\frac{1}{\sqrt{2}\begin{pmatrix}\tan\left(\sqrt{\frac{\sigma}{a^2}}\left(ax+by-\frac{b^2\eta t}{a}\right)\right)\\+\sec\left(\sqrt{\frac{\sigma}{a^2}}\left(ax+by-\frac{b^2\eta t}{a}\right)\right)\end{pmatrix}},\tag{3.8}$$

$$V_{4,11}(x,y,t)=-\frac{\sqrt{2}\left(1-\tan\left(\sqrt{\frac{\sigma}{4a^2}}\left(ax+by-\frac{b^2\eta t}{a}\right)\right)\right)}{2\left(1+\tan\left(\sqrt{\frac{\sigma}{4a^2}}\left(ax+by-\frac{b^2\eta t}{a}\right)\right)\right)}-\frac{\left(1+\tan\left(\sqrt{\frac{\sigma}{4a^2}}\left(ax+by-\frac{b^2\eta t}{a}\right)\right)\right)}{\sqrt{2}\left(1-\tan\left(\sqrt{\frac{\sigma}{4a^2}}\left(ax+by-\frac{b^2\eta t}{a}\right)\right)\right)},$$

$$V_{4,12}(x,y,t)=\frac{\sqrt{2}\left(4-5\cos\left(\sqrt{\frac{\sigma}{a^2}}\left(ax+by-\frac{b^2\eta t}{a}\right)\right)\right)}{2\left(3+5\sin\left(\sqrt{\frac{\sigma}{a^2}}\left(ax+by-\frac{b^2\eta t}{a}\right)\right)\right)}+\frac{\left(3+5\sin\left(\sqrt{\frac{\sigma}{a^2}}\left(ax+by-\frac{b^2\eta t}{a}\right)\right)\right)}{\sqrt{2}\left(4-5\cos\left(\sqrt{\frac{\sigma}{a^2}}\left(ax+by-\frac{b^2\eta t}{a}\right)\right)\right)},$$

$$\begin{aligned}V_{4,13}(x,y,t)=&\frac{\sqrt{\frac{2a^2}{\sigma}}\left(\sqrt{\frac{\sigma(a^2-b^2)}{4a^2}}-\sqrt{\frac{\sigma}{4a^2}}\cos\left(\sqrt{\frac{\sigma}{a^2}}\left(ax+by-\frac{b^2\eta t}{a}\right)\right)\right)}{\sqrt{2\sigma}\left(a\sin\left(\sqrt{\frac{\sigma}{a^2}}\left(ax+by-\frac{b^2\eta t}{a}\right)\right)+b\right)}\\&+\frac{\sqrt{2\sigma}\left(a\sin\left(\sqrt{\frac{\sigma}{a^2}}\left(ax+by-\frac{b^2\eta t}{a}\right)\right)+b\right)}{4a\left(\sqrt{\frac{\sigma(a^2-b^2)}{4a^2}}-\sqrt{\frac{\sigma}{4a^2}}\cos\left(\sqrt{\frac{\sigma}{a^2}}\left(ax+by-\frac{b^2\eta t}{a}\right)\right)\right)},\end{aligned}$$

$$\begin{aligned}V_{4,14}(x,y,t)=&\sqrt{-\frac{1}{2}}\left(1-\frac{2a}{a+\cos\left(\sqrt{\frac{\sigma}{a^2}}\left(ax+by-\frac{b^2\eta t}{a}\right)\right)-i\sin\left(\sqrt{\frac{\sigma}{a^2}}\left(ax+by-\frac{b^2\eta t}{a}\right)\right)}\right)\\&-\sqrt{\frac{1}{2}}\left(1-\frac{2a}{a+\cos\left(\sqrt{\frac{\sigma}{a^2}}\left(ax+by-\frac{b^2\eta t}{a}\right)\right)-i\sin\left(\sqrt{\frac{\sigma}{a^2}}\left(ax+by-\frac{b^2\eta t}{a}\right)\right)}\right).\end{aligned}$$

Family 3: When $\varphi=0$, then Eq. (1.1) has rational solution:

$$V_{4,15}(x,y,t)=-\frac{\sqrt{\frac{2a^2}{\sigma}}}{\left(ax+by-\frac{b^2\eta t}{a}\right)}-\frac{\sqrt{2\sigma}}{4a}\left(ax+by-\frac{b^2\eta t}{a}\right).$$

For **Set 5**, substituting the values of constants into Eq. (3.4) and Eq. (3.1) along with Eq. (2.6)-Eq. (2.20) and simplifying, yields following solutions, respectively:
Family 1: When $\varphi<0$ and the hyperbolic solutions of Eq. (1.1) are given as follows:

$$\begin{aligned}V_{5,1}(x,y,t)=&\frac{1}{2}-\frac{1}{4}\tanh\left(\sqrt{\frac{\sigma}{32a^2}}\left(ax+by-\frac{3a^2\sqrt{2\sigma}+2b^2\eta t}{2a}\right)\right)\\&-\frac{1}{4\tanh\left(\sqrt{\frac{\sigma}{32a^2}}\left(ax+by-\frac{3a^2\sqrt{2\sigma}+2b^2\eta t}{2a}\right)\right)},\end{aligned}$$

$$\begin{aligned}V_{5,2}(x,y,t)=&\frac{1}{2}-\frac{1}{4}\coth\left(\sqrt{\frac{\sigma}{32a^2}}\left(ax+by-\frac{3a^2\sqrt{2\sigma}+2b^2\eta t}{2a}\right)\right)\\&-\frac{1}{4\coth\left(\sqrt{\frac{\sigma}{32a^2}}\left(ax+by-\frac{3a^2\sqrt{2\sigma}+2b^2\eta t}{2a}\right)\right)},\end{aligned}$$

$$\begin{aligned}V_{5,3}(x,y,t)=&\frac{1}{2}-\frac{1}{4}\begin{pmatrix}\tanh\left(\sqrt{\frac{\sigma}{8a^2}}\left(ax+by-\frac{3a^2\sqrt{2\sigma}+2b^2\eta t}{2a}\right)\right)\\+i\,\mathrm{sech}\left(\sqrt{\frac{\sigma}{8a^2}}\left(ax+by-\frac{3a^2\sqrt{2\sigma}+2b^2\eta t}{2a}\right)\right)\end{pmatrix}\\&-\frac{1}{4\begin{pmatrix}\tanh\left(\sqrt{\frac{\sigma}{8a^2}}\left(ax+by-\frac{3a^2\sqrt{2\sigma}+2b^2\eta t}{2a}\right)\right)\\+i\,\mathrm{sech}\left(\sqrt{\frac{\sigma}{8a^2}}\left(ax+by-\frac{3a^2\sqrt{2\sigma}+2b^2\eta t}{2a}\right)\right)\end{pmatrix}},\end{aligned}\tag{3.9}$$

$$V_{5,4}(x,y,t)=\frac{1}{2}-\frac{\sqrt{\frac{2a^2}{\sigma}}\left(-\frac{\sigma}{32a^2}-\sqrt{\frac{\sigma}{32a^2}}\tanh\left(\sqrt{\frac{\sigma}{32a^2}}\left(ax+by-\frac{3a^2\sqrt{2\sigma}+2b^2\eta t}{2a}\right)\right)\right)}{\left(1+\sqrt{\frac{\sigma}{32a^2}}\tanh\left(\sqrt{\frac{\sigma}{32a^2}}\left(ax+by-\frac{3a^2\sqrt{2\sigma}+2b^2\eta t}{2a}\right)\right)\right)}$$
$$-\frac{\sqrt{2\sigma}\left(1+\sqrt{\frac{\sigma}{32a^2}}\tanh\left(\sqrt{\frac{\sigma}{32a^2}}\left(ax+by-\frac{3a^2\sqrt{2\sigma}+2b^2\eta t}{2a}\right)\right)\right)}{32a\left(-\frac{\sigma}{32a^2}-\sqrt{\frac{\sigma}{32a^2}}\tanh\left(\sqrt{\frac{\sigma}{32a^2}}\left(ax+by-\frac{3a^2\sqrt{2\sigma}+2b^2\eta t}{2a}\right)\right)\right)},$$

$$V_{5,5}(x,y,t)=\frac{1}{2}+\frac{\left(5-4\cosh\left(\sqrt{\frac{\sigma}{8a^2}}\left(ax+by-\frac{3a^2\sqrt{2\sigma}+2b^2\eta t}{2a}\right)\right)\right)}{4\left(3+4\sinh\left(\sqrt{\frac{\sigma}{8a^2}}\left(ax+by-\frac{3a^2\sqrt{2\sigma}+2b^2\eta t}{2a}\right)\right)\right)}$$
$$+\frac{\left(3+4\sinh\left(\sqrt{\frac{\sigma}{8a^2}}\left(ax+by-\frac{3a^2\sqrt{2\sigma}+2b^2\eta t}{2a}\right)\right)\right)}{4\left(5-4\cosh\left(\sqrt{\frac{\sigma}{8a^2}}\left(ax+by-\frac{3a^2\sqrt{2\sigma}+2b^2\eta t}{2a}\right)\right)\right)},$$

$$V_{5,6}(x,y,t)=\frac{1}{2}+\frac{\sqrt{\frac{2a^2}{\sigma}}\left(\sqrt{\frac{\sigma(a^2+b^2)}{32a^2}}-\sqrt{\frac{\sigma}{32}}\cosh\left(\sqrt{\frac{\sigma}{8a^2}}\left(ax+by-\frac{3a^2\sqrt{2\sigma}+2b^2\eta t}{2a}\right)\right)\right)}{\left(a\sinh\left(\sqrt{\frac{\sigma}{8a^2}}\left(ax+by-\frac{3a^2\sqrt{2\sigma}+2b^2\eta t}{2a}\right)\right)+b\right)}$$
$$+\frac{\sqrt{2\sigma}\left(a\sinh\left(\sqrt{\frac{\sigma}{8a^2}}\left(ax+by-\frac{3a^2\sqrt{2\sigma}+2b^2\eta t}{2a}\right)\right)+b\right)}{32a\left(\sqrt{\frac{\sigma(a^2+b^2)}{32a^2}}-\sqrt{\frac{\sigma}{32}}\cosh\left(\sqrt{\frac{\sigma}{8a^2}}\left(ax+by-\frac{3a^2\sqrt{2\sigma}+2b^2\eta t}{2a}\right)\right)\right)},$$

$$V_{5,7}(x,y,t)=\frac{1}{2}+\frac{1}{4}-\frac{\frac{1}{2}a}{\left(\begin{array}{l}a+\cosh\left(\sqrt{\frac{\sigma}{8a^2}}\left(ax+by-\frac{3a^2\sqrt{2\sigma}+2b^2\eta t}{2a}\right)\right)\\-\sinh\left(\sqrt{\frac{\sigma}{8a^2}}\left(ax+by-\frac{3a^2\sqrt{2\sigma}+2b^2\eta t}{2a}\right)\right)\end{array}\right)}$$
$$-\frac{1}{4-\frac{8a}{\left(\begin{array}{l}a+\cosh\left(\sqrt{\frac{\sigma}{8a^2}}\left(ax+by-\frac{3a^2\sqrt{2\sigma}+2b^2\eta t}{2a}\right)\right)\\-\sinh\left(\sqrt{\frac{\sigma}{8a^2}}\left(ax+by-\frac{3a^2\sqrt{2\sigma}+2b^2\eta t}{2a}\right)\right)\end{array}\right)}}.$$

Family 2: Given $\varphi>0$, using the method obtained the following trigonometric solutions for Eq. (1.1):

$$V_{5,8}(x,y,t)=\frac{1}{2}+\sqrt{-\frac{1}{16}}\tan\left(\sqrt{-\frac{\sigma}{32a^2}}\left(ax+by-\frac{3a^2\sqrt{2\sigma}+2b^2\eta t}{2a}\right)\right)$$
$$-\frac{1}{4i\tan\left(\sqrt{-\frac{\sigma}{32a^2}}\left(ax+by-\frac{3a^2\sqrt{2\sigma}+2b^2\eta t}{2a}\right)\right)},$$

$$V_{5,9}(x,y,t)=\frac{1}{2}+\sqrt{-\frac{1}{16}}\cot\left(\sqrt{-\frac{\sigma}{32a^2}}\left(ax+by-\frac{3a^2\sqrt{2\sigma}+2b^2\eta t}{2a}\right)\right)$$
$$-\frac{1}{4i\cot\left(\sqrt{-\frac{\sigma}{32a^2}}\left(ax+by-\frac{3a^2\sqrt{2\sigma}+2b^2\eta t}{2a}\right)\right)},$$

$$V_{5,10}(x,y,t)=\frac{1}{2}+\sqrt{-\frac{1}{16}}\left(\begin{array}{l}\tan\left(\sqrt{-\frac{\sigma}{8a^2}}\left(ax+by-\frac{3a^2\sqrt{2\sigma}+2b^2\eta t}{2a}\right)\right)\\+\sec\left(\sqrt{-\frac{\sigma}{8a^2}}\left(ax+by-\frac{3a^2\sqrt{2\sigma}+2b^2\eta t}{2a}\right)\right)\end{array}\right)$$
$$+\frac{1}{4i\left(\begin{array}{l}\tan\left(\sqrt{-\frac{\sigma}{8a^2}}\left(ax+by-\frac{3a^2\sqrt{2\sigma}+2b^2\eta t}{2a}\right)\right)\\+\sec\left(\sqrt{-\frac{\sigma}{8a^2}}\left(ax+by-\frac{3a^2\sqrt{2\sigma}+2b^2\eta t}{2a}\right)\right)\end{array}\right)},$$

$$V_{5,11}(x,y,t)=\frac{1}{2}-\frac{\sqrt{-\frac{1}{16}}\left(1+\tan\left(\sqrt{-\frac{\sigma}{32a^2}}\left(ax+by-\frac{3a^2\sqrt{2\sigma}+2b^2\eta t}{2a}\right)\right)\right)}{\left(1-\tan\left(\sqrt{-\frac{\sigma}{32a^2}}\left(ax+by-\frac{3a^2\sqrt{2\sigma}+2b^2\eta t}{2a}\right)\right)\right)}-\frac{\left(1-\tan\left(\sqrt{-\frac{\sigma}{32a^2}}\left(ax+by-\frac{3a^2\sqrt{2\sigma}+2b^2\eta t}{2a}\right)\right)\right)}{4i\left(1-\tan\left(\sqrt{-\frac{\sigma}{32a^2}}\left(ax+by-\frac{3a^2\sqrt{2\sigma}+2b^2\eta t}{2a}\right)\right)\right)},$$

$$V_{5,12}(x,y,t)=\frac{1}{2}+\frac{\sqrt{-\frac{1}{16}}\left(4-5\cos\left(\sqrt{-\frac{\sigma}{8a^2}}\left(ax+by-\frac{3a^2\sqrt{2\sigma}+2b^2\eta t}{2a}\right)\right)\right)}{\left(3+5\sin\left(\sqrt{-\frac{\sigma}{8a^2}}\left(ax+by-\frac{3a^2\sqrt{2\sigma}+2b^2\eta t}{2a}\right)\right)\right)}+\frac{\left(3+5\sin\left(\sqrt{-\frac{\sigma}{8a^2}}\left(ax+by-\frac{3a^2\sqrt{2\sigma}+2b^2\eta t}{2a}\right)\right)\right)}{4i\left(4-5\cos\left(\sqrt{-\frac{\sigma}{8a^2}}\left(ax+by-\frac{3a^2\sqrt{2\sigma}+2b^2\eta t}{2a}\right)\right)\right)},$$

$$V_{5,13}(x,y,t)=\frac{1}{2}+\frac{\sqrt{\frac{2a^2}{\sigma}}\left(\sqrt{-\frac{\sigma(a^2-b^2)}{32a^2}}-\sqrt{-\frac{\sigma}{32}}\cos\left(\sqrt{-\frac{\sigma}{8a^2}}\left(ax+by-\frac{3a^2\sqrt{2\sigma}+2b^2\eta t}{2a}\right)\right)\right)}{\left(a\sin\left(\sqrt{-\frac{\sigma}{8a^2}}\left(ax+by-\frac{3a^2\sqrt{2\sigma}+2b^2\eta t}{2a}\right)\right)+b\right)}+\frac{\sqrt{2\sigma}\left(a\sin\left(\sqrt{-\frac{\sigma}{8a^2}}\left(ax+by-\frac{3a^2\sqrt{2\sigma}+2b^2\eta t}{2a}\right)\right)+b\right)}{32a\left(\sqrt{-\frac{\sigma(a^2-b^2)}{32a^2}}-\sqrt{-\frac{\sigma}{32}}\cos\left(\sqrt{-\frac{\sigma}{8a^2}}\left(ax+by-\frac{3a^2\sqrt{2\sigma}+2b^2\eta t}{2a}\right)\right)\right)},$$

$$V_{5,14}(x,y,t)=\frac{1}{2}-\sqrt{\frac{1}{16}}+\frac{\frac{1}{2}a}{\left(\begin{array}{l}a+\cos\left(\sqrt{-\frac{\sigma}{8a^2}}\left(ax+by-\frac{3a^2\sqrt{2\sigma}+2b^2\eta t}{2a}\right)\right)\\-i\sin\left(\sqrt{-\frac{\sigma}{8a^2}}\left(ax+by-\frac{3a^2\sqrt{2\sigma}+2b^2\eta t}{2a}\right)\right)\end{array}\right)}-\frac{1}{4-\frac{8a}{\left(\begin{array}{l}a+\cos\left(\sqrt{-\frac{\sigma}{8a^2}}\left(ax+by-\frac{3a^2\sqrt{2\sigma}+2b^2\eta t}{2a}\right)\right)\\-i\sin\left(\sqrt{-\frac{\sigma}{8a^2}}\left(ax+by-\frac{3a^2\sqrt{2\sigma}+2b^2\eta t}{2a}\right)\right)\end{array}\right)}}.$$

Family 3: When $\varphi=0$, then Eq. (1.1) has rational solution:

$$V_{5,15}(x,y,t)=\frac{1}{2}-\frac{\sqrt{\frac{2a^2}{\sigma}}}{\left(ax+by-\frac{3a^2\sqrt{2\sigma}+2b^2\eta t}{2a}\right)}-\frac{\sqrt{2\sigma}}{32a}\left(ax+by-\frac{3a^2\sqrt{2\sigma}+2b^2\eta t}{2a}\right).$$

For **Set 6**, substituting the values of constants into Eq. (3.4) and Eq. (3.1) along with Eq. (2.6)-Eq. (2.20) and simplifying, yields following solutions, respectively:
Family 1: When $\varphi<0$ and the hyperbolic solutions of Eq. (1.1) are given as follows:

$$V_{6,1}(x,y,t)=-\frac{1}{2}-\frac{1}{2\tanh\left(\sqrt{\frac{\sigma}{8a^2}}\left(ax+by-\frac{(-3\sqrt{2}a^2\sigma+2\sqrt{\sigma}b^2\eta)t}{2\sqrt{\sigma}a}\right)\right)},$$

$$V_{6,2}(x,y,t)=-\frac{1}{2}-\frac{1}{2\coth\left(\sqrt{\frac{\sigma}{8a^2}}\left(ax+by-\frac{(-3\sqrt{2}a^2\sigma+2\sqrt{\sigma}b^2\eta)t}{2\sqrt{\sigma}a}\right)\right)},$$

$$V_{6,3}(x,y,t)=-\frac{1}{2}-\frac{1}{2\left(\begin{array}{l}\tanh\left(\sqrt{\frac{\sigma}{2a^2}}\left(ax+by-\frac{\left(-3\sqrt{2}a^2\sigma+2\sqrt{\sigma}b^2\eta\right)t}{2\sqrt{\sigma}a}\right)\right)\\+i\,\mathrm{sech}\left(\sqrt{\frac{\sigma}{2a^2}}\left(ax+by-\frac{\left(-3\sqrt{2}a^2\sigma+2\sqrt{\sigma}b^2\eta\right)t}{2\sqrt{\sigma}a}\right)\right)\end{array}\right)},$$

$$V_{6,4}(x,y,t)=-\frac{1}{2}+\frac{\sqrt{2\sigma}\left(1+\sqrt{\frac{\sigma}{8a^2}}\tanh\left(\sqrt{\frac{\sigma}{8a^2}}\left(ax+by-\frac{\left(-3\sqrt{2}a^2\sigma+2\sqrt{\sigma}b^2\eta\right)t}{2\sqrt{\sigma}a}\right)\right)\right)}{8a\left(-\frac{\sigma}{8a^2}-\sqrt{\frac{\sigma}{8a^2}}\tanh\left(\sqrt{\frac{\sigma}{8a^2}}\left(ax+by-\frac{\left(-3\sqrt{2}a^2\sigma+2\sqrt{\sigma}b^2\eta\right)t}{2\sqrt{\sigma}a}\right)\right)\right)},$$

$$V_{6,5}(x,y,t)=-\frac{1}{2}+\frac{\left(3+4\sinh\left(\sqrt{\frac{\sigma}{4a^2}}\left(ax+by-\frac{\left(-3\sqrt{2}a^2\sigma+2\sqrt{\sigma}b^2\eta\right)t}{2\sqrt{\sigma}a}\right)\right)\right)}{2\left(5-4\cosh\left(\sqrt{\frac{\sigma}{4a^2}}\left(ax+by-\frac{\left(-3\sqrt{2}a^2\sigma+2\sqrt{\sigma}b^2\eta\right)t}{2\sqrt{\sigma}a}\right)\right)\right)},$$

$$V_{6,6}(x,y,t)=-\frac{1}{2}+\frac{\sqrt{2\sigma}\left(a\sinh\left(\sqrt{\frac{\sigma}{2a^2}}\left(ax+by-\frac{\left(-3\sqrt{2}a^2\sigma+2\sqrt{\sigma}b^2\eta\right)t}{2\sqrt{\sigma}a}\right)\right)+b\right)}{8a\left(\sqrt{\frac{\sigma(a^2+b^2)}{8a^2}}-\sqrt{\frac{\sigma}{8}}\cosh\left(\sqrt{\frac{\sigma}{2a^2}}\left(ax+by-\frac{\left(-3\sqrt{2}a^2\sigma+2\sqrt{\sigma}b^2\eta\right)t}{2\sqrt{\sigma}a}\right)\right)\right)},$$

$$V_{6,7}(x,y,t)=-\frac{1}{2}+\frac{1}{2-\frac{4a}{\left(\begin{array}{l}a+\cosh\left(\sqrt{\frac{\sigma}{2a^2}}\left(ax+by-\frac{\left(-3\sqrt{2}a^2\sigma+2\sqrt{\sigma}b^2\eta\right)t}{2\sqrt{\sigma}a}\right)\right)\\ \quad-\sinh\left(\sqrt{\frac{\sigma}{2a^2}}\left(ax+by-\frac{\left(-3\sqrt{2}a^2\sigma+2\sqrt{\sigma}b^2\eta\right)t}{2\sqrt{\sigma}a}\right)\right)\end{array}\right)}}.$$

Family 2: Given $\varphi>0$, using the method obtained the following trigonometric solutions for Eq. (1.1):

$$V_{6,8}(x,y,t)=-\frac{1}{2}+\frac{1}{2i\tan\left(\sqrt{-\frac{\sigma}{8a^2}}\left(ax+by-\frac{\left(-3\sqrt{2}a^2\sigma+2\sqrt{\sigma}b^2\eta\right)t}{2\sqrt{\sigma}a}\right)\right)},$$

$$V_{6,9}(x,y,t)=-\frac{1}{2}-\frac{1}{2i\cot\left(\sqrt{-\frac{\sigma}{8a^2}}\left(ax+by-\frac{\left(-3\sqrt{2}a^2\sigma+2\sqrt{\sigma}b^2\eta\right)t}{2\sqrt{\sigma}a}\right)\right)},$$

$$V_{6,10}(x,y,t)=-\frac{1}{2}+\frac{1}{2i\left(\begin{array}{l}\tan\left(\sqrt{-\frac{\sigma}{2a^2}}\left(ax+by-\frac{\left(-3\sqrt{2}a^2\sigma+2\sqrt{\sigma}b^2\eta\right)t}{2\sqrt{\sigma}a}\right)\right)\\ +\sec\left(\sqrt{-\frac{\sigma}{2a^2}}\left(ax+by-\frac{\left(-3\sqrt{2}a^2\sigma+2\sqrt{\sigma}b^2\eta\right)t}{2\sqrt{\sigma}a}\right)\right)\end{array}\right)},$$

$$V_{6,11}(x,y,t)=-\frac{1}{2}-\frac{\left(1+\tan\left(\sqrt{-\frac{\sigma}{8a^2}}\left(ax+by-\frac{\left(-3\sqrt{2}a^2\sigma+2\sqrt{\sigma}b^2\eta\right)t}{2\sqrt{\sigma}a}\right)\right)\right)}{2i\left(1-\tan\left(\sqrt{-\frac{\sigma}{8a^2}}\left(ax+by-\frac{\left(-3\sqrt{2}a^2\sigma+2\sqrt{\sigma}b^2\eta\right)t}{2\sqrt{\sigma}a}\right)\right)\right)},$$

$$V_{6,12}(x,y,t)=-\frac{1}{2}+\frac{\left(3+5\sin\left(\sqrt{-\frac{\sigma}{2a^2}}\left(ax+by-\frac{\left(-3\sqrt{2}a^2\sigma+2\sqrt{\sigma}b^2\eta\right)t}{2\sqrt{\sigma}a}\right)\right)\right)}{2i\left(4-5\cos\left(\sqrt{-\frac{\sigma}{2a^2}}\left(ax+by-\frac{\left(-3\sqrt{2}a^2\sigma+2\sqrt{\sigma}b^2\eta\right)t}{2\sqrt{\sigma}a}\right)\right)\right)},$$

$$V_{6,13}(x,y,t)=-\frac{1}{2}+\frac{\sqrt{2\sigma}\left(a\sin\left(\sqrt{-\frac{\sigma}{2a^2}}\left(ax+by-\frac{\left(-3\sqrt{2}a^2\sigma+2\sqrt{\sigma}b^2\eta\right)t}{2\sqrt{\sigma}a}\right)\right)+b\right)}{2a\left(\sqrt{-\frac{\sigma(a^2-b^2)}{8a^2}}-\sqrt{-\frac{\sigma}{8}}\cos\left(\sqrt{-\frac{\sigma}{2a^2}}\left(ax+by-\frac{\left(-3\sqrt{2}a^2\sigma+2\sqrt{\sigma}b^2\eta\right)t}{2\sqrt{\sigma}a}\right)\right)\right)},$$

$$V_{6,14}(x,y,t)=-\frac{1}{2}-\frac{1}{2-\dfrac{4a}{\left(\begin{array}{l} a+\cos\left(\sqrt{-\dfrac{\sigma}{2a^2}}\left(ax+by-\dfrac{\left(-3\sqrt{2}a^2\sigma+2\sqrt{\sigma}b^2\eta\right)t}{2\sqrt{\sigma}a}\right)\right) \\ -i\sin\left(\sqrt{-\dfrac{\sigma}{2a^2}}\left(ax+by-\dfrac{\left(-3\sqrt{2}a^2\sigma+2\sqrt{\sigma}b^2\eta\right)t}{2\sqrt{\sigma}a}\right)\right)\end{array}\right)}}.$$

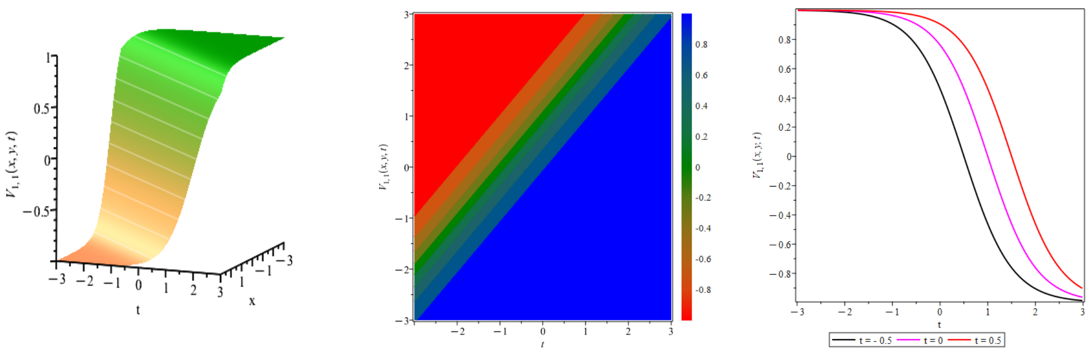
Family 3: When $\varphi=0$, then Eq. (1.1) has rational solution:

$$V_{6,15}(x,y,t)=-\frac{1}{2}-\frac{\sqrt{2\sigma}}{8a\left(ax+by-\frac{\left(-3\sqrt{2}a^2\sigma+2\sqrt{\sigma}b^2\eta\right)t}{2\sqrt{\sigma}a}\right)}.$$

4. Discussion of the results achieved and graphical representation

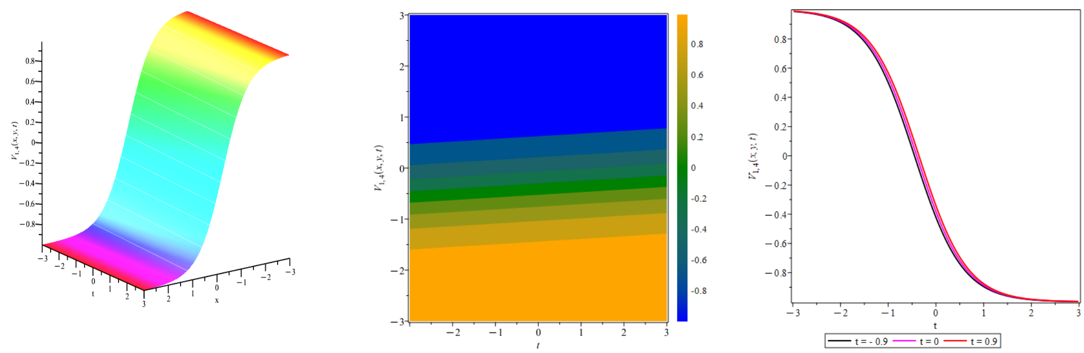
The (2+1)-dimensional CIE is solved using the canonical-like transformation approach and the trial equation method [24]. Sulaiman et al.'s is focused on finding new lump solutions to significant equations with variable coefficients, specifically the (2+1)-dimensional CIE [19]. In this study, METEM is utilized for obtaining several soliton solutions for the (2+1)-dimensional CIE, like trigonometric, hyperbolic, and rational solutions. METEM offers deeper analysis and gives an easier and powerful framework for researching complex occurrences like impact interactions. 3D, contour, and 2D graphs are also provided to help comprehend the patterns of these solutions.

A kink soliton is usually a type of soliton that moves at a certain speed and conserves its energy. Its graph shows a sharp transition from an initial low value to a high value (or vice versa). This can show the movement of a particle in a potential trough or a phase change in a field. Bright solitons are solutions in which the intensity is sharply higher, with a pronounced peak in the centre, while dark solitons are solutions in which the intensity decreases in the centre. The difference between the two types of soliton is the behaviour of their density in the centre. Singular solitons are typically more complex and sometimes theoretically significant solutions. Such solutions usually have infinite values in a given region. Solutions with a singularity are usually cases where a wave or field grows very rapidly, indicating a breakdown point at the centre.



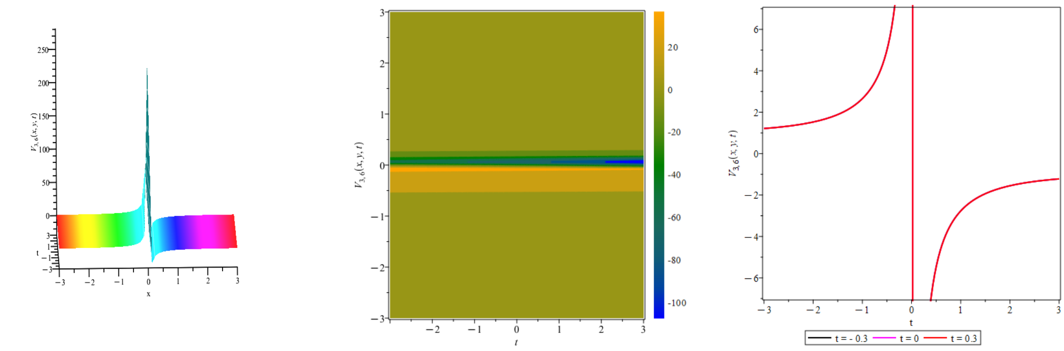
(a) 3D (b) Contour (c) 2D

Figure 4.1: The graphical explanation for $V_{1,1}(x,y,t)$ to Eq. (3.5) when $\sigma=2$, $a=1$, $b=-1$, $\eta=1$, $y=1$.



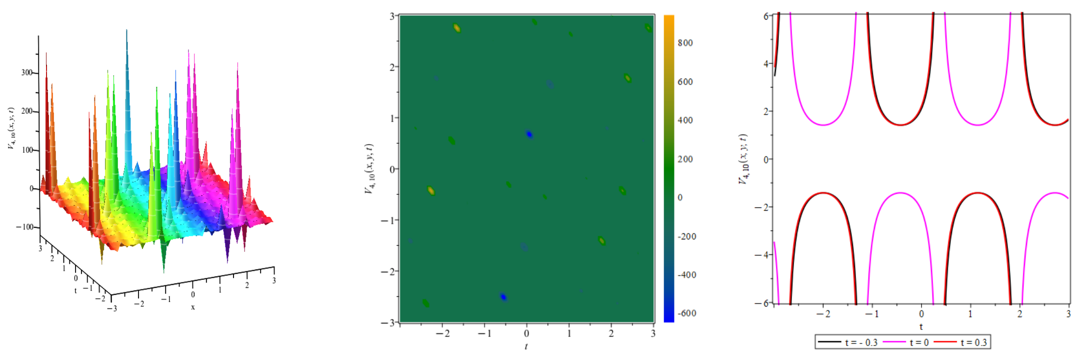
(a) 3D (b) Contour (c) 2D

Figure 4.2: The physical explanation for $V_{1,4}(x,y,t)$ to Eq. (3.6) when $\sigma=2$, $a=5$, $b=1$, $\eta=1.3$, $y=1$.



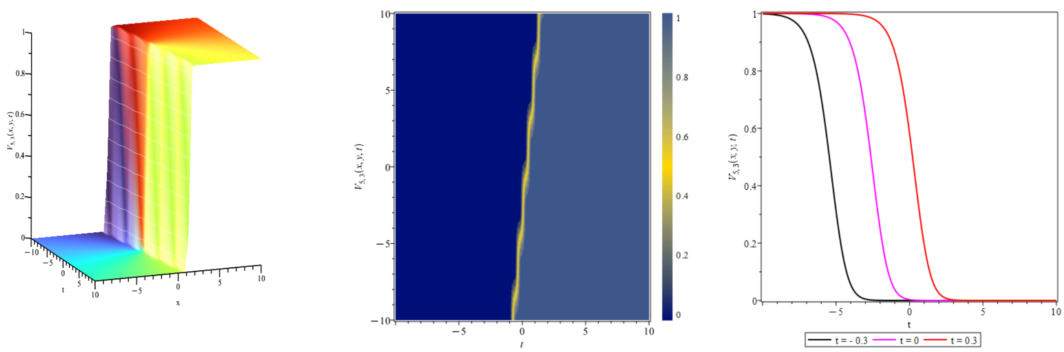
(a) 3D (b) Contour (c) 2D

Figure 4.3: The physical explanation for $V_{3,6}(x,y,t)$ to Eq. (3.7) when $\sigma = 0.3$, $a = 0.5$, $b = 0.04$, $\eta = 0.73$, $y = 1$.



(a) 3D (b) Contour (c) 2D

Figure 4.4: The graphical representation for $V_{4,10}(x,y,t)$ to Eq. (3.8) when $\sigma = 4$, $a = 1$, $b = 2$, $\eta = 1.3$, $y = 1$.



(a) 3D (b) Contour (c) 2D

Figure 4.5: The physical representation for $V_{5,3}(x,y,t)$ to Eq. (3.9) when $\sigma = 4$, $a = 1$, $b = 2$, $\eta = 1.3$, $y = 1$.

5. Conclusion

In this paper, various analytical solutions for (2+1)-dimensional CIE were found using an efficient methodology. In contrast to conventional methods used up to this point, METEM exhibits its capacity to generate novel and broadly applicable precise solutions. This manifestation demonstrates the great promise and effectiveness of the technique in solving difficult single-wave issues that are frequently encountered in mathematical physics. The approach employed here yields analytical solutions, including trigonometric, rational, and hyperbolic function solutions, to the (2+1)-dimensional CIE. Numerous phenomena, including periodic waves, kink-wave patterns, and bright and dark solitons, have been reported in relation to the (2+1)-dimensional CIE. To further explain the dynamic behavior of resource solutions, graphical representations have been created. The unique dynamic structures and features of these solutions can be fully understood through the use of 3D, contour, and 2D graphs. In the field of mathematical physics, these functions are useful to solve PDEs and offer a handy way to illustrate periodic solutions [25–28] . This specific approach has the ability to address a multitude of higher-dimensional nonlinear issues that arise in the fields of mathematics and the applied sciences [23] . As a result, it is expected to contribute to the comprehensive study and investigation of future research. The new results obtained from a wide range of dynamical structures and arbitrary parameters are expected to provide important new insights into the behavior of the gas diffusion equations in a homogeneous medium. The accuracy of these results has been ensured and extensively verified using Maple symbolic computing software.

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