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Pseudo Symmetric and Pseudo Ricci Symmetric $N(k)$-Contact Metric Manifolds

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Abstract

The purpose of the present paper is to study the existence of pseudo symmetric, pseudo Ricci symmetric and generalized Ricci recurrent $N(k)$-contact metric manifolds.

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1. Introduction

From the curvature point of view, the most simple nonflat spaces are the spaces of constant sectional curvature. These spaces were considered by Riemann and Helmholtz as the model spaces for the universe because of their property of free mobility. Cartan [6] generalized these spaces to the locally symmetric spaces. A Riemannian manifold (M^n, g) is said to be locally symmetric due to Cartan if its curvature tensor R satisfies the relation $\nabla R = 0$, where ∇ denotes the operator of covariant differentiation with respect to the metric tensor g.

During the last eight decades the process of generalization of locally symmetric spaces have been carried out by many authors around the globe in different directions, for instance, recurrent manifolds by Walker [31], semi-symmetric manifold by Szabo [21], pseudosymmetric manifold by Chaki [8], pseudo-symmetric manifold by Deszcz [14], weakly symmetric manifold by Tamassy and Binh [22], weakly symmetric manifold by Selberg [20]. However, the notion of pseudo-symmetry by Chaki and Deszcz are different and that of weak symmetry by Selberg and Tamassy and Binh are also different.

A non-flat Riemannian manifold $(M^n, g)(n \geq 2)$ is said to be pseudo symmetric [8] if its curvature tensor R satisfies

$$\begin{aligned}(\nabla_X R)(Y,Z)W = {} & 2A(X)R(Y,Z)W + A(Y)R(X,Z)W + A(Z)R(Y,X)W \\ & + A(W)R(Y,Z)X + g(R(Y,Z)W,X)\rho, \end{aligned} \tag{1.1}$$

where A is a non-zero associated 1-form, ρ is a vector field defined by $g(X,\rho) = A(X)$, for every vector field X and ∇ denotes the operator of the covariant differentiation with respect to the metric tensor g. If $A = 0$, then the manifold reduces to a symmetric manifold in the sense of Cartan.

Again a Riemannian manifold is said to be Ricci symmetric if the condition $\nabla S = 0$ holds, where S is the Ricci tensor of type $(0,2)$. A Riemannian manifold is called Ricci recurrent [16] if the condition $\nabla S = A \otimes S$ holds, where A is a nonzero 1-form. Every locally symmetric manifold is Ricci symmetric but not conversely and every recurrent manifold is Ricci recurrent but the converse does not hold, in general. Again every Ricci symmetric manifold is Ricci recurrent but not conversely. A non-flat Riemannian manifold which is called generalized Ricci recurrent [11] realizing the following relation

$$(\nabla_X S)(Y,Z) = A(X)S(Y,Z) + B(X)g(Y,Z), \tag{1.2}$$

where A and B are two non-zero 1-forms. If the 1-form $B = 0$ then the generalized Ricci recurrent manifold reduces to a Ricci recurrent manifold.

In 1988, Chaki [9] introduced pseudo Ricci symmetric manifold and is defined as a non-flat Riemannian manifold (M^n, g) $(n \geq 3)$ whose Ricci tensor S of type $(0,2)$ satisfies the condition

$$(\nabla_X S)(Y,Z) = 2A(X)S(Y,Z) + A(Y)S(X,Z) + A(Z)S(Y,X), \tag{1.3}$$

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where A and ∇ are stated as in the definition of pseudosymmetric manifold.
Tarafdar and De extensively studied pseudo symmetric and pseudo Ricci symmetric conditions on different manifolds [[26], [27], [28]]. Motivated by the above studies in this article we study pseudo symmetric and pseudo Ricci symmetric conditions on $N(k)$-contact metric manifold.
The paper is organized as follows: In Section 2 we report some basic information on $N(k)$-contact metric manifold. The existence of pseudo symmetric and pseudo Ricci symmetric $N(k)$-contact metric manifolds are respectively studied in section 3 and 4. Section 5 deals with the generalized Ricci recurrent $N(k)$-contact metric manifolds and shown that in a generalized Ricci recurrent $N(k)$-contact metric manifold M^{2n+1} the associated 1-forms are linearly dependent and the vector fields of the associated 1-forms are in opposite direction. In the last section, we construct an example which verifies the result in section 5.

2. Preliminaries

An odd-dimensional manifold M^{2n+1} is said to have an almost contact structure (ϕ,ξ,η) if it admits a tensor field ϕ of type $(1,1)$, a characteristic vector field ξ and a global 1-form η satisfying

$$\phi^2 = -I+\eta\otimes\xi, \quad \eta(\xi)=1, \quad \phi\xi=0, \quad \eta\circ\phi=0. \tag{2.1}$$

An almost contact structure is said to be normal if the induced almost complex structure J on the product manifold $M^{2n+1}\times R$ defined by

$$J\left(X,f\frac{d}{dt}\right)=\left(\phi X-f\xi,\eta(X)\frac{d}{dt}\right),$$

is integrable, where X is tangent to M^{2n+1}, t is the coordinate of R and f a smooth function on $M^{2n+1}\times R$.
Let g be the compatible Riemannian metric with almost contact structure (ϕ,ξ,η) such that,

$$g(\phi X,\phi Y)=g(X,Y)-\eta(X)\eta(Y), \quad g(X,\xi)=\eta(X), \quad g(\phi X,Y)=g(X,\phi Y), \tag{2.2}$$

then the structure (ϕ,ξ,η,g) on M^{2n+1} is said to be an almost contact metric structure. A manifold M^{2n+1} together with this almost contact metric structure is called an almost contact metric manifold and it is denoted by $M^{2n+1}(\phi,\xi,\eta,g)$.
An almost contact metric structure becomes a contact metric structure if $g(X,\phi Y)=d\eta(X,Y)$, for all vector fields X,Y. The 1-form η is called a contact form and ξ is characteristic vector field. We define a $(1,1)$-tensor field h by $h=\frac{1}{2}\mathscr{L}_\xi\phi$, where $\mathscr{L}$ denotes the Lie differentiation. Then the tensor h is symmetric and satisfies

$$h\xi=0, \quad h\phi+\phi h=0, \quad \nabla_X\xi=-\phi X-\phi hX, \tag{2.3}$$

where ∇ denotes the Riemannian connection of g.
The (k,μ)-nullity distribution on a contact metric manifold was introduced by Blair et al [4] and is defined by

$$N(k,\mu): p\to N_p(k,\mu)=\{U\in T_pM \mid R(X,Y)U=(kI+\mu h)(g(Y,U)X-g(X,U)Y)\},$$

for all $X,Y\in TM$, where $(k,\mu)\in R^2$. A contact metric manifold with $\xi\in N(k,\mu)$ is called a (k,μ)-contact metric manifold. If $\mu=0$, the (k,μ)-nullity distribution reduces to k-nullity distribution. The k-nullity distribution $N(k)$ of a Riemannian manifold is defined by [25]

$$N(k): p\to N_p(k)=\{U\in T_pM \mid R(X,Y)U=k(g(Y,U)X-g(X,U)Y)\},$$

k being a constant. If the characteristic vector field $\xi\in N(k)$, then we call a contact metric manifold as $N(k)$-contact metric manifold [5]. If $k=1$, then the manifold is Sasakian and if $k=0$, then the manifold is locally isometric to the product $E^{n+1}(0)\times S^n(4)$ for $n>1$ and flat for $n=1$ [3].

In a $N(k)$-contact metric manifold, following relations hold [5]:

$$h^2=(k-1)\phi^2, \tag{2.4}$$
$$(\nabla_X\eta)(Y)=g(X+hX,\phi Y), \tag{2.5}$$
$$(\nabla_X\phi)(Y)=g(X+hX,Y)\xi-\eta(Y)(X+hX), \tag{2.6}$$
$$R(X,Y)\xi=k[\eta(Y)X-\eta(X)Y], \quad R(\xi,X)Y=k[g(X,Y)\xi-\eta(Y)X], \tag{2.7}$$
$$S(X,Y)=2(n-1)g(X,Y)+2(n-1)g(hX,Y)+[2nk-2(n-1)]\eta(X)\eta(Y), \quad n\geq 1. \tag{2.8}$$
$$S(\phi X,\phi Y)=S(X,Y)-2nk\eta(X)\eta(Y)-4(n-1)g(hX,Y), \tag{2.9}$$
$$S(X,\xi)=2nk\eta(X). \tag{2.10}$$

3. Pseudo symmetric $N(k)$-contact metric manifold

In this section we suppose that an (2n+1)-dimensional $N(k)$-contact metric manifold M^{2n+1} is pseudo symmetric.

Theorem 1. *A pseudo symmetric $N(k)$-contact metric manifold M^{2n+1} is either locally isometric to the product $E^{n+1}\times S^n(4)$ for $n>1$ and flat for $n=1$ (for $k=0$) or M^{2n+1} $(n\geq 1)$ can not exist (for $k\neq 0$).*

Proof. Let $N(k)$-contact metric manifold be pseudo symmetric. Then from (1.1) we get

$$(\nabla_X S)(Y,Z) = 2A(X)S(Y,Z) + A(Y)S(X,Z) + A(R(X,Y)Z) + A(Z)S(Y,X) + A(R(X,Z)Y). \tag{3.1}$$

Putting $Z=\xi$ in (3.1) and by virtue (2.7) and (2.10), (3.1) yields

$$(\nabla_X S)(Y,\xi) = 4nkA(X)\eta(Y) + 2nkA(Y)\eta(X) + k\eta(Y)A(X) - k\eta(X)A(Y) + A(\xi)S(Y,X) + k\eta(Y)A(X) - kg(X,Y)A(\xi). \tag{3.2}$$

We know that

$$(\nabla_X S)(Z,\xi) = \nabla_X S(Z,\xi) - S(\nabla_X Z,\xi) - S(Z,\nabla_X \xi).$$

In view of (2.3), (2.5) and (2.10) above equation takes the form

$$(\nabla_X S)(Z,\xi) = 2nkg(X+hX,\phi Z) + S(Z,\phi X + \phi hX). \tag{3.3}$$

Replacing X by ξ in the above equation and taking in to account of (2.1) and (3.2), we have

$$(6nk+k)A(\xi)\eta(Y) + (2nk-k)A(Y) = 0. \tag{3.4}$$

Taking $Y=\xi$ in (3.4), it follows that for $n>0$ that

$$k=0, \tag{3.5}$$

or

$$A(\xi)=0. \tag{3.6}$$

If $k\neq 0$ then $A(\xi)=0$. So, equations (3.4) and (3.6) imply that

$$A(X)=0. \tag{3.7}$$

This completes the proof of the theorem 1. □

We know that an $N(k)$-contact metric manifold with $k=1(\neq 0)$ is a Sasakian manifold. Hence, from theorem 1 we can state the followings:

Corollary 1. *A pseudo symmetric Sasakian manifold M^{2n+1} $(n\geq 1)$ can not exist.*

Corollary 3.2 has already been proved for n-dimensional Sasakian manifold and n-dimensional K-contact manifold by Tarafdar [26] and Tarafdar and De [27].

4. Pseudo Ricci symmetric $N(k)$-contact metric manifold

Suppose that $N(k)$-contact metric manifold is pseudo symmetric. From (1.3) we have

$$(\nabla_X S)(Y,\xi) = 2A(X)S(Y,\xi) + A(Y)S(X,\xi) + A(\xi)S(Y,X). \tag{4.1}$$

By virtue of (2.10) and (3.3), we obtain

$$2nkg(X+hX,\phi Y) + S(Y,\phi X+\phi hX) = 4nkA(X)\eta(Y) + 2nkA(Y)\eta(X) + A(\xi)S(Y,X). \tag{4.2}$$

Setting $X=\xi$ in (4.2) and using (2.1) and (2.10), one can get

$$6nkA(\xi)\eta(Y) + 2nkA(Y) = 0. \tag{4.3}$$

Putting $Y=\xi$ in the above equation, it follows for $n>0$ that

$$k=0, \tag{4.4}$$

or

$$A(\xi)=0. \tag{4.5}$$

Thus, we can state the following:

Theorem 2. *A pseudo Ricci symmetric $N(k)$-contact metric manifold M^{2n+1} is either locally isometric to the product $E^{n+1}\times S^n(4)$ for $n>1$ and flat for $n=1$ or M^{2n+1} satisfies $A(\xi)=0$.*

Suppose $k\neq 0$ which implies $A(\xi)=0$. Substituting Y by ξ in (4.3) and then using (4.5), equation (4.3) gives

$$A(X)=0. \tag{4.6}$$

This leads to the following assertion:

Theorem 3. *There is no pseudo Ricci symmetric $N(k)$-contact metric manifold M^{2n+1} $(n\geq 1)$ for $k\neq 0$.*

Corollary 2. *There is no pseudo Ricci symmetric Sasakian manifold M^{2n+1} $(n\geq 1)$.*

Corollary 4.2 is proved in [[26], [27]].

5. Generalized Ricci recurrent $N(k)$-contact metric manifolds

Definition 1. *A $N(k)$-contact metric manifold M^{2n+1} is said to be generalized Ricci recurrent if its Ricci tensor S is non vanishing and satisfies the condition*

$$(\nabla_X S)(Y,Z) = A(X)S(Y,Z) + B(X)g(Y,Z), \tag{5.1}$$

where A and B are two non-zero 1-forms such that $A(X) = g(X,P)$ and $B(X) = g(X,L)$, P and L being the associated vector fields of the 1-forms.

Taking $Z = \xi$ in (5.1) and using (3.3), we have

$$2nkg(X+hX,\phi Y) + S(Y,\phi X + \phi hX) = 2nkA(X)\eta(Y) + B(X)\eta(Y). \tag{5.2}$$

On substituting Y by ξ in the above equation, one can obtain

$$2nkA(X) + B(X) = 0. \tag{5.3}$$

Thus we have:

Theorem 4. *In a generalized Ricci recurrent $N(k)$-contact metric manifold M^{2n+1} the associated 1-forms are linearly dependent and the vector fields of the associated 1-forms are of opposite direction.*

Theorem 5. *A generalized Ricci recurrent $N(k)$-contact metric manifold M^{2n+1} is a η-Einstein manifold.*

Proof. Replacing Y by ϕY in (5.2) and using (2.1), (2.9) and the symmetric property of h, we have

$$S(X,Y) = Ag(X,Y) + B\eta(X)\eta(Y), \tag{5.4}$$

where $A = [\frac{2(n-1)^2(k-1)+2nk(n-1)+2(n-1)^2}{nk}]$ and $B = \frac{-4nk(n-1)^2(k-1)+\{2nk+2(n-1)\}\{2nk-2(n-1)\}}{2nk}$. This completes the proof. □

6. Examples

In this section we construct an examples which verifies the results of section 5. We consider a 3-dimensional manifold $M = \{(x,y,z) \in R^3, (x,y,z) \neq 0\}$, where (x,y,z) are the standard coordinate in R^3. Let e_1, e_2, e_3 be three linearly independent vector fields in R^3 which satisfies

$$[e_1,e_2] = (1+\lambda)e_3, \quad [e_1,e_3] = -(1-\lambda)e_2, \quad [e_2,e_3] = 2e_1,$$

where λ is a real number.
Let g be the Riemannian metric defined by

$$g(e_i,e_j) = \begin{cases} 0 & for\ i \neq j \\ 1 & for\ i = j \end{cases} \quad 1 \leq i,j \leq 3.$$

Let η be the 1-form defined by $\eta(Z) = g(Z,e_1)$ for any vector field $Z \in \chi(M)$.
Let ϕ be the (1, 1)-tensor field defined by

$$\phi(e_1) = 0, \quad \phi(e_2) = e_3, \quad \phi(e_3) = -e_2.$$

Using the linearity of ϕ and g, we have

$$\begin{aligned} &\eta(e_1) = 1, \\ &\phi^2 Z = -Z + \eta(Z)e_1, \\ &g(\phi Z, \phi W) = g(Z,W) - \eta(Z)\eta(W), \end{aligned}$$

for any $Z, W \in \chi(M)$.
Moreover

$$he_i = \begin{cases} 0 & for\ \ i = 1 \\ \lambda e_i & for\ \ i = 2,3 \end{cases}$$

The Riemannian connection ∇ of the metric g known as Koszul's formula and is given by

$$\begin{aligned} 2g(\nabla_X Y, Z) = {} & Xg(Y,Z) + Yg(Z,X) - Zg(X,Y) - g(X,[Y,Z]) \\ & - g(Y,[X,Z]) + g(Z,[X,Y]). \end{aligned}$$

Using Koszul's formula for Riemannian metric g, we can easily calculate

$$\left(\ \nabla_{e_i} e_j \ \right) = \begin{pmatrix} 0 & 0 & 0 \\ -(1+\lambda)e_3 & 0 & (1+\lambda)e_1 \\ (1-\lambda)e_2 & -(1-\lambda)e_1 & 0 \end{pmatrix}, \qquad 1 \leq i,j \leq 3.$$

So, above relations tells us that the manifold satisfies the equation (2.3) for any vector field X in $\chi(M)$ and $\xi = e_1$. Hence the manifold is a contact metric manifold.
Using the above relations it can be verified that

$$\begin{array}{lll} R(e_1,e_2)e_3 = 0, & R(e_2,e_3)e_3 = -(1-\lambda^2)e_2, & R(e_1,e_3)e_3 = (1-\lambda^2)e_1, \\ R(e_1,e_2)e_2 = (1-\lambda^2)e_1, & R(e_2,e_3)e_2 = (1-\lambda^2)e_3, & R(e_1,e_3)e_2 = 0, \\ R(e_1,e_2)e_1 = -(1-\lambda^2)e_2, & R(e_2,e_3)e_1 = 0, & R(e_1,e_3)e_1 = -(1-\lambda^2)e_3. \end{array}$$

In view of the expressions of the curvature tensors we conclude that the manifold is a $N(1-\lambda^2)$-contact metric manifold.
Using this, we find the values of the Ricci tensors as follows:

$$S(e_1,e_1) = 2(1-\lambda^2), \quad S(e_2,e_2) = 0, \quad S(e_3,e_3) = 0.$$

Since $\{e_1,e_2,e_3\}$ forms a basis of M^3, any vector fields $X,Y \in \chi(M)$ can be written as

$$X = a_1e_1 + b_1e_2 + c_1e_3,$$
$$Y = a_2e_1 + b_2e_2 + c_2e_3,$$

where $a_i,b_i,c_i \in R^+$ (the set of all positive real numbers), $i = 1,2$. This implies that

$$S(X,Y) = 2(1-\lambda^2)a_1a_2 \quad \textit{and} \quad g(X,Y) = a_1a_2 + b_1b_2 + c_1c_2.$$

By virtue of above, we have the following:

$$(\nabla_{e_1}S)(X,Y) = 0,$$
$$(\nabla_{e_2}S)(X,Y) = -2(1+\lambda)(1-\lambda^2)\{c_1a_2 + a_1c_2\},$$
$$(\nabla_{e_3}S)(X,Y) = 2(1-\lambda)(1-\lambda^2)\{a_1b_2 + b_1a_2\}.$$

This means that manifold under the consideration is not Ricci symmetric. Let us now consider the 1-forms

$$\begin{array}{ll} A(e_1) = 0, & B(e_1) = 0, \\ A(e_2) = \dfrac{(1+\lambda)(c_1a_2 + a_1c_2)}{b_1b_2 + c_1c_2}, & B(e_2) = \dfrac{-2(1+\lambda)(1-\lambda^2)(c_1a_2 + a_1c_2)}{b_1b_2 + c_1c_2}, \\ A(e_3) = \dfrac{(1-\lambda)(c_1a_2 + a_1c_2)}{b_1b_2 + c_1c_2}, & B(e_3) = \dfrac{-2(1-\lambda)(1-\lambda^2)(a_1b_2 + b_1a_2)}{b_1b_2 + c_1c_2}, \end{array}$$

at any point $X \in M$. From (5.1) we have

$$(\nabla_{e_i}S)(Y,Z) = A(e_i)S(Y,Z) + B(e_i)g(Y,Z), \qquad i = 1,2,3. \tag{6.1}$$

It can be easily shown that the manifold with these 1-forms satisfies the relation (6.1). Hence manifold under consideration is a generalized Ricci recurrent $N(k)$-contact metric manifold. Also with the help of these 1-forms we can easily verify the equation (5.3) and theorem 4 for three dimensional case.

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