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Oscillatory Behavior for Certain Theorems and Examples of Higher order Nonlinear Delay Differential Equations

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Abstract

In this paper the oscillatory behaviour of higher order nonlinear delay differential equation theorems and examples are investigated. Some new oscillatory main results of higher order nonlinear delay differential equations are given. We discuss the relation of Riccati transformation of the nonlinear delay differential equation to studying properties of the two higher order differential equations. Furthermore, an average integrating method is introduced as a asymptotic approach to study the oscillatory behavior. Some results are extended to nonlinear delay differential equations of any order. An example is also discussed, to illustrate the efficiency of the results obtained.

Keywords: *oscillatory, higher order nonlinear delay differential equations*
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1. Introduction

The theory of impulsive delay differential equations is promising as an important role of investigation, since it is better than the corresponding theory of delay differential equation without impulse effects [13, 15, 17, 19, 21, 23, 25, 27, 29, 31, 33, 35, 37, 41]. Furthermore, such equations may demonstrate several real-world phenomena in physics,chemistry, biology, engineering, etc. In the last few years the theory of periodic solutions and delay differential equations with impulses has been studied by many authors, respectively [14, 16, 18, 20, 22, 24, 26]. There are several books and a many of papers dealing with the periodic solution of delay differential equations [28, 30, 32, 34, 36, 38, 40]. Periodic solutions of impulsive delay differential equations is a new research area and there are many publications in this field. The paper deals with impulsive equations with constant delay and Fredholm operator of index zero. We obtain the theorems of existence of periodic solution based on the following Mawhin's continuation theorem.

In recent years, there has been much research work concerning the oscillation theory and applications of nonlinear higher order delay differential equations; see [5, 31, 39]. Therefore, the oscillatory criteria of higher order differential equations theorems and examples gave many new results. In this paper, the study of oscillatory criteria of nonlinear higher order delay differential equations is detail, but most of them are about delay differential equation; there are many results dealing with the oscillation of the solutions of nonlinear higher order delay differential equations with any order in [1, 3, 7]. A regular function which is defined for all large t is called oscillatory if it has no last zero, otherwise it is called nonoscillatory.

The differential equation itself is called oscillatory if all its assumptions are oscillatory [2, 4, 6]. In recent research, it has been used the oscillation solution and applications of nonlinear delay differential equations and examples; see [8, 10, 12].The authors have worked different solutions of the nonlinear equations [13, 17, 23, 31].

R. P. Agarwal [29, 30, 31] obtained second order and third order conditions for the oscillation of solutions, under the equation that it is also higher order. Then we worked previous results. We follow the same condition as in [9, 11], but with different results in examples and theorems. We shall introduce the properties of the nonlinear delay differential equations. We obtain some previous works known for the nonlinear delay differential equations that are in basically compared on the parameters of nonlinear delay differential equations.

Our work is based on the Riccati transformation and average integrating method for comparing the nonlinear delay with a set of the nonlinear delay differential equations. The oscillation and asymptotic behavior have extensive applications in the real world. See the monographs [30] for more details. The problem of obtaining the oscillation and asymptotic behavior of certain higher-order nonlinear functional differential equations has been studied by a number of authors, see [39] and the references cited therein. The oscillations of these equations are oscillates and converges to zero. Moreover, our results can be easily extend to cover the neutral differential equations in any order.

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2. Oscillatory Behaviour for Theorems

In this section we shall state and prove the theorems of oscillatory behaviour.

Theorem 2.1. Let $\alpha > 1$. Let $f'(u)$ be nondecreasing on $(-\infty, -t)$ and nonincreasing on (t, ∞), $t \geq 0$. We assume that

$$\int^{\infty} p(s)\left|f[c(s-\tau)]\right| \mathrm{ds} = \infty \qquad \text{,for all } c \neq 0 \tag{2.1}$$

and moreover

$$\int^{\infty} \left(\tau^2(s)p(s) - \frac{\tau'(s)}{f'[\lambda(s-\tau)]}\right) \mathrm{ds} = \infty \qquad \text{for some } \lambda > 0. \tag{2.2}$$

Proof. Assume that it has a positive solution of $u(t)$.Then

$$(z)' = -p(t)f\left[u(t-\tau)\right] < 0.$$

Hence, the function $|u'(t)|u'(t)$ is decreasing. Therefore, either $u'(t) > 0$, or $u'(t) < 0$. Since

$$0 > (z)' = 2|u'(t)|u''(t),$$

we assume that $u''(t) < 0$. Let $u(t) \to -\infty$ as $t \to \infty$. This is a contradiction. So we conclude that $u(t) > 0$, $u'(t) > 0$, $u''(t) < 0$ and

$$\left[\left(u'(t)\right)^2\right]' = -p(t)f\left[u(t-\tau)\right]. \tag{2.3}$$

We define

$$w(t) = \tau^2(t)\frac{[u'(t)]^2}{f[u(t-\tau)]}. \tag{2.4}$$

Then $w(t) > 0$ and

$$\begin{aligned} w'(t) &= 2\tau\tau'(t)\frac{[u'(t)]^2}{f[u(t-\tau)]} + \tau^2(t)\frac{\left[\left(u'(t)\right)^2\right]'}{f[u(t-\tau)]} \\ &\quad -\tau^2(t)\frac{[u'(t)]^2 f'[u(t-\tau)]u'(\tau(t))\tau'(t)}{f^2[u(t-\tau)]} \\ &= 2\frac{\tau'(t)}{\tau(t)}w(t) - \tau^2(t)p(t) - w(t)\frac{f'[u(t-\tau)]u'(\tau(t))\tau'(t)}{f[u(t-\tau)]}. \end{aligned} \tag{2.5}$$

We claim that $u'(t) \to 0$ as $t \to \infty$. To prove it is contradiction, that is $u'(t) \to 2c$ as $t \to \infty$, $c > 0$. Then $u'(t) \geq 2c$ which on integration from t_1 to t implies

$$u(t) \geq u(t_1) + 2c(t-t_1) \geq ct. \tag{2.6}$$

Integrating (2.3) from t_1 to t and using (2.6)

$$-\left[u'(t)\right]^2 + \left[u'(t_1)\right]^2 = \int_{t_1}^{t} p(s)f[u(s-\tau)]\,\mathrm{ds} > \int_{t_1}^{t} p(s)f[c(s-\tau)]\,\mathrm{ds}.$$

Putting $t \to \infty$ we have

$$\int_{t_1}^{\infty} p(s)f[c(s-\tau)]\,\mathrm{ds} < \infty.$$

It shows that $u'(t) \to 0$ as $t \to \infty$. Therefore, for any $\lambda > 0$ there exists a t_1 such that $\lambda/2 > u'(t)$, $t \geq t_1$. Integrating the functional inequality from t_1 to t we have

$$u(t) \leq u(t_1) + \frac{\lambda}{2}(t-t_1) \leq \lambda t, \qquad t \geq t_2 \geq t_1$$

and so for any $\lambda > 0$ and t large enough

$$f'[u(t-\tau)] \geq f'[\lambda(t-\tau)]. \tag{2.7}$$

Conversely, since $u'(t)$ is decreasing and $u'(t) \to 0$ as $t \to \infty$ it follows that

$$u'(t-\tau) \geq u'(t) \geq \left(u'(t)\right)^2. \tag{2.8}$$

Combining (2.7) and (2.8) together with (2.5) we have

$$\begin{aligned} w'(t) &\leq -\tau^2(t)p(t) + 2\frac{\tau'(t)}{\tau(t)}w(t) - \frac{\tau'(t)f'[\lambda(t-\tau)]}{\tau^2(t)}w^2(t) = -\tau^2(t)p(t) \\ &\quad -\frac{\tau'(t)f'[\lambda(t-\tau)]}{\tau^2(t)}\left[\left(w(t) - \frac{\tau(t)}{f'[\lambda(t-\tau)]}\right)^2 - \frac{\tau^2(t)}{(f'[\lambda(t-\tau)])^2}\right] \\ &\leq -\tau(t)p(t) + \frac{\tau(t)\tau'(t)}{f'[\lambda(t-\tau)]}. \end{aligned} \tag{2.9}$$

Integrating the above inequality from t_2 to t we conclude in the point of (2.2) that $w(t) \to -\infty$ as $t \to \infty$. This is a contradiction. Hence the proof is complete.

Theorem 2.2. Let $\alpha > 1$. Let $f'(u)$ be nonincreasing on $(-\infty, -t)$ and nondecreasing on (t, ∞), $t \geq 0$. We assume that (2.1) holds for any $c \neq 0$. If

$$\int^{\infty} \left(\tau^2(s)p(s) - M\tau(s)\tau'(s)\right) \mathrm{d}s = \infty \qquad \text{for some } M > 0. \tag{2.10}$$

Proof. We assume that $M > 0$ is such that (2.10) holds. Here $u(t)$ is a positive solution. In the proof of Theorem 2.1 we can verify that $u'(t) > 0$, $u''(t) < 0$ and $u'(t) \to 0$ as $t \to \infty$. Then there exists $c > 0$ such that $u(t-\tau) > c$, exactly. If $w(t)$ be defined by (2.4), then $w(t) > 0$ and (2.5) is determined. It is easy to verify that

$$f'[u(t-\tau)]u'(t-\tau) \geq f'(c)\,u'(t) = f'(c)\left(u'(t)\right)^{-1}\left(u'(t)\right)^2. \tag{2.11}$$

Since $u'(t) \to 0$ then for any $\lambda > 0$ we have $u'(t) < \lambda$, exactly. We see from (2.11) that

$$f'[u(t-\tau)]u'(t-\tau) \geq f'(c)\lambda^{-1}\left(u'(t)\right)^2 = K\left(u'(t)\right)^2,$$

where λ is taken such that $f'(c)\lambda^{-1} = 1/(M)$. We have

$$\begin{aligned} w'(t) &\leq -\tau^2(t)p(t) + 2\frac{\tau'(t)}{\tau(t)}w(t) - K\frac{\tau'(t)}{\tau^2(t)}w^2(t) = -\tau^2(t)p(t) \\ &\quad -K\frac{\tau'(t)}{\tau^2(t)}\left[\left(w(t) - \frac{\tau(t)}{K}\right)^2 - \frac{\tau^2(t)}{K^2}\right] \\ &\leq -\tau^2(t)p(t) + \frac{1}{K}\tau(t)\tau'(t). \end{aligned} \tag{2.12}$$

We Integrate the inequality from t_1 to t, and then putting $t \to \infty$.This is contradiction. Hence the proof is complete.

Theorem 2.3. We assume that

$$\int_{t_0}^{\infty} \frac{\mathrm{d}u}{|f(u)|^{1/2}} < \infty$$

and

$$\int_{t_0}^{\infty} \tau'(s)\left(\int_s^{\infty} p(x)\,\mathrm{d}x\right)^{1/2} \mathrm{d}s = \infty$$

are oscillatory.

Proof. We assume that $u(t)$ is a positive solution. Similarly as in the proof of Theorem 2.1 it can be shown that $u'(t) > 0$ and $u''(t) < 0$. Integrating from t to s we have

$$-\left[u'(s)\right]^2 + \left[u'(t)\right]^2 = \int_t^s p(x)f[u(x-\tau)]\,\mathrm{d}x \geq f[u(t-\tau)]\int_t^s p(s)\,\mathrm{d}s.$$

Using conditions of $u'(t)$ and putting $s \to \infty$ we have

$$\left(u'[t-\tau]\right)^2 \geq \left(u'(t)\right)^2 \geq f[u(t-\tau)]\int_t^{\infty} p(s)\,\mathrm{d}s. \tag{2.13}$$

It follows from (2.13) that

$$\frac{u'[t-\tau]\tau'(t)}{f^{1/2}[u(t-\tau)]} \geq \tau'(t)\left(\int_t^{\infty} p(x)\,\mathrm{d}x\right)^{1/2}$$

which on integration from t_1 to t gives

$$\int_{u[(t_1)-\tau]}^{u[t-\tau]} \frac{\mathrm{d}s}{f^{1/2}(s)} \geq \int_{t_1}^{t} \tau'(s)\left(\int_s^{\infty} p(x)\,\mathrm{d}x\right)^{1/2} \mathrm{d}s. \tag{2.14}$$

The left side of (2.14) is bounded, on the other hand the right side of (2.14) tends to ∞ as $t \to \infty$. Hence the proof is complete.

Theorem 2.4. Assume that $u(t) > 0$, $u'(t) > 0$, $u''(t) > 0$, $(a(t)(u''(t))^n)' \leq 0$ on $[t_0, \infty)$. Then for each $\ell \in (0,1)$ there exists $T_\ell \geq t_0$ such that

$$\frac{u(\tau(t))}{a(\tau(t))} \geq \ell\frac{u(t)}{a(t)} \qquad for t \geq T_\ell.$$

Proof. We set $a(t)(u''(t))^n$ is non-increasing. Then we define $a^{1/n}(t)(u''(t))$.

$$u(t) - u(\tau(t)) = \int_{\tau(t)}^{t} a^{1/n}(s)(u''(s))\frac{1}{a^{1/n}(s)}\,ds \leq a^{1/n}(\tau(t))u''(\tau(t))\left(a(t) - a(\tau(t))\right). \tag{2.15}$$

$$\begin{aligned} u(\tau(t)) &\geq u(\tau(t)) - u(t_0) \\ &\geq a^{1/n}(\tau(t))u''(\tau(t))\left(a(\tau(t)) - a(t_0)\right). \end{aligned}$$

It means that $\lim_{t\to\infty}\frac{a(\tau)-a(t_0)}{a(\tau)}=1$, for each $\ell\in(0,1)$ there exists $T_\ell\geq t_0$ such that $\big(a(\tau(t))-a(t_0)\big)>\ell a(\tau(t))$ for $t\geq T_\ell$. From the above (2.15),

$$\frac{u'(\tau(t))}{u''(\tau(t))}\geq \ell a^{1/n}(\tau(t))a(\tau(t)),\quad t\geq T_\ell. \tag{2.16}$$

Combining (2.15) together with (2.16), we have

$$\frac{u(t)}{u(\tau(t))}\leq 1+\frac{a(t)-a(\tau(t))}{\ell a(\tau(t))}\leq\frac{a(t)}{\ell a(\tau(t))},$$

which completes the proof.

Theorem 2.5. Assume that on (T_ℓ,∞) and then $\frac{z(t)}{z'(t)}\geq\frac{a^{1/n}(t)a(t)}{2}$ $for\quad t\geq T_\ell$.

Proof. We set $a(t)(z'''(t))^n$ is positive and non-increasing. Then, we define $a^{1/n}(t)z'''(t)$. Let $z''(t)>0$, $z'(t)>0$, $a(t)>0$, we have

$$z''(t)\geq z''(t)-z''(\tau(t))\geq\int_{T_\ell}^t\frac{a^{1/n}(s)z'''(s)}{a^{1/n}(s)}ds\geq a^{1/n}(t)a(t)z'''(t). \tag{2.17}$$

Let us denote $a'(t)=a^{-1/n}(t)$ and $a(T_\ell)z''(T_\ell)>0$,

$$a'(t)z''(t)\geq a(t)z'''(t),\quad t\geq T_\ell. \tag{2.18}$$

Integrating both sides of the above inequality, we have

$$\int_{T_\ell}^t a'(s)z''(s)ds\geq a(t)z''(t)-\int_{T_\ell}^t a'(s)z''(s)ds.$$

Which implies that

$$\int_{T_\ell}^t a'(s)z''(s)ds\geq\frac{1}{2}a(t)z''(t). \tag{2.19}$$

Therefore $a(t)$ is non-increasing, then we have $a(t)>0$, $a'(t)>0$, $a''(t)\geq 0$. and denote

$$(a'(t)z(t))'=a'(t)z'(t)+a''(t)z(t)\geq a'(t)z'(t). \tag{2.20}$$

At the end, integrating on both sides of the above equation (2.19), one have

$$a'(t)z(t)\geq\frac{1}{2}a(t)z''(t),\quad t\geq T_\ell,$$

which completes the proof.

Theorem 2.6. Let $x(t)$ be a positive solution. Let $A<\infty$, $B<\infty$ and $z(t)$ satisfy $A\leq r-r^{1+\frac{1}{n}}$ and $A+B\leq 1$. If $A=\infty$ or $B=\infty$, then $z(t)$ does not have any other oscillatory conditions.

Proof. Since that $x(t)$ is a positive solution, then the corresponding function $z(t)$ satisfies that

$$x(t)=z(t)-p(t)x(\tau_1(t))>z(t)-p(t)z(\tau_1(t))\geq(1-p)z(t). \tag{2.21}$$

Using this equation (2.21), we have

$$(a(t)(z'''(t))^n)'\leq-(1-p)^n q(t)z^n(\tau(t))\leq 0. \tag{2.22}$$

we see that $w(t)$ is a positive solution of

$$\begin{aligned}w'(t)&=\frac{1}{(z''(t))^n}\big(a(t)(z'''(t))^n\big)'-na(t)\Big(\frac{z'''(t)}{z''(t)}\Big)^{n+1}\\&\leq -q(t)(1-p)^n\frac{z^n(\tau(t))}{(z''(t))^n}-\frac{n}{a^{1/n}(t)}w^{1+\frac{1}{n}}(t).\end{aligned}$$

From Lemma 2.4 with $u(t)=z'(t)$, we can verify that

$$\frac{1}{z''(t)}\geq\ell\frac{a(\tau(t))}{a(t)}\frac{1}{z''(\tau(t))},\quad t\geq T_\ell,$$

where ℓ is equal to A_ℓ. Now (2.22) provides

$$w''(t)\leq-\ell^n q(t)(1-p)^n\Big(\frac{a(\tau(t))}{a(t)}\Big)^n\frac{z^n(\tau(t))}{(z''(\tau(t)))^n}-\frac{n}{a^{1/n}(t)}w^{1+\frac{1}{n}}(t).$$

Since $z(t)\geq\frac{a^{1/n}(t)a(t)}{2}z''(t)$, we denote

$$w'(t)+A_\ell(t)+\frac{n}{a^{1/n}(t)}w^{1+\frac{1}{n}}(t)\leq 0. \tag{2.23}$$

Then $A_\ell(t)>0$ and $w(t)>0$ for $t\geq T_\ell$. We see that $w''(t)\leq 0$ and $-w''(t)\geq nw^{1+(1/n)}(t)/a^{1/n}(t)$, one have

$$\Big(\frac{1}{w^{1/n}(t)}\Big)' > \frac{1}{a^{1/n}(t)}.$$

where $w^{-1/n}(T_\ell)>0$. Integrating from T_ℓ to t, we have

$$w(t) < \frac{1}{\left(\int_{T_\ell}^t a^{-1/n}(s)\,ds\right)^n},$$

It means that $\lim_{t\to\infty} w(t)=0$.
On the other hand, from the equation in view of provides $w(t)$,

$$a^n(t)w(t) = a(t)\Big(\frac{a(t)z'''(t)}{z''(t)}\Big)^n = \Big(\frac{a(t)z'''(t)}{a'(t)z''(t)}\Big)^n \leq 1^n.$$

Furthermore,

$$0\leq r\leq R\leq 1. \tag{2.24}$$

Let $\varepsilon>0$. Then from A and r, we can use $t\geq T_\ell$, such that

$$a^n(t)\int_t^\infty A_\ell(s)ds \geq A-\varepsilon \quad and \quad a^n(t)w(t)\geq r-\varepsilon \quad for\ t\geq T_\ell.$$

Integrating (2.23) from t to ∞, we have

$$w(t)\geq \int_t^\infty A_\ell(s)ds + n\int_t^\infty \frac{w^{1+\frac{1}{n}}(s)}{a^{1/n}(s)}ds \quad for\ t\geq T_\ell. \tag{2.25}$$

Multiplying the above equation (2.25) by $a^n(t)$ and simplifying, we have

$$a^n(t)w(t)\geq a^n(t)\int_t^\infty A_\ell(s)ds + na^n(t)\int_t^\infty \frac{a^{n+1}(s)w^{1+\frac{1}{n}}(s)}{a^{n+1}(s)a^{1/n}(s)}ds$$

$$\geq (A-\varepsilon)+(r-\varepsilon)^{1+\frac{1}{n}}a^n(t)\int_t^\infty \frac{na'(s)}{a^{n+1}(s)}ds,$$

$$a^n(t)w(t)\geq (A-\varepsilon)+(r-\varepsilon)^{1+\frac{1}{n}}.$$

Taking the limit on both sides as $t\to\infty$, we have

$$r\geq (A-\varepsilon)+(r-\varepsilon)^{1+\frac{1}{n}}.$$

Since $\varepsilon>0$ is arbitrary, we have the required result

$$A\leq r-r^{1+\frac{1}{n}}.$$

Multiplying (2.23) by $a^{n+1}(t)$ and integrating it from t_2 to t, we have

$$\int_{t_2}^t a^{n+1}(s)w''(s)ds \leq -\int_{t_2}^t a^{n+1}(s)A_\ell(s)ds - n\int_{t_2}^t \frac{(a^n(s)w(s))^{(n+1)/n}}{a^{1/n}(s)}ds.$$

$$a^{n+1}(t)w(t)\leq a^{n+1}(t_2)w(t_2) - \int_{t_2}^t a^{n+1}(s)A_\ell(s)ds$$

$$-n\int_{t_2}^t \frac{(a^n(s)w(s))^{(n+1)/n}}{a^{1/n}(s)}ds + \int_{t_2}^t w(s)\left(a^{n+1}(s)\right)'ds.$$

Which implies that

$$a^{n+1}(t)w(t)\leq a^{n+1}(t_2)w(t_2) - \int_{t_2}^t a^{n+1}(s)A_\ell(s)ds$$

$$+\int_{t_2}^t \Big[\frac{(n+1)a^n(s)w(s)}{a^{1/n}(s)} - \frac{n(a^n(s)w(s))^{(n+1)/n}}{a^{1/n}(s)}\Big]ds.$$

Using the notation

$$Eu-Du^{(n+1)/n}\leq \frac{n^n}{(n+1)^{n+1}}\frac{E^{n+1}}{D^n} \tag{2.26}$$

and $u=a^n(t)w(t)$, $D=\frac{n}{a^{1/n}(t)}$, $E=\frac{n+1}{a^{1/n}(t)}$, we set

$$a^{n+1}(t)w(t)\leq a^{n+1}(t_2)w(t_2) - \int_{t_2}^t a^{n+1}(s)A_\ell(s)ds + a(t)-a(t_2).$$

It means that

$$a^n(t)w(t) \le \frac{1}{a(t)}a^{n+1}(t_2)w(t_2) - \frac{1}{a(t)}\int_{t_2}^{t} a^{n+1}(s)A_\ell(s)ds + 1 - \frac{a(t_2)}{a(t)}.$$

Taking the limit on both sides as $t \to \infty$, we have

$$R \le -B+1. \tag{2.27}$$

Combining this inequality (2.27) with (2.24), one have

$$A \le r - r^{1+\frac{1}{n}} \le r \le R \le -B+1,$$

We assume that $x(t)$ is a positive solution. We will prove that $z(t)$ can not satisfy. On the contradiction, $A = \infty$. From (2.25),

$$a^n(t)w(t) \ge a^n(t)\int_t^\infty A_\ell(s)ds.$$

Then the equation (2.24) is equal to 1. But the limit is $A = \infty$. This leads to a contradiction.
Next, we assume that $B = \infty$. Then combining equation (2.27), $R = -\infty$, which leads to a contradiction $0 \le R \le 1$ in (2.24). We can assume that $x(t)$ is a non-oscillatory solution. We can use without loss of generality that $x(t)$ is positive solution. If $A = \infty$, then $z(t)$ does not have any other oscillatory conditions . Hence, $z(t)$ satisfies, therefore, we have $\lim_{t\to\infty} x(t) = 0$.
Furthermore, we obtain $z(t)$ satisfies that set $A \le r - r^{(n+1)/n}$. Using (2.26) with $E = D = 1$, we obtain

$$A \le \frac{n^n}{(n+1)^{n+1}},$$

which leads to a contradiction. The proof is complete.

Theorem 2.7. Assume further that there exists a $\rho \in C^1(I, R^+)$ such that

$$\limsup_{t\to\infty} \int_T^t \left[K\rho(s)q(s) - \frac{2^{l-3}(l-1)!(n-l+2)!(\rho'(s))^2}{g^{l-1}(s)(s-g(s))^{n-l+2}g'(s)\rho(s)}\right]ds = +\infty \tag{2.28}$$

holds for every $T \ge a$ and for all $l = 2,4,\dots,n+2$ when n is even and for all $l = 1,3,\dots,n+2$ when n is odd. Then every solution x is oscillatory, or satisfies $x(t) \to 0$ as $t \to \infty$.

Proof. Let x be a non-oscillatory solution. Without loss of generality, we may assume that $x(t) > 0$ and $x(g(t)) > 0$ for $t \ge a$. There exists a constant $T \ge a$ such that $x^{(n)}(t) > 0$ or $x^{(n)}(t) < 0$ for $t \ge T$.
Consider firstly the case that $x^{(n)}(t) > 0, t \ge T$. We know that $x^{(n+3)}(t) < 0, t \ge T$. Therefore, it follows that there exists $l \in \{1,3,\dots,n+2\}$ when n is odd such that for all sufficiently large t, $x^{(j)}(t) > 0$ for $j = 0,1,\dots,l$ and $(-1)^{n+j}x^{(j)}(t) > 0$ for $j = l+1, l+2,\dots,n+2$.
If $l \ge 1$, then we consider the function w defined by

$$w(t) = \frac{\rho(t)x^{(n+2)}(t)}{x(g(t))}, \quad t \in I. \tag{2.29}$$

$$\begin{aligned}
w'(t) &= \frac{\rho'(t)}{\rho(t)}w(t) - \frac{\rho(t)q(t)f(x(g(t)))}{x(g(t))} - \frac{\rho(t)p(t)x^{(n)}(t)}{x(g(t))} - \frac{\rho(t)x^{(n+2)}(t)x'(g(t))g'(t)}{x^2(g(t))} \\
&\le \frac{\rho'(t)}{\rho(t)}w(t) - K\rho(t)q(t) - \frac{g^{l-1}(t)(t-g(t))^{n-l+2}g'(t)\rho(t)(x^{(n+2)}(t))^2}{2^{l-1}(l-1)!(n-l+2)!x^2(g(t))} \\
&= \frac{\rho'(t)}{\rho(t)}w(t) - K\rho(t)q(t) - w^2(t)\frac{g^{l-1}(t)(t-g(t))^{n-l+2}g'(t)}{2^{l-1}(l-1)!(n-l+2)!\rho(t)} \\
&= -K\rho(t)q(t) - \frac{g^{l-1}(t)(t-g(t))^{n-l+2}g'(t)}{2^{l-1}(l-1)!(n-l+2)!\rho(t)}\Big(w(t) \\
&\quad - \frac{2^{l-1}(l-1)!(n-l+2)!\rho(t)\rho'(t)}{2\rho(t)g^{l-1}(t)(t-g(t))^{n-l+2}g'(t)}\Big)^2 + \frac{2^{l-3}(l-1)!(n-l+2)!\rho'^2(t)}{\rho(t)g^{l-1}(t)(t-g(t))^{n-l+2}g'(t)}.
\end{aligned}$$

Thus

$$w'(t) \le -K\rho(t)q(t) + \frac{2^{l-3}(l-1)!(n-l+2)!\rho'^2(t)}{\rho(t)g^{l-1}(t)(t-g(t))^{n-l+2}g'(t)}.$$

Integration yields

$$\int_T^t \left(K\rho(s)q(s) - \frac{2^{l-3}(l-1)!(n-l+2)!\rho'^2(s)}{\rho(s)g^{l-1}(s)(s-g(s))^{n-l+2}g'(s)}\right)ds \le w(T) - w(t), \quad t > T,$$

which contradicts (2.28).
If $l = 0$, then

$$\begin{gathered}
x'(t) < 0, \quad x''(t) > 0, \quad x'''(t) < 0, \quad \dots, \\
x^{(n)}(t) > 0, \quad x^{(n+1)}(t) < 0, \quad x^{(n+2)}(t) > 0
\end{gathered}$$

for sufficiently large t, namely, for $t \geq T_1$. Let $\lim_{t\to\infty} x(t) = \mu$. If $\mu \neq 0$, then there exists a constant $T_2 \geq T_1$ such that $x(g(t)) \geq x(t) > \mu > 0, t \geq T_2$. We obtain

$$x^{(n+2)}(t) \leq x^{(n+2)}(T_2) - K\int_{T_2}^{t} x(g(u))q(u)du \leq x^{(n+2)}(T_2) - K\mu\int_{T_2}^{t} q(u)du, \tag{2.30}$$

for $t \geq T_2$. We know that $\int_{T_2}^{\infty} q(u)du = +\infty$. Thus inequality (2.30) implies that $x^{(n+2)}(t)$ is eventually negative, a contradiction to (2.30).
Consider next the case that $x^{(n)}(t) < 0$ for $t \geq T$. By $x(t)$ is eventually monotonous and $x^{(n-1)}(t)$ is eventually positive. Let

$$\lim_{t\to+\infty} x(t) = \alpha_1, \quad \lim_{t\to+\infty} x^{(n-1)}(t) = \alpha_2.$$

We claim that $\alpha_1 = 0$. If this is not true, then there exist constants $\beta_1, \beta_2 > 0$ such that

$$x(g(t)) > \beta_1, \quad 0 < x^{(n-1)}(t) < \beta_2, \quad t \geq T_3 \tag{2.31}$$

for some constant $T_3 > 0$.
Integrating from T_3 to t yields

$$\begin{aligned} x^{(n+2)}(t) + \textstyle\int_{T_3}^{t}[(p(u)x^{(n-1)}(u))' - p'(u)x^{(n-1)}(u)]du \\ + \textstyle\int_{T_3}^{t} x(g(u))q(u)\frac{f(x(g(u)))}{x(g(u))}du \\ = x^{(n+2)}(T_3). \end{aligned}$$

from (2.31) we obtain

$$\begin{aligned} x^{(n+2)}(t) &\leq x^{(n+2)}(T_3) + p(T_3)x^{(n-1)}(T_3) + \textstyle\int_{T_3}^{t} p'(u)x^{(n-1)}(u)du - \int_{T_3}^{t}\beta_1 Kq(u)du \\ &\leq x^{(n+2)}(T_3) + p(T_3)x^{(n-1)}(T_3) + \textstyle\int_{T_3}^{t} x^{(n-1)}p'_+(u)du - \int_{T_3}^{t}\beta_1 Kq(u)du \\ &\leq x^{(n+2)}(T_3) + p(T_3)x^{(n-1)}(T_3) + \textstyle\int_{T_3}^{t}\beta_2 p'_+(u)du - \int_{T_3}^{t}\beta_1 Kq(u)du \\ &= x^{(n+2)}(T_3) + p(T_3)x^{(n-1)}(T_3) - \beta_1 K\textstyle\int_{T_3}^{t}[q(u) - \frac{\beta_2}{\beta_1 K}p'_+(u)]du. \end{aligned}$$

By letting $t \to +\infty$, we have from $x^{(n+2)}(t) \to -\infty$. Consequently, there is a constant $T_4 \geq T_3$ such that $x^{(n+2)}(t) \leq -1$ for $t \geq T_4$. Hence $x^{(n+1)}(t) \leq x^{(n+1)}(T_4) - (t - T_4) \to -\infty$ as $t \to +\infty$. By the same way, it follows that $x^{(n)}(t), x^{(n-1)}(t), \ldots, x'(t), x(t) \to -\infty$ as $t \to +\infty$. This contradict the assumption that $x(t)$ is eventually positive.

Theorem 2.8. Assume further that there exist functions $H \in \Re$ and $\rho \in C^1(I, R^+)$ such that

$$\limsup_{t\to\infty} \frac{1}{H(t,T)}\int_T^t \Big[K\rho(s)H(t,s)q(s) - \frac{(\rho(s)h(t,s) - \sqrt{H(t,s)}\rho'(s))^2}{\rho^2(s)G_l(s)}\Big]ds = +\infty, \tag{2.32}$$

where

$$G_l(t) = \frac{g^{l-1}(t)(t-g(t))^{n-l+2}g'(t)}{a(l)\rho(t)} \quad a(l) = 2^{l-3}(l-1)!(n-l+2)!,$$

where $l = 2, 4, \ldots, n+2$ when n is even, and $l = 1, 3, \ldots, n+2$ when n is odd. Then every solution x is oscillatory, or satisfies $x(t) \to 0$ as $t \to \infty$.

Proof. Let x be a non-oscillatory solution. Without loss of generality, we may assume that $x(t) > 0$ and $x(g(t)) > 0$ for $t \geq a$. There exists a constant $T \geq a$ such that $x^{(n)}(t) > 0$ or $x^{(n)}(t) < 0$ for $t \geq T$.
Assume firstly that $x^{(n)}(t) > 0$ for $t \geq T$. It follows that $x^{(n+3)}(t) < 0$ and hence there exists $l \in \{1, 3, \ldots, n+2\}$ when n is odd such that for all sufficiently large t, $x^{(j)}(t) > 0$ for $j = 0, 1, \ldots, l$ and $(-1)^{n+j}x^{(j)}(t) > 0$ for $j = l+1, l+2, \ldots, n+2$.
Defining again the function w as in (2.29). If $l \neq 0$, then we have from (2.30) that

$$K\rho(t)q(t) \leq -w'(t) + \frac{\rho'(t)}{\rho(t)}w(t) - \frac{1}{4}w^2(t)G_l(t). \tag{2.33}$$

Thus

$$\begin{aligned} K\textstyle\int_T^t H(t,s)\rho(s)q(s)ds \\ \leq \textstyle\int_T^t \Big[-w'(s)H(t,s) + \Big(\frac{\rho'(s)}{\rho(s)}w(s) - \frac{1}{4}w^2(s)G_l(s)\Big)H(t,s)\Big]ds. \end{aligned}$$

Using integration by parts and noting that $H \in \Re$, we find

$$\begin{aligned} -\int_T^t w'(s)H(t,s)ds \quad &= w(T)H(t,T) + \textstyle\int_T^t w(s)\frac{\partial H(t,s)}{\partial s}ds \\ &= w(T)H(t,T) - \textstyle\int_T^t w(s)h(t,s)\sqrt{H(t,s)}ds. \end{aligned}$$

Let

$$Q(t,s) = h(t,s) - \sqrt{H(t,s)}\frac{\rho'(s)}{\rho(s)},$$

then

$$\begin{aligned} &K\int_T^t H(t,s)\rho(s)q(s)ds \\ &\leq w(T)H(t,T)-\int_T^t\left[w(s)\sqrt{H(t,s)}Q(t,s)+\tfrac{1}{4}G_l(s)H(t,s)w^2(s)\right]ds \\ &= w(T)H(t,T)-\tfrac{1}{4}\int_T^t G_l(s)H(t,s)\left(w(s)+\tfrac{2Q(t,s)}{G_l(s)\sqrt{H(t,s)}}\right)^2ds+\int_T^t\tfrac{Q^2(t,s)}{G_l(s)}ds \\ &\leq w(T)H(t,T)+\int_T^t\tfrac{Q^2(t,s)}{G_l(s)}ds. \end{aligned}$$

It turns out that

$$\frac{1}{H(t,T)}\int_T^t\left[KH(t,s)\rho(s)q(s)-\frac{Q^2(t,s)}{G_l(s)}\right]ds\leq w(T). \tag{2.34}$$

This contradicts 2.32. The rest of the proof is the same as in Theorem 2.7, and hence it is omitted.

Theorem 2.9. Suppose the following conditions hold:

(i) It has an eventually positive increasing solution;
(ii) there are integer $m>1$ and constant $\alpha>0$ such that $\lim_{t\to\infty}q(t)/t^{m-1}\geq\alpha$;
(iii) $g(t)=at-\tau$ with $0<a\leq 1$ and $\tau>0$.

Then every solution x is oscillatory, or satisfies $x(t)\to 0$ as $t\to\infty$.

Proof. we only give the proof of the case that $a=1$. Obviously, condition (ii) implies that $q(t)/(t-\tau)^{m-1}>\alpha/2, t\geq T_1$ for some constant $T_1>a$. Hence

$$\begin{aligned} &\int_{T_1}^t\left(\tfrac{g(t)}{g(s)}-1\right)^m q(s)ds \\ &=\int_{T_1}^t\tfrac{(t-s)^m}{s-\tau}\cdot\tfrac{q(s)}{(s-\tau)^{m-1}}ds \\ &\geq\tfrac{\alpha}{2}\int_{T_1}^t\tfrac{(t-s)^m}{s-\tau}ds \\ &=\tfrac{\alpha}{2}\int_{T_1}^t\tfrac{((t-\tau)-(s-\tau)))^m}{s-\tau}ds \\ &=\tfrac{\alpha}{2}\sum_{k=0}^m C_m^k(-1)^k(t-\tau)^{m-k}\int_{T_1}^t(s-\tau)^{k-1}ds \\ &=\tfrac{\alpha}{2}\left((t-\tau)^m ln\tfrac{t-\tau}{T_1-\tau}+\sum_{k=1}^m C_m^k(-1)^k\tfrac{(t-\tau)^m-(t-\tau)^{m-k}(T_1-\tau)^k}{k}\right), \end{aligned}$$

where $C_m^k=\frac{m!}{(m-k)!k!}$. It turns out that

$$\begin{aligned} &\lim_{t\to\infty}\tfrac{1}{(g(t)-g(T))^m}\int_T^t\left(\tfrac{g(t)}{g(s)}-1\right)^m q(s)ds \\ &\geq\lim_{t\to\infty}\tfrac{\alpha}{2}\left(\tfrac{(t-\tau)^m}{(t-T)^m}ln\tfrac{t-\tau}{T_1-\tau}+\sum_{k=1}^m C_m^k(-1)^k\tfrac{(t-\tau)^m-(t-\tau)^{m-k}(T_1-\tau)^k}{k(t-T)^m}\right) \\ &=+\infty. \end{aligned}$$

$$\begin{aligned} &\tfrac{1}{(g(t)-g(T))^m}\int_T^t\tfrac{(g(t)-g(s))^{m-2}g^2(t)g'(s)}{g^{l+m+1}(s)(s-g(s))^{n-l+2}}ds \\ &=\tfrac{(t-\tau)^2}{(t-T)^m}\int_T^t\tfrac{((t-\tau)-(s-\tau))^{m-2}}{(s-\tau)^{l+m+1}\tau^{n-l+2}}ds \\ &=\tfrac{1}{\tau^{n-l+2}}I_l(t), \end{aligned}$$

where

$$I_l(t)=\frac{(t-\tau)^2}{(t-T)^m}\int_T^t\frac{((t-\tau)-(s-\tau))^{m-2}}{(s-\tau)^{l+m+1}}ds.$$

If $m=2$, then

$$I_l(t)=\left(\frac{t-\tau}{t-T}\right)^2\int_T^t\frac{1}{(s-\tau)^{l+3}}ds<M_1, \tag{2.35}$$

where M_1 is a constant.
If $m>2$, then

$$\begin{aligned} I_l(t)\quad &=\tfrac{(t-\tau)^2}{(t-T)^m}\sum_{k=0}^{m-2}C_{m-2}^k(-1)^k(t-\tau)^{m-2-k}\int_T^t(s-\tau)^{k-l-m-1}ds \\ &=\left(\tfrac{t-\tau}{t-T}\right)^m\sum_{k=0}^{m-2}C_{m-2}^k(-1)^k(t-\tau)^{-k}\tfrac{(T-\tau)^{k-l-m}-(t-\tau)^{k-l-m}}{m+l-k} \\ &=\left(\tfrac{t-\tau}{t-T}\right)^m\sum_{k=0}^{m-2}C_{m-2}^k(-1)^k\tfrac{(T-\tau)^{k-l-m}(t-\tau)^{-k}-(t-\tau)^{-l-m}}{m+l-k}<M_2, \end{aligned}$$

where M_2 is a constant.
By (2.35), (2.35) and (2.36), it is easy to see that

$$\limsup_{t\to\infty}\frac{1}{[g(t)-g(T)]^m}\int_T^t\frac{m^2a(l)(g(t)-g(s))^{m-2}g^2(t)g'(s)}{g^{l+m+1}(s)(s-g(s))^{n-l+2}}ds<+\infty. \tag{2.36}$$

This completes the proof.

3. Oscillatory Behaviour for Examples

Example 1
We consider the third order nonlinear delay differential Equation

$$\Big(\frac{1}{\sqrt{t}}|u'(t)|^{1/2}u'(t)\Big)' + \frac{u}{t^{5/2}}\Big(\frac{3}{2}+\frac{3}{2\ln t}+\frac{1}{\ln^2 t}\Big)|u(2t)|^{1/2}u(2t) = 0, \quad t>1,$$

where

$$u>0, \quad \alpha=\frac{3}{2}, \quad r(t)=\frac{1}{\sqrt{t}}, \quad h(t)=2t, \quad p(t)=\frac{u}{t^{5/2}}(\frac{3}{2}+\frac{3}{2\ln t}+\frac{1}{\ln^2 t}).$$

Here

$$\pi_0(t)=P(t)=\frac{u}{t^{3/2}}(1+\frac{1}{\ln t}), \quad \pi_1(t) > \frac{9u^{5/3}}{7t^{7/6}}+\frac{u}{t^{3/2}}(1+\frac{1}{\ln t}).$$

Then

$$\lim_{t\to\infty}\pi_1(t)\exp\Big(\alpha\int_1^t(\frac{P(s)}{r(s)})^{1/\alpha}ds\Big)$$
$$\geq \lim_{t\to\infty}\Big(\frac{9u^{5/3}}{7t^{7/6}}+\frac{u}{t^{3/2}}(1+\frac{1}{\ln t})\Big)\exp\Big(\frac{3}{2}\int_1^t(\frac{u(1+\frac{1}{\ln s})}{s})^{2/3}ds\Big)$$
$$\geq \lim_{t\to\infty}\Big(\frac{9u^{5/3}}{7t^{7/6}}+\frac{u}{t^{3/2}}\Big)\exp\Big(\frac{3}{2}\int_1^t(\frac{u}{s})^{2/3}ds\Big) \geq \lim_{t\to\infty}\frac{u_1}{t^{3/2}}e^{u_2 t^{1/3}} = \infty,$$

where $u_1 = ue^{-9/2u^{2/3}}$ and $u_2 = 9u^{2/3}/2$. Thus, Theorem 2.1 is satisfied for $\alpha = 2$. Hence, it is oscillatory.

Example 2
Consider the fourth-order nonlinear delay differential equation

$$x^{(4)}(t)+\frac{3(\ln^2 t-2)}{t^3\ln^3 t}x'(t)+\frac{t+1}{t^2+1}x\Big((1+\sin\frac{1}{t^2+1})\frac{t}{2}\Big)=0, \quad t\geq 1. \tag{3.1}$$

The delay function $g(t)=(1+\sin\frac{1}{t^2+1})\frac{t}{2}$ satisfies $0<g(t)<t$, $\lim_{t\to+\infty}g(t)=+\infty$ and $t/g(t)\geq 2/(1+\sin(1/2))>1$. It is not hard to check that the equation $u'''+p(t)u=0$, with $p(t)=\frac{3(\ln^2 t-2)}{t^3\ln^3 t}$, has a positive and strictly increasing solution $u(t)=t\ln^3 t$. Moreover, since

$$p'(t)=\frac{3}{t^4\ln^4 t}(6+6\ln t-ln^2 t-3\ln^3 t),$$

and $p'_+(t)=0$ is for all t. Clearly, $\int_1^\infty q(t)dt \geq \int_1^\infty \frac{t+1}{2t^2}dt=+\infty$, which implies that it is true. Thus, any solution of (3.1) is oscillatory, or satisfies $x(t)\to 0$ as $t\to\infty$.
Example 3
Consider the fifth-order nonlinear delay differential equation

$$x^{(5)}(t)+\frac{2}{t^3(1+2\ln t)}x''(t)+(5+e^{-t}\cos t)tx(at-\tau)(2+\exp[-x(at-\tau)])=0, \tag{3.2}$$

for $t\geq 1$, where $a\in(0,1], \tau>0$. Obviously, the function $f(x)=x(2+e^{-x})$ satisfies that $f(x)/x\geq 2$ for $x\neq 0$. It is easy to check that the equation $u'''+p(t)u=0$ has a positive and strictly increasing solution $u(t)=t(2\ln t+1)$. Moreover, since $p'(t)\leq 0$ and $\int_1^\infty q(t)dt=\int_1^\infty(5+e^{-t}\cos t)tdt=+\infty$ are satisfied. Clearly, $\lim_{t\to\infty}q(t)/t=5$. Thus, any solution of (3.2) is oscillatory, or satisfies $x(t)\to 0$ as $t\to\infty$.
Example 4
Consider the eighth-order nonlinear delay differential equation

$$x^{(8)}(t)+\frac{1}{(1+2t)^2}\Big(\frac{t^2+t-2}{(1+t)^3\ln(1+t)}+\frac{3}{(1+2t)}\Big)x^{(5)}(t)+\frac{3t+\sin t}{t^2-2}x(t-\ln t)=0, \tag{3.3}$$

for $t\geq 2$. Here $n=5$,

$$p(t)=\frac{1}{(1+2t)^2}\Big(\frac{t^2+t-2}{(1+t)^3\ln(1+t)}+\frac{3}{(1+2t)}\Big), \quad q(t)=\frac{3t+\sin t}{t^2-2}$$

with $K=1$.
The equation $u'''+p(t)u=0$ has a positive and strictly increasing solution $u(t)=(2t+1)^{3/2}\ln(1+t)$. It is easy to see that $\int_2^\infty q(t)dt=+\infty$, $p'(t)$ is eventually negative and hence that it is true. Let $\rho(t)=t$, then it is easy to see that for $l=1,3,5,7$,

$$\lim_{t\to\infty} sup\Big(\int_2^t\Big[K\rho(s)q(s)-\frac{2^{l-3}(l-1)!(n-l+2)!(\rho'(s))^2}{g^{l-1}(s)(s-g(s))^{n-l+2}g'(s)\rho(s)}\Big]ds\Big)$$
$$=\lim_{t\to\infty} sup\Big(\int_2^t\Big[\frac{3s^2+s\sin s}{s^2-2}-\frac{2^{l-3}(l-1)!(7-l)!}{(s-\ln s)^{l-1}(\ln s)^{7-l}(s-1)}\Big]ds\Big)=+\infty.$$

Consequently, any solution of (3.3) is oscillatory, or satisfies $x(t)\to 0$ as $t\to\infty$.

4. Conclusion

Some new oscillatory principle consequences of higher order nonlinear delay differential equations are given. We discuss about the connection of Riccati change of the nonlinear differential equations to examining properties of the higher order differential equations. Besides, a normal coordinating strategy is presented as an asymptotic way to deal with consider the oscillatory behaviour. A few comes about are reached out to nonlinear delay differential equations of any order. A case is additionally examined, to represent the effectiveness of the outcomes got.

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