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Research Article

Semiprime Ideal of Rings with Symmetric Bi-Derivations

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Abstract: Let Ω be a ring with $\wp$ a semiprime ideal of Ω, $\Im$ an ideal of Ω, $\Delta : \Omega \times \Omega \to \Omega$ a symmetric bi-derivation and δ be the trace of Δ. In the present paper, we shall prove that δ is a $\wp$-commuting map on $\Im$ if any one of the following holds: i. $\delta(\sigma) \circ \kappa \in \wp$, ii. $\delta([\sigma,\kappa]) \pm [\delta(\sigma),\kappa] \in \wp$, iii. $\delta(\sigma \circ \kappa) \pm (\delta(\sigma) \circ \kappa) \in \wp$, iv. $\delta([\sigma,\kappa]) \pm \delta(\sigma) \circ \kappa \in \wp$, v. $\delta(\sigma \circ \kappa) \pm [\delta(\sigma),\kappa] \in \wp$, vi. $\delta(\sigma) \circ \kappa \pm [\delta(\kappa),\sigma] \in \wp$, vii. $\delta([\sigma,\kappa]) \pm \delta(\sigma) \circ \kappa - [\delta(\kappa),\sigma] \in \wp$, viii. $\delta([\sigma,\kappa]) \pm [\delta(\sigma),\kappa] + [\delta(\kappa),\sigma] \in \wp$, ix. $\Delta(\sigma,\kappa\kappa_3) \pm \Delta(\sigma,\kappa)\kappa_3 \in \wp$, x. $\Delta(\delta(\sigma),\sigma) \in \wp$, xi. $\delta(\delta(\sigma)) = g(\sigma)$, xii. $\delta(\sigma)\kappa \pm \sigma g(\kappa) \in \wp$, xiii. $[\delta(\sigma),\kappa] \pm [g(\kappa),\sigma] \in \wp$, xiv. $\delta(\sigma) \circ \kappa \pm (\sigma \circ g(\kappa)) \in \wp$, xv. $[\delta(\sigma),\kappa] \pm (\sigma \circ g(\kappa)) \in \wp$, xvi. $\delta(\sigma) \circ \kappa \pm [g(\kappa),\sigma] \in \wp$ for all $\sigma,\kappa \in \Im$ where $G : \aleph \times \aleph \to \aleph$ is a symmetric bi-derivation such that g is the trace of G.

Keywords: Rings, ideals, semiprime ideals, derivations, symmetric bi-derivations.

1. Introduction

Let Ω be an associative ring with center Z. A proper ideal $\wp$ of Ω is termed prime if for any elements $\sigma,\kappa \in \Omega$, the inclusion $\sigma\Omega\kappa \subseteq \wp$ implies that either $\sigma \in \wp$ or $\kappa \in \wp$. Equivalently, the ring Ω is said to be prime if (0), the zero ideal, is a prime ideal. In addition to prime ideals, the concept of semiprime ideals is also fundamental in ring theory. A proper ideal $\wp$ is semiprime if for any $\sigma \in \Omega$, the condition $\sigma\Omega\sigma \subseteq \wp$ implies $\sigma \in \wp$. The ring Ω is semiprime if (0) is a semiprime ideal. While every prime ideal is semiprime, the converse is not generally true. For any $\sigma,\kappa \in \Omega$, the symbol $[\sigma,\kappa]$ stands for the commutator $\sigma\kappa - \kappa\sigma$ and the symbol $\sigma \circ \kappa$ stands for the commutator $\sigma\kappa + \kappa\sigma$. An additive mapping $\delta : \Omega \to \Omega$ is called a derivation if $\delta(\sigma\kappa) = \delta(\sigma)\kappa + \sigma\delta(\kappa)$ holds for all $\sigma,\kappa \in \Omega$. A mapping $\Delta(.,.) : \Omega \times \Omega \to \Omega$ is said to be symmetric if $\Delta(\sigma,\kappa) = \Delta(\kappa,\sigma)$ for all $\sigma,\kappa \in \Omega$. A mapping $\delta : \Omega \to \Omega$ is called the trace of $\Delta(.,.)$ if $\delta(\sigma) = \Delta(\sigma,\sigma)$ for all $\sigma \in \Omega$.It is obvious that if $\Delta(.,.)$ is bi-additive (i.e., additive in both arguments), then the trace δ of $\Delta(.,.)$

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satisfies the identity $\delta(\sigma+\kappa)=\delta(\sigma)+\delta(\kappa)+2\Delta(\sigma,\kappa)$ for all $\sigma,\kappa\in\Omega$. If $\Delta(.,.)$ is bi-additive and satisfies the identities

$$\Delta(\sigma\kappa,\varsigma)=\Delta(\sigma,\varsigma)\kappa+\sigma\Delta(\kappa,\varsigma)$$

and

$$\Delta(\sigma,\kappa\varsigma)=\Delta(\sigma,\kappa)\varsigma+\kappa\Delta(\sigma,\varsigma)$$

for all $\sigma,\kappa,\varsigma\in\Omega$, then $\Delta(.,.)$ is called a symmetric bi-derivation.

Example 1.1 *Suppose the ring*

$$\Omega=\left\{\left(\begin{array}{cc} a & b\\ 0 & 0\end{array}\right)\middle|\, a,b\in\mathbb{R}\right\}.$$

Define maps $\Delta:\Omega\times\Omega\to\Omega$ *as follows:*

$$\Delta\left(\left(\begin{array}{cc} a & b\\ 0 & 0\end{array}\right),\left(\begin{array}{cc} c & \delta\\ 0 & 0\end{array}\right)\right)=\left(\left(\begin{array}{cc} 0 & ac\\ 0 & 0\end{array}\right)\right).$$

Then it is easy to verify that Δ *is a symmetric bi-derivation on* Ω. *Also, the trace of* Δ *is*

$$\delta\left(\begin{array}{cc} a & b\\ 0 & 0\end{array}\right)=\left(\begin{array}{cc} 0 & a^2\\ 0 & 0\end{array}\right).$$

Let S be a nonempty subset of Ω. A mapping T from Ω to Ω is called commuting on S if $[T(\sigma),\sigma]=0$ for all $\sigma\in S$. This definition has been generalized such as: A map $T:\Omega\to\Omega$ is called a U-commuting map on S if $[T(\sigma),\sigma]\in U$ for all $\sigma\in S$ and some $U\subseteq\Omega$. In particular, if $U=0$, then T is called a commuting map on S if $[T(\sigma),\sigma]=0$. Note that every commuting map is a $U-$commuting map (put $0=U$). But the converse is not true in general. Take U some a set of Ω has no zero such that $[T(\sigma),\sigma]\in U$, then T is a $U-$commuting map but it is not a commuting map. The notion of additive commuting mapping is closely connected with the notion of bi-derivation. Every additive commuting mapping $T:\Omega\to\Omega$ gives rise to a bi-derivation on Ω. Namely, linearizing $[T(\sigma),\sigma]\in\wp$, we get $[T(\sigma),\kappa]=[\sigma,T(\kappa)]$ and we note that the map $(\sigma,\kappa)\longmapsto[T(\sigma),\kappa]$ is a bi-derivation. The concept of bi-derivation was introduced by Maksa in [8]. It is shown in [9] that symmetric bi-derivations are related to general solution of some functional expressions.

The property of interchangeability of prime or semiprime rings with derivation was first discussed by Posner [10]. Later, many authors studied the commutativity conditions in prime and semiprime rings. In recent years, the effects of these conditions on the derivation of prime and semiprime ideals have begun to be examined.

In 1992, Daif and Bell showed that if the derivativon δ on a semiprime ring Ω satisfies the condition $\sigma\kappa \pm \delta(\sigma\kappa) = \kappa\sigma \pm \delta(\kappa\sigma)$ for each $\sigma, \kappa \in \Omega$, then the ring Ω is commutative [5]. In 1999, Ashraf considered the same condition for the symmetric bi-derivation on a prime ring [1]. In 2015, Reddy, Rao, and Reddy generalized this theorem for semiprime rings [11]. In 2001, Ashraf and Rehman showed that if the derivation δ on an ideal $\Im$ of a prime ring Ω satisfies one of the conditions $\delta(\sigma\kappa) - \sigma\kappa \in \varsigma$ or $\delta(\sigma\kappa) - \kappa\sigma \in \varsigma$ for each $\sigma, \kappa \in \Omega$, then $\Im$ is commutative [2]. These conditions were investigated by Koç Sögütcü and Gölbaşı in 2021 for inverse bi-derivation Lie ideals [6].

On the other hand, Vukman proved in 1990 that if Ω is a semiprime ring, Δ is the symmetric bi-derivation on the ring Ω and δ, Δ is the trace of the symmetric bi-derivation, then $\Delta = 0$ if $\Delta(\delta(\sigma), \sigma) = 0$ and $\delta(\delta(\sigma)) = g(\sigma)$ for all $\sigma \in \Omega$ [12]. In 2017, Reddy and Naik considered the above conditions for the symmetric reverse bi-derivation. It was investigated by Koç Sögütcü and Gölbaşı in 2021 for reverse bi-derivation Lie ideals on the semiprime ring [7].

Ashraf et al. in 2005 studied the commutativity of a prime ring Ω, which allows a generalized derivation T and the associated derivation δ, satisfying any of these properties: $\delta(\sigma) \circ T(\kappa) = 0$ or $[\delta(\sigma), T(\kappa)] = 0$ for all $\sigma, \kappa \in \Omega$ [3]. In 2024, Çelik and Koç Sögütcü considered these conditions for multiplicative derivation with semiprime ideal [4].

In this paper, we investigate the algebraic identities mentioned above for symmetric bi-derivation acting on a semiprime ideal without making any assumptions on the ideal of the ring. We will make some extensive use of the basic commutator identities: 1) $[\sigma, \kappa\varsigma] = \kappa[\sigma, \varsigma] + [\sigma, \kappa]\varsigma$,
2) $[\sigma\kappa, \varsigma] = [\sigma, \varsigma]\kappa + \sigma[\kappa, \varsigma]$,
3) $\sigma\kappa \circ \varsigma = (\sigma \circ \varsigma)\kappa + \sigma[\kappa, \varsigma] = \sigma(\kappa \circ \varsigma) - [\sigma, \varsigma]\kappa$,
4) $\sigma \circ \kappa\varsigma = \kappa(\sigma \circ \varsigma) + [\sigma, \kappa]\varsigma = (\sigma \circ \kappa)\varsigma + \kappa[\varsigma, \sigma]$.

2. Main Results

Lemma 2.1 *Let Ω be a ring with $\wp$ a semiprime ideal of R, $\Im$ an ideal of Ω, $char(\Omega/\wp) \neq 2$ and $\Delta : \Omega \times \Omega \to \Omega$ a symmetric bi-derivation and δ be the trace of Δ. If $\delta(\sigma) \circ \kappa \in \wp$ for all $\sigma, \kappa \in \Im$, then δ is $\wp-$commuting map on $\Im$.*

Proof By the hypothesis, we get

$$\delta(\sigma) \circ \kappa \in \wp, \text{ for all } \sigma, \kappa \in \Im.$$

Taking κ by $\kappa\varsigma$ in the last expression, we obtain that

$$\kappa(\delta(\sigma) \circ \varsigma) + [\delta(\sigma), \kappa]\varsigma \in \wp \text{ for all } \sigma, \kappa, \varsigma \in \Im.$$

Using the hypothesis, we have

$$[\delta(\sigma),\kappa]\varsigma \in \wp \text{ for all } \sigma,\kappa,\varsigma \in \Im.$$

Taking ς by $t[\delta(\sigma),\kappa], t \in \Omega$ in the above expression, we get

$$[\delta(\sigma),\kappa]t[\delta(\sigma),\kappa] \in \wp \text{ for all } \sigma,\kappa \in \Im, t \in \Omega.$$

Since $\wp$ is a semiprime ideal of Ω, we conclude that

$$[\delta(\sigma),\kappa] \in \wp \text{ for all } \sigma,\kappa \in \Im.$$

Replacing κ by σ in the last expression, we get

$$[\delta(\sigma),\sigma] \in \wp \text{ for all } \sigma \in \Im.$$

Hence we conclude that δ is $\wp-$commuting on $\Im$. □

Theorem 2.2 *Let Ω be a ring with $\wp$ a semiprime ideal of Ω, $\Im$ an ideal of Ω, $char(\Omega/\wp) \neq 2$ and $\Delta : \Omega \times \Omega \to \Omega$ a symmetric bi-derivation and δ be the trace of Δ. If any of the following conditions is satisfied for all $\sigma,\kappa \in \Omega$, then δ is $\wp-$commuting map on $\Im$.*
i) $\delta(\sigma \circ \kappa) \pm (\delta(\sigma) \circ \kappa) \in \wp$,
ii) $\delta([\sigma,\kappa]) \pm (\delta(\sigma) \circ \kappa) \in \wp$,
iii) $\delta(\sigma \circ \kappa) \pm [\delta(\sigma),\kappa] \in \wp$,
iv) $\delta(\sigma) \circ \kappa \pm [\delta(\kappa),\sigma] \in \wp$.

Proof i) By the hypothesis, we get

$$\delta(\sigma \circ \kappa) \pm (\delta(\sigma) \circ \kappa) \in \wp \text{ for all } \sigma,\kappa \in \Im.$$

Taking κ by $\kappa+\varsigma, \varsigma \in \Im$, we have

$$\delta(\sigma \circ \kappa) + \delta(\sigma \circ \varsigma) + 2\Delta(\sigma \circ \kappa, \sigma \circ \varsigma) \pm \delta(\sigma) \circ \kappa \pm \delta(\sigma) \circ \varsigma \in \wp.$$

Using the hypothesis, we arrive at

$$2\Delta(\sigma \circ \kappa, \sigma \circ \varsigma) \in \wp.$$

Since $char(\Omega/\wp) \neq 2$, we have

$$\Delta(\sigma \circ \kappa, \sigma \circ \varsigma) \in \wp.$$

Replacing ς by κ in this expression, we find that

$$\Delta(\sigma \circ \kappa, \sigma \circ \kappa) \in \wp \text{ for all } \sigma,\kappa \in \Im$$

and so

$$\delta(\sigma \circ \kappa) \in \wp \text{ for all } \sigma, \kappa \in \Im.$$

Again using the hypothesis, we obtain that

$$\delta(\sigma) \circ \kappa \in \wp \text{ for all } \sigma, \kappa \in \Im.$$

We see that δ is $\wp-$commuting map on $\Im$ by Lemma 2.1.

ii) By the hypothesis, we have

$$\delta([\sigma,\kappa]) \pm (\delta(\sigma) \circ \kappa) \in \wp \text{ for all } \sigma, \kappa \in \Im.$$

Taking κ by $\kappa + \varsigma, \varsigma \in \Im$ in the hypothesis, we get

$$\delta([\sigma,\kappa]) + \delta([\sigma,\varsigma]) + 2\Delta([\sigma,\kappa],[\sigma,\varsigma]) \pm \delta(\sigma) \circ \kappa \pm \delta(\sigma) \circ \varsigma \in \wp.$$

Using the hypothesis and $char(\Omega/\wp) \neq 2$, we find that

$$\Delta([\sigma,\kappa],[\sigma,\varsigma]) \in \wp.$$

Replacing ς by κ in the last expression, we see that

$$\Delta([\sigma,\kappa],[\sigma,\kappa]) \in \wp \text{ for all } \sigma, \kappa \in \Im.$$

That is

$$\delta([\sigma,\kappa]) \in \wp \text{ for all } \sigma, \kappa \in \Im.$$

Using this expression in our hypothesis, we get

$$\delta(\sigma) \circ \kappa \in \wp \text{ for all } \sigma, \kappa \in \Im.$$

By Lemma 2.1, we obtain that δ is $\wp-$commuting on $\Im$.

iii) By the hypothesis, we have

$$\delta(\sigma \circ \kappa) \pm [\delta(\sigma), \kappa] \in \wp \text{ for all } \sigma, \kappa \in \Im.$$

Taking κ by $\kappa + \varsigma, \varsigma \in \Im$, we get

$$\delta(\sigma \circ \kappa) + \delta(\sigma \circ \varsigma) + 2\Delta(\sigma \circ \kappa, \sigma \circ \varsigma) \pm [\delta(\sigma), \kappa] \pm [\delta(\sigma), \varsigma] \in \wp.$$

Using the hypothesis, we see that

$$\Delta(\sigma \circ \kappa, \sigma \circ \varsigma) \in \wp.$$

Replacing ς by κ in the last expression, we get

$$\Delta(\sigma \circ \kappa, \sigma \circ \kappa) \in \wp \text{ for all } \sigma, \kappa \in \Im$$

and so

$$\delta\left(\sigma \circ \kappa\right) \in \wp \text{ for all } \sigma, \kappa \in \Im.$$

Using the hypothesis, we have

$$[\delta(\sigma), \kappa] \in \wp \text{ for all } \sigma, \kappa \in \Im.$$

Replacing κ by σ in the last expression, we obtain that δ is $\wp-$commuting on $\Im$.

iv) We have

$$\delta(\sigma) \circ \kappa \pm [\delta(\kappa), \sigma] \in \wp \text{ for all } \sigma, \kappa \in \Im.$$

Replacing κ by $\kappa + \varsigma, \varsigma \in \Im$, we get

$$\delta(\sigma) \circ \kappa + \delta(\sigma) \circ \varsigma \pm 2[\Delta(\kappa, \varsigma), \sigma] \pm [\delta(\kappa), \sigma] \pm [\delta(\varsigma), \sigma] \in \wp.$$

By the hypothesis, we have

$$2[\Delta(\kappa, \varsigma), \sigma] \in \wp.$$

Using $char(\Omega/\wp) \neq 2$, we find that

$$[\Delta(\kappa, \varsigma), \sigma] \in \wp.$$

Replacing ς by κ in this expression, we get

$$[\Delta(\kappa, \kappa), \sigma] \in \wp \text{ for all } \sigma, \kappa \in \Im$$

and so

$$[\delta(\kappa), \sigma] \in \wp \text{ for all } \sigma, \kappa \in \Im.$$

Using the hypothesis, we have

$$\delta(\sigma) \circ \kappa \in \wp \text{ for all } \sigma, \kappa \in \Im.$$

By Lemma 2.1, we conclude that δ is $\wp-$commuting on $\Im$. □

Theorem 2.3 *Let Ω be a ring with $\wp$ a semiprime ideal of Ω, $\Im$ an ideal of Ω, $char(\Omega/\wp) \neq 2$ and $\Delta : \Omega \times \Omega \to \Omega$ a symmetric bi-derivation and δ be the trace of Δ. If any of the following conditions is satisfied for all $\sigma, \kappa \in \Omega$, then δ is $\wp-$commuting map on $\Im$.*

i) $\delta([\sigma, \kappa]) \pm (\delta(\sigma) \circ \kappa) + [\delta(\kappa), \sigma] \in \wp$,

ii) $\delta([\sigma, \kappa]) \pm [\delta(\sigma), \kappa] + [\delta(\kappa), \sigma] \in \wp$.

Proof i) By the hypothesis, we obtain that

$$\delta([\sigma, \kappa]) \pm (\delta(\sigma) \circ \kappa) + [\delta(\kappa), \sigma] \in \wp \text{ for all } \sigma, \kappa \in \Im.$$

Replacing κ by $\kappa + \varsigma, \varsigma \in \Im$ in the last expression, we get

$$\delta\left([\sigma, \kappa]\right) + \delta([\sigma, \varsigma]) + 2\Delta([\sigma, \kappa], [\sigma, \varsigma]) \pm \delta(\sigma) \circ \kappa \pm \delta(\sigma) \circ \varsigma + [\delta(\kappa), \sigma] + [\delta(\varsigma), \sigma] + 2[\Delta(\kappa, \varsigma), \sigma] \in \wp.$$

Using the hypothesis and $char(\Omega/\wp) \neq 2,$ we have

$$\Delta([\sigma,\kappa],[\sigma,\varsigma]) + [\Delta(\kappa,\varsigma),\sigma] \in \wp.$$

Replacing κ by ς in the above expression, we see that

$$\Delta([\sigma,\kappa],[\sigma,\kappa]) + [\Delta(\kappa,\kappa),\sigma] \in \wp \text{ for all } \sigma,\kappa \in \Im.$$

That is

$$\delta([\sigma,\kappa]) + [\delta(\kappa),\sigma] \in \wp \text{ for all } \sigma,\kappa \in \Im.$$

We can write this expression such as

$$\delta([\sigma,\kappa]) + [\delta(\kappa),\sigma] \pm \delta(\kappa)\circ\sigma \mp \delta(\kappa)\circ\sigma \in \wp.$$

Using the hypothesis, we arrive at

$$\delta(\kappa)\circ\sigma \in \wp \text{ for all } \sigma,\kappa \in \Im.$$

By Lemma 2.1, we conclude that δ is $\wp-$commuting on $\Im$.

ii) By the hypothesis, we get

$$\delta([\sigma,\kappa]) \pm [\delta(\sigma),\kappa] + [\delta(\kappa),\sigma] \in \wp \text{ for all } \sigma,\kappa \in \Im.$$

Writing κ by $\kappa+\varsigma,\ \varsigma \in \Im$ in this expression, we obtain that

$$\delta([\sigma,\kappa]) + \delta([\sigma,\varsigma]) + 2\Delta([\sigma,\kappa],[\sigma,\varsigma]) \pm [\delta(\sigma),\kappa] \pm [\delta(\sigma),\varsigma] + [\delta(\kappa),\sigma] + [\delta(\varsigma),\sigma] + 2[\Delta(\kappa,\varsigma),\sigma] \in \wp.$$

Using the hypothesis and $char(\Omega/\wp) \neq 2,$ we see that

$$\Delta([\sigma,\kappa],[\sigma,\varsigma]) + [\Delta(\kappa,\varsigma),\sigma] \in \wp \text{ for all } \sigma,\kappa,\varsigma \in \Im.$$

Replacing ς by κ in the last expression, we have

$$\delta([\sigma,\kappa]) + [\delta(\kappa),\sigma] \in \wp \text{ for all } \sigma,\kappa \in \Im.$$

The rest of the proof is the same as above. This completes proof. □

Theorem 2.4 *Let Ω be a ring with $\wp$ a semiprime ideal of Ω, $char(\Omega/\wp) \neq 2$ and $\Delta : \Omega\times\Omega \to \Omega$ a symmetric bi-derivation, δ be the trace of Δ. If $\Delta(\sigma,\kappa\varsigma) - \Delta(\sigma,\kappa)\varsigma \in \wp$ for all $\sigma,\kappa,\varsigma \in \Omega$, then δ is $\wp-$commuting on Ω.*

Proof By the hypothesis, we get

$$\Delta(\sigma,\kappa\varsigma) - \Delta(\sigma,\kappa)\varsigma \in \wp \text{ for all } \sigma,\kappa,\varsigma \in \Omega.$$

Since Δ is a bi-derivation, we can write

$$\Delta(\sigma,\kappa\varsigma)=\Delta(\sigma,\kappa)\varsigma+\kappa\Delta(\sigma,\varsigma).$$

Using the hypothesis, we obtain that

$$\kappa\Delta(\sigma,\varsigma)\in\wp \text{ for all } \sigma,\kappa,\varsigma\in\Omega.$$

Replacing κ by $\Delta(\sigma,\varsigma)\kappa$ in the last expression, we get

$$\Delta(\sigma,\varsigma)\kappa\Delta(\sigma,\varsigma)\in\wp \text{ for all } \sigma,\kappa,\varsigma\in\Omega.$$

Since $\wp$ is semiprime ideal of Ω, we obtain that $\Delta(\sigma,\varsigma)\in\wp$ for all $\sigma,\varsigma\in\Omega$. Replacing ς by σ, we get $\delta(\sigma)\in\wp$, and so $[\delta(\sigma),\sigma]\in\wp$ for all $\sigma\in\Im$. Hence we conclude that δ is $\wp-$commuting map.

□

Theorem 2.5 *Let Ω be a ring with $\wp$ a semiprime ideal of Ω, $char(\Omega/\wp)\neq 2$ and $\Delta:\Omega\times\Omega\to\Omega$ a symmetric bi-derivation, δ be the trace of Δ. If $\Delta(\delta(\sigma),\sigma)\in\wp$, for all $\sigma\in\Omega$, then δ is $\wp-$commuting on Ω.*

Proof By the hypothesis, we have

$$\Delta(\delta(\sigma),\sigma)\in\wp \text{ for all } \sigma\in\Omega.$$

Replacing σ by $\sigma+\kappa$, $\kappa\in\Omega$ in the last expression, we get

$$\Delta(\delta(\sigma),\sigma)+\Delta(\delta(\sigma),\kappa)+\Delta(\delta(\kappa),\sigma)+\Delta(\delta(\kappa),\kappa)+2\Delta(\Delta(\sigma,\kappa),\sigma)+2\Delta(\Delta(\sigma,\kappa),\kappa)\in\wp.$$

Using our hypothesis, we see that

$$\Delta(\delta(\sigma),\kappa)+\Delta(\delta(\kappa),\sigma)+2\Delta(\Delta(\sigma,\kappa),\sigma)+2\Delta(\Delta(\sigma,\kappa),\kappa)\in\wp.$$

Replacing σ by $-\sigma$ in the above expression, we obtain that

$$\Delta(\delta(\sigma),\kappa)-\Delta(\delta(\kappa),\sigma)+2\Delta(\Delta(\sigma,\kappa),\sigma)-2\Delta(\Delta(\sigma,\kappa),\kappa)\in\wp.$$

We obtained from the last two expressions

$$\Delta(\delta(\sigma),\kappa)+2\Delta(\Delta(\sigma,\kappa),\sigma)\in\wp. \tag{1}$$

Taking κ by $\sigma\kappa$ in (1) and using the hypothesis, we get

$$\sigma\Delta(\delta(\sigma),\kappa)+2\delta(\sigma)\Delta(\sigma,\kappa)+2\sigma\Delta(\Delta(\sigma,\kappa),\sigma)+2\delta(\sigma)\Delta(\kappa,\sigma)\in\wp. \tag{2}$$

Multipliying in (1) by σ on left hand side, we see that

$$\sigma\Delta(\delta(\sigma),\kappa)+2\sigma\Delta(\Delta(\sigma,\kappa),\sigma)\in\wp. \tag{3}$$

Subtracting (2) from (3), we arrive at

$$4\delta(\sigma)\Delta(\sigma,\kappa) \in \wp.$$

Using $char(\Omega/\wp) \neq 2,$ we get

$$\delta(\sigma)\Delta(\sigma,\kappa) \in \wp.$$

Replacing κ by $\kappa\sigma$ in the last expression, we have

$$\delta(\sigma)\kappa\delta(\sigma) \in \wp \text{ for all } \sigma,\kappa \in \Omega.$$

Since $\wp$ is semiprime ideal of R, we get $\delta(\sigma) \in \wp,$ and so $[\delta(\sigma),\sigma] \in \wp$ for all $\sigma \in \Im.$ Hence we conclude that δ is $\wp-$commuting map. □

Theorem 2.6 *Let Ω be a ring with $\wp$ a semiprime ideal of Ω $char(\Omega/\wp) \neq 2,,$ $char\Omega/\wp \neq 3$ and $\Delta : \Omega \times \Omega \to \Omega,$ $G : \Omega \times \Omega \to \Omega$ two symmetric reverse bi-derivations where δ is the trace of Δ and g is the trace of G. If $\delta(\delta(\sigma)) \pm g(\sigma) \in \wp$ for all $\sigma \in \Omega,$ then g is $\wp-$commuting on Ω.*

Proof By our hypothesis, we have

$$\delta(\delta(\sigma)) \pm g(\sigma) \in \wp \text{ for all } \sigma \in \Omega.$$

Replacing σ by $\sigma+\kappa,$ $\kappa \in \Omega,$ we get

$$\delta(\delta(\sigma)) + \delta(\delta(\kappa)) + 2\Delta(\delta(\sigma),\delta(\kappa)) + 4\delta(\Delta(\sigma,\kappa)) + 4\Delta(\delta(\sigma),\Delta(\sigma,\kappa)) + 4\Delta(\delta(\kappa),\Delta(\sigma,\kappa)) \\ \pm g(\sigma) \pm g(\kappa) \pm 2G(\sigma,\kappa) \in \wp.$$

Using the hypothesis and $char(\Omega/\wp) \neq 2,$ we obtain that

$$\Delta(\delta(\sigma),\delta(\kappa)) + 2\delta(\Delta(\sigma,\kappa)) + 2\Delta(\delta(\sigma),\Delta(\sigma,\kappa)) + 2\Delta(\delta(\kappa),\Delta(\sigma,\kappa)) \pm G(\sigma,\kappa) \in \wp. \quad (4)$$

Replacing σ by $-\sigma$ in the above expression, we see that

$$\Delta(\delta(\sigma),\delta(\kappa)) + 2\delta(\Delta(\sigma,\kappa)) - 2\Delta(\delta(\sigma),\Delta(\sigma,\kappa)) - 2\Delta(\delta(\kappa),\Delta(\sigma,\kappa)) \mp G(\sigma,\kappa) \in \wp. \quad (5)$$

Subtracting (4) from (5), we arrive at

$$4\Delta(\delta(\sigma),\Delta(\sigma,\kappa)) + 4\Delta(\delta(\kappa),\Delta(\sigma,\kappa)) \pm 2G(\sigma,\kappa) \in \wp.$$

Since $char(\Omega/\wp) \neq 2,$ we get

$$2\Delta(\delta(\sigma),\Delta(\sigma,\kappa)) + 2\Delta(\delta(\kappa),\Delta(\sigma,\kappa)) \pm G(\sigma,\kappa) \in \wp \text{ for all } \sigma,\kappa \in \Omega. \quad (6)$$

Replacing σ by 2σ in the last expression, we see that

$$16\Delta(\delta(\sigma),\Delta(\sigma,\kappa)) + 4\Delta(\delta(\kappa),\Delta(\sigma,\kappa)) \pm 2G(\sigma,\kappa) \in \wp. \quad (7)$$

Using (6), we have

$$4\Delta(\delta(\sigma),\Delta(\sigma,\kappa))+4\Delta(\delta(\kappa),\Delta(\sigma,\kappa))\pm 2G(\sigma,\kappa)\in\wp \text{ for all } \sigma,\kappa\in\Omega. \tag{8}$$

Subtracting (7) from (8), we arrive at

$$12\Delta(\delta(\sigma),\Delta(\sigma,\kappa))\in\wp.$$

Using $char(\Omega/\wp)\neq 2$ and $char\Omega/\wp\neq 3$, we get

$$\Delta(\delta(\sigma),\Delta(\sigma,\kappa))\in\wp \text{ for all } \sigma,\kappa\in\Omega.$$

Replacing κ by σ in the above expression, we see that

$$\Delta(\delta(\sigma),\Delta(\sigma,\sigma))\in\wp.$$

That is

$$\delta(\delta(\sigma))\in\wp \text{ for all } \sigma\in\Omega.$$

Returning our hypothesis and using this, we get $g(\sigma)\in\wp$, and so $[g(\sigma),\sigma]\in\wp$ for all $\sigma\in\Im$. Hence we conclude that g is $\wp-$commuting map. This completes proof. □

Theorem 2.7 *Let Ω be a ring with $\wp$ a semiprime ideal of Ω, $\Im$ an ideal of Ω, $char(\Omega/\wp)\neq 2$ and $\Delta:\Omega\times\Omega\to\Omega$, $G:\Omega\times\Omega\to\Omega$ two symmetric bi-derivations where δ is the trace of Δ and g is the trace of G. If $\delta(\sigma)\kappa\pm\sigma g(\kappa)\in\wp$ for all $\sigma,\kappa\in\Im$, then δ is $\wp-$commuting map.*

Proof Let assume that

$$\delta(\sigma)\kappa\pm\sigma g(\kappa)\in\wp \text{ for all } \sigma,\kappa\in\Im.$$

Writing κ by $\kappa+\varsigma,\varsigma\in\Im$, we have

$$\delta(\sigma)\kappa+\delta(\sigma)\varsigma\pm\sigma g(\kappa)\pm\sigma g(\varsigma)\pm 2\sigma G(\kappa,\varsigma)\in\wp.$$

Using the hypothesis, we get

$$2\sigma G(\kappa,\varsigma)\in\wp.$$

Since $char(\Omega/\wp)\neq 2$ and replacing ς by κ, we see that

$$\sigma G(\kappa,\kappa)\in\wp \text{ for all } \sigma,\kappa\in\Im.$$

That is

$$\sigma g(\kappa)\in\wp \text{ for all } \sigma,\kappa\in\Im.$$

By the hypothesis, we get

$$\delta(\sigma)\kappa\in\wp \text{ for all } \sigma,\kappa\in\Im.$$

Thus we can write $\delta(\sigma)\sigma\in\wp$ for all $\sigma\in\Im$, and so $[\delta(\sigma),\sigma]\in\wp$ for all $\sigma\in\Im$. Hence we conclude that δ is $\wp-$commuting map. □

Theorem 2.8 *Let Ω be a ring with $\wp$ a semiprime ideal of Ω, $\Im$ an ideal of Ω, $char(\Omega/\wp) \neq 2$ and $\Delta : \Omega \times \Omega \to \Omega$, $G : \Omega \times \Omega \to \Omega$ two symmetric bi-derivations where δ is the trace of Δ and g is the trace of G. If any of the following conditions is satisfied for all $\sigma, \kappa \in \Omega$, then δ is $\wp-$ commuting map on $\Im$.*

i) $[\delta(\sigma), \kappa] \pm [g(\kappa), \sigma] \in \wp$,

ii) $\delta(\sigma) \circ \kappa \pm (\sigma \circ g(\kappa)) \in \wp$,

iii) $[\delta(\sigma), \kappa] \pm (\sigma \circ g(\kappa)) \in \wp$,

iv) $\delta(\sigma) \circ \kappa \pm [g(\kappa), \sigma] \in \wp$.

Proof i) By the hypothesis, we have

$$[\delta(\sigma), \kappa] \pm [g(\kappa), \sigma] \in \wp \text{ for all } \sigma, \kappa \in \Im.$$

Taking κ by $\kappa + \varsigma, \varsigma \in \Im$, we get

$$[\delta(\sigma), \kappa] + [\delta(\sigma), \varsigma] \pm [g(\kappa), \sigma] \pm [g(\varsigma), \sigma] \pm 2[G(\kappa, \varsigma), \sigma] \in \wp.$$

Using the hypothesis, we have

$$2[G(\kappa, \varsigma), \sigma] \in \wp.$$

Since $\Omega/\wp$ is characteristic not two ring, we obtain that

$$[G(\kappa, \varsigma), \sigma] \in \wp.$$

Replacing ς by κ in the last expression, we have

$$[G(\kappa, \kappa), \sigma] \in \wp \text{ for all } \sigma, \kappa \in \Im.$$

That is,

$$[g(\kappa), \sigma] \in \wp \text{ for all } \sigma, \kappa \in \Im.$$

Using this in our hypothesis, we obtain that $[\delta(\sigma), \kappa] \in \wp$, and so $[\delta(\sigma), \sigma] \in \wp$ for all $\sigma \in \Im$. Hence δ is $\wp-$commuting on $\Im$.

ii) By the hypothesis, we get

$$\delta(\sigma) \circ \kappa \pm (\sigma \circ g(\kappa)) \in \wp \text{ for all } \sigma, \kappa \in \Im.$$

Replacing κ by $\kappa + \varsigma$, $\varsigma \in \Im$, we obtain that

$$\delta(\sigma) \circ \kappa + \delta(\sigma) \circ \varsigma \pm (\sigma \circ g(\kappa)) \pm (\sigma \circ g(\varsigma)) \pm 2(\sigma \circ G(\kappa, \varsigma)) \in \wp \text{ for all } \sigma, \kappa, \varsigma \in \Im.$$

Using the hypothesis and $char(\Omega/\wp) \neq 2$, we see that

$$(\sigma \circ G(\kappa, \varsigma)) \in \wp \text{ for all } \sigma, \kappa, \varsigma \in \Im.$$

Taking ς by κ in this expression, we have

$$\sigma \circ g(\kappa) \in \wp \text{ for all } \sigma, \kappa \in \Im.$$

Hence we get from our hypothesis,

$$\delta(\sigma) \circ \kappa \in \wp \text{ for all } \sigma, \kappa \in \Im.$$

Replacing κ by $\kappa\varsigma,\ \varsigma \in \Im$, we find that

$$\kappa[\varsigma, \delta(\sigma)] \in \wp \text{ for all } \sigma, \kappa, \varsigma \in \Im.$$

Replacing κ by $[\varsigma, \delta(\sigma)]\Omega$ in the last expression, we have

$$[\varsigma, \delta(\sigma)]t[\varsigma, \delta(\sigma)] \in \wp \text{ for all } \sigma, \varsigma \in \Im,\ t \in \Omega.$$

Since $\wp$ is semiprime ideal of Ω, we get

$$[\varsigma, \delta(\sigma)] \in \wp \text{ for all } \sigma, \varsigma \in \Im.$$

In particular we have $[\sigma, \delta(\sigma)] \in \wp$ for all $\sigma \in \Im$, and so δ is $\wp-$commuting on $\Im$.

iii) By the hypothesis, we have

$$[\delta(\sigma), \kappa] \pm \sigma \circ g(\kappa) \in \wp \text{ for all } \sigma, \kappa \in \Im.$$

Taking κ by $\kappa + \varsigma, \varsigma \in \Im$, we get

$$[\delta(\sigma), \kappa] + [\delta(\sigma), \varsigma] \pm (\sigma \circ g(\varsigma)) \pm 2(\sigma \circ G(\kappa, \varsigma)) \in \wp.$$

Using $char(\Omega/\wp) \neq 2$, we see that

$$(\sigma \circ G(\kappa, \varsigma)) \in \wp \text{ for all } \sigma, \kappa, \varsigma \in \Im.$$

Taking ς by κ in this expression, we have

$$\sigma \circ g(\kappa) \in \wp \text{ for all } \sigma, \kappa \in \Im.$$

Hence we get from our hypothesis,

$$\delta(\sigma) \circ \kappa \in \wp \text{ for all } \sigma, \kappa \in \Im.$$

By Lemma 2.1, we conclude that δ is $\wp-$commuting on $\Im$.

iv) By the hypothesis, we get

$$\delta(\sigma) \circ \kappa \pm [g(\kappa), \sigma] \in \wp, \text{ for all } \sigma, \kappa \in \Im.$$

Taking κ by $\kappa+\varsigma, \varsigma \in \Im$, we get

$$\delta(\sigma)\circ\kappa+\delta(\sigma)\circ\varsigma \pm [g(\kappa),\sigma] \pm [g(\varsigma),\sigma] \pm 2[G(\kappa,\varsigma),\sigma] \in \wp.$$

Using the hypothesis, we have

$$2[G(\kappa,\varsigma),\sigma] \in \wp.$$

Since $\Omega/\wp$ is characteristic not two ring, we obtain that

$$[G(\kappa,\varsigma),\sigma] \in \wp.$$

Replacing ς by κ in the last expression, we have

$$[G(\kappa,\kappa),\sigma] \in \wp \text{ for all } \sigma,\kappa \in \Im.$$

That is,

$$[g(\kappa),\sigma] \in \wp \text{ for all } \sigma,\kappa \in \Im.$$

Using this in our hypothesis, we obtain that $[\delta(\sigma),\kappa] \in \wp$, and so $[\delta(\sigma),\sigma] \in \wp$ for all $\sigma \in \Im$. Hence δ is $\wp-$commuting on $\Im$. □

3. Conclusion

In this study, the subject of symmetric bi-derivations of a ring under the influence of a semiprime ideal is considered without imposing a condition on the ring. Here, the conditions used in the literature to prove that a ring is commutative are examined for symmetric bi-derivations and semiprime ideals. The results obtained provide a new perspective on the structural properties of rings with derivations. Based on this research, further research can be conducted by considering different algebraic structures such as generalized derivations instead of symmetric bi-derivations, homoderivations and alternating rings instead of rings, near rings, operator algebras, Banach algebras and other areas where ring theory plays a fundamental role.

Declaration of Ethical Standards

The authors declare that the materials and methods used in their study do not require ethical committee and/or legal special permission.

Authors Contributions

Author [Emine Koç Sögütcü]: Thought and designed the research/problem, contributed to research method or evaluation of data, wrote the manuscript (%60).

Author [Öznur Gölbaşı]: Collected the data, contributed to research method or evaluation of data (%40).

Conflicts of Interest

The authors declare no conflict of interest.

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