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A Note About the Trace Functions on Mantaci–Reutenauer Algebra

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Abstract

In this paper, we obtain the trace functions of Mantaci-Reutenauer algebra $\mathscr{MR}(W_n)$, where (W_n, S_n) is a Coxeter system of type B_n. We also show for every $\lambda \in \mathscr{DP}(n)$ that each characteristic class function e_λ of the group W_n is a trace function of Mantaci-Reutenauer algebra, where $\mathscr{DP}(n)$ stands for the set of all double partitions of n. Since the dimension of the trace function space on the Mantaci-Reutenauer algebra is $|\mathscr{DP}(n)|$, it exactly coincides with the algebra $\mathbb{Q}\mathrm{Irr}W_n$ generated by the irreducible characters of the group W_n. Although the multiplication of basis elements d_A and $d_{A'}$ is not commutative in Mantaci-Reutenauer algebra, the images of $d_A d_{A'}$ and $d_{A'} d_A$ under e_λ are equal to each other.

Keywords: *Mantaci-Reutenauer algebra, Orthogonal primitive idempotent, Trace function.*
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1. Introduction

Mantaci-Reutenauer algebra $\mathscr{MR}(W_n)$ introduced by Mantaci and Reutenauer [6], is a subalgebra of the group algebra $\mathbb{Q}W_n$ and is also a generalization of Solomon's descent algebra for Coxeter group of type B_n. Bonnafé and Hohlweg [2] elegantly reconstructed Mantaci-Reutenauer algebra by using the group structure of W_n. Bonnafé [3] determined the ring multiplication rule and studied the representation theory of this algebra, more explicitly. In [1], we found all characteristic class function of the group W_n and gave a formula to calculate the size of each conjugacy class $\mathscr{C}_\lambda$ of W_n, $\lambda \in \mathscr{DP}(n)$. In this article, at first we show for every double partition λ of n that every characteristic class function e_λ of Coxeter group W_n of type B_n is a trace function on Mantaci-Reutenauer algebra $\mathscr{MR}(W_n)$ and then give an explicit construction of the commutator subspace of Mantaci-Reutenauer algebra $\mathscr{MR}(W_n)$. The following theorem is the main result of this paper:

Theorem 1.1. *The space of trace functions on $\mathscr{MR}(W_n)$ is $\mathbb{Q}IrrW_n$, where $\mathbb{Q}IrrW_n$ denotes the algebra generated by all irreducible characters of the group W_n.*

The trace functions on Iwahori-Hecke algebra obtained by Geck [4]. Iwahori-Hecke algebra is defined as a deformation of the group algebra of the corresponding Coxeter group W. There are some different properties between Iwahori-Hecke algebra of Coxeter group W_n and Mantaci-Reutenauer algebra. For instance, though all the basis elements of Iwahori-Hecke algebra of Coxeter group W_n are invertible, this is not the case in the Mantaci-Reutenauer algebra.

2. Preliminaries

Let (W_n, S_n) denote a Coxeter system of type B_n and write its generating set as $S_n = \{t, s_1, \cdots, s_{n-1}\}$. Any w element of W_n acts by the permutation on the set $I_n = \{-n, \cdots, -1, 1, \cdots, n\}$ such that for every $i \in I_n$, $w(-i) = -w(i)$. The Dynkin diagram for (W_n, S_n) is as follows:

$$B_n \ : \quad \overset{t}{\circ} \Leftarrow \overset{s_1}{\circ} - \overset{s_2}{\circ} - \cdots - \overset{s_{n-1}}{\circ}.$$

For $J \subset S_n$, if W_J is generated by J, then it is called a *standard parabolic subgroup* of W_n. A *parabolic subgroup* P of W_n is a subgroup conjugate to W_J for some $J \subset S_n$. Put $t_1 := t$ and $t_{i+1} := s_i t_i s_i$ for each i, $1 \leq i \leq n-1$. Thus all elements t_i conjugate to t_1. Setting $T_n := \{t_1, t_2, \cdots, t_n\}$, then the defining relations between the elements of S_n and T_n are stated as follows:

1. $t_i^2 = 1, s_j^2 = 1$ for all i, j, $1 \leq i \leq n$, $1 \leq j \leq n-1$;
2. $(s_1 t_1)^4 = 1$;

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3. $(s_i s_{i+1})^3 = 1$ for all $i,\ 1 \le i \le n-2$;
4. $(s_i t_1)^2 = 1,$ for all $i,\ 2 \le i \le n-1$;
5. $(s_i s_j)^2 = 1$ for all i, j, $|i-j| \ge 2$;
6. $(t_i t_j)^2 = 1$ for all i, j, $1 \le i, j \le n$.

Let $l : W_n \to \mathbb{N}$ is the length function on (W_n, S_n) and let $\mathscr{T}_n$ be the reflection subgroup of W_n generated by the reflection set T_n. It is well-known that $\mathscr{T}_n$ is a normal subgroup of W_n. Now let $S_{-n} = \{s_1, \cdots, s_{n-1}\}$. The reflection subgroup of W_n generated by S_{-n} is denoted by W_{-n} and isomorphic to the symmetric group Ξ_n of degree n. Thus W_n is a split group extension of $\mathscr{T}_n$ by W_{-n}. In other words, $W_n = W_{-n} \ltimes \mathscr{T}_n$. Therefore, the order of W_n is $2^n.n!$. For further information about Coxeter groups of type B_n, one may apply [5].
Let K be any commutative ring and let H be an associative free K-algebra with finite generators. If a K-linear map $\tau : H \to K$ satisfies the relation

$$\tau(hh') = \tau(h'h)$$

for every $h, h' \in H$, then τ is called a *trace function* on H and the set of all trace functions defined on H is a K-module [4].
For any finite Coxeter group W, Solomon introduced a remarkable subalgebra $\sum W$ of the group algebra $\mathbb{Q}W$, called the *Solomon's descent algebra* [7]. In [2], Bonnafé and Hohlweg reconstructed Mantaci-Reutenauer algebra $\mathscr{MR}(W_n)$ by using the methods which depend more on the group structure of W_n.
Now, we mention the structure of Mantaci-Reutenauer algebra due to [2]:
For a positive integer n, a *signed composition* of n is an expression of n as a finite sequence $A = (a_1, \cdots, a_k)$ whose each part consists of non-zero integers such that the summation of the absolute value of all parts equals n. It is denoted by $\mathscr{SC}(n)$ the set of all signed compositions of n. Note here that the size of $\mathscr{SC}(n)$ is 2.3^{n-1}. Now let $\mathscr{DP}(n)$ be the set of double partitions of n. A *double partition* $\lambda = (\lambda^+; \lambda^-)$ of n consists of a pair of partitions λ^+ and λ^- such that $|\lambda| = |\lambda^+| + |\lambda^-| = n$. If the length of λ^+ (resp. the length of λ^-) is equal to zero, then we write $\emptyset$ instead of λ^+ (resp. λ^-). For a $\lambda = (\lambda^+; \lambda^-)$ double partition of n, $\hat{\lambda}$ denotes the signed composition of n obtained by concatenating λ^+ and $-\lambda^-$, that is, $\hat{\lambda} = \lambda^+ \sqcup -\lambda^-$ is a signed composition obtained by appending the sequence of components of λ^+ to that of $-\lambda^-$. Let S_n' be the set $\{s_1 \cdots s_{n-1}, t_1, t_2, \cdots, t_n\}$.
In [2], Bonnafé and Hohlweg introduced some reflection subgroup of W_n for any signed composition of n as follows: For $A = (a_1, \cdots, a_k) \in \mathscr{SC}(n)$, the set S_A is defined as

$$\begin{aligned} S_A = &\{s_p \in S_{-n} \ :\ |a_1| + \cdots + |a_{i-1}| + 1 \le p \le |a_1| + \cdots + |a_i| - 1\} \\ &\cup \{t_{|a_1| + \cdots + |a_{j-1}| + 1} \in T_n \mid a_j > 0\} \subset S_n'. \end{aligned}$$

The reflection subgroup W_A of W_n is generated by S_A and (W_A, S_A) is a Coxeter system [2]. For any $A \in \mathscr{SC}(n)$, the set of all distinguished coset representatives of W_A in W_n is defined in the following way:

$$D_A = \{x \in W_n \ :\ \forall\, s \in S_A,\ \ l(xs) > l(x)\}.$$

For $A, B \in \mathscr{SC}(n)$, the set $D_{AB} = D_A^{-1} \cap D_B$ stands for the collection of elements with minimal length in (W_A, W_B)-double cosets. For $A \in \mathscr{SC}(n)$, set

$$d_A = \sum_{w \in D_A} w \in \mathbb{Q}W_n.$$

Then by [2], Mantaci-Reutenauer algebra is described explicitly as follows:

$$\mathscr{MR}(W_n) = \bigoplus_{A \in \mathscr{SC}(n)} \mathbb{Q}d_A.$$

Moreover, $\dim_{\mathbb{Q}} \mathscr{MR}(W_n) = |2.3^{n-1}|$. Let the map $\Phi_n\ :\ \mathscr{MR}(W_n) \to \mathbb{Q}\mathrm{Irr}W_n$ be the unique $\mathbb{Q}$-linear map such that $\Phi_n(d_A) = \mathrm{Ind}_{W_A}^{W_n} 1_A$ for every $A \in \mathscr{SC}(n)$, where 1_A stands for the trivial character of W_A. This map is a surjective algebra morphism as well. Furthermore, it is well-known from [2] that the radical of $\mathscr{MR}(W_n)$ is $\mathrm{Ker}\Phi_n = \sum_{A \equiv_n A'} \mathbb{Q}(d_A - d_{A'})$, where $A \equiv_n A'$ means that W_A is W_n-conjugate to $W_{A'}$.
The multiplication structure in $\mathscr{MR}(W_n)$ is given in the following proposition due to [3]:

Proposition 2.1. ([3, Proposition C]) *Let A and B be any two signed composition of n.*

(a) *Then, there is a map $f_{AB} : D_{AB} \to \mathscr{SC}(n)$ satisfying the following conditions:*

- *For every $x \in D_{AB}$, $f_{AB}(x) \subset B$ and $f_{AB}(x) \equiv_B {}^{x^{-1}}A \cap B$.*
- *$d_A d_B - \sum_{x \in D_{AB}} d_{f_{AB}(x)} \in \mathscr{MR}_{\subsetneq_\lambda A}(W_n) \cap \mathscr{MR}_{\prec B}(W_n) \cap Ker\Phi_n$.*

(b) *If A parabolic or B semi-positive, then $f_{AB}(d) = {}^{d^{-1}}A \cap B$ and $d_A d_B = \sum_{d \in D_{AB}} d_{{}^{d^{-1}}A \cap B}$ for every $d \in D_{AB}$.*

In the Proposition 2.1, the inclusion $f_{AB}(x) \subset B$ means that $W_{f_{AB}(x)}$ is a subgroup of W_B. By [3], the symbols $\subset_\lambda$ and $\prec$ are a pre-order and an ordering defined on $\mathscr{SC}(n)$, respectively.

3. Trace functions on Mantaci–Reutenauer algebra

Let $\phi_\lambda = \mathrm{Ind}_{W_{\hat{\lambda}}}^{W_n} 1_{\hat{\lambda}}$ for each $\lambda \in \mathscr{DP}(n)$. For a double partition μ of n, let denote by cox_μ a Coxeter element of $W_{\hat{\mu}}$ in terms of generating set $S_{\hat{\mu}}$. Since the matrix $(\phi_\lambda(\mathrm{cox}_\mu))_{\lambda,\mu}$ is upper diagonal and has positive diagonal entries, then $(\phi_\lambda(\mathrm{cox}_\mu))_{\lambda,\mu}$ is invertible in $\mathbb{Q}$. Thus the inverse of $(\phi_\lambda(\mathrm{cox}_\mu))_{\lambda,\mu}$ will be denoted by $(u_{\lambda\mu})_{\lambda,\mu}$. We have obtained in [1] that for each $\lambda \in \mathscr{DP}(n)$, orthogonal primitive idempotent of $\mathbb{Q}\mathrm{Irr}W_n$

$$e_\lambda = \sum_{\mu \in \mathscr{DP}(n)} u_{\lambda\mu} \phi_\mu$$

is also the characteristic class function on W_n corresponding to the conjugate class $\mathscr{C}_\lambda$. More explicitly, we have $e_\lambda(\mathrm{cox}_\mu) = \delta_{\lambda,\mu}$ for any $\lambda,\mu \in \mathscr{DP}(n)$. For every $\lambda \in \mathscr{DP}(n)$ and $A \in \mathscr{SC}(n)$, if e_λ is extended by linearity to group algebra $\mathbb{Q}W_n$, then we have

$$e_\lambda(d_A) = \sum_{x \in D_A} e_\lambda(x) = |\mathscr{C}_\lambda \cap D_A|.$$

Lemma 3.1. *For every $\lambda \in \mathscr{DP}(n)$, the orthogonal primitive idempotent e_λ is not an irreducible character of $\mathbb{Q}IrrW_n$.*

Proof. We first note that for every $\lambda \in \mathscr{DP}(n)$, the cardinalities of $\mathscr{C}_\lambda$ and W_n are different. By definition of the inner product of characters, we have

$$\begin{aligned}\langle e_\lambda, e_\gamma\rangle &= \frac{1}{|W_n|}\sum_{w\in W_n} e_\lambda(w)e_\gamma(w^{-1})\\ &= \frac{|\mathscr{C}_\lambda|}{|W_n|}\delta_{\lambda,\gamma} \neq 1.\end{aligned}$$

Thus, the proof is completed. □

Morever, in [1], we obtained that for any $\lambda \in \mathscr{DP}(n)$ the size of the conjugacy class $\mathscr{C}_\lambda$ is equal to $|W_n| \cdot \sum_{\mu\in\mathscr{DP}(n)} u_{\lambda,\mu}$.
For any $\lambda \in \mathscr{DP}(n)$, by [2] the algebra homomorphism $\pi_\lambda : \mathscr{MR}(W_n) \to \mathbb{Q},\ x \mapsto \Phi_n(x)(\mathrm{cox}_\lambda)$ is an irreducible character of $\mathscr{MR}(W_n)$, where cox_λ denotes a Coxeter element of $W_{\hat{\lambda}}$.
If H is a K-algebra, the commutator of any two elements h,h' of H is defined as $[h,h'] = hh' - h'h$ and commutator subspace $[H,H]$ of H is a K-subspace generated by all commutators [4]. The commutator subspace $[H,H]$ lies in the kernel of every trace function on H and conversely, if $\tau : H \to K$ is any K-linear map which is identically zero on the subspace $[H,H]$, then τ is a trace function on H [4]. Thus, from [4], there is one to one corresponding between the space of all trace functions on H and $\mathrm{Hom}_K(H/[H,H],K)$ dual space of quotient module $H/[H,H]$. Therefore, we can now give the following proposition without proof.

Proposition 3.1. *As the set of all irreducible characters of $\mathscr{MR}(W_n)$ is $\{\pi_\lambda : \lambda \in \mathscr{DP}(n)\}$, the dimension of the space of trace functions on $\mathscr{MR}(W_n)$*

$$dim_{\mathbb{Q}}\mathscr{MR}(W_n)/[\mathscr{MR}(W_n),\mathscr{MR}(W_n)] = |\mathscr{DP}(n)|.$$

Since all elements of $\mathrm{Ker}(\Phi_n)$ are nilpotent and e_λ is a characteristic class function of W_n, then e_λ vanishes on the $\mathrm{Ker}(\Phi_n)$.

Lemma 3.2. *For any two signed compositions A and A' of n, we have $e_\lambda(d_A d_{A'}) = e_\lambda(d_{A'} d_A)$.*

Proof. Taking into consideration Proposition 2.1, since the expression $d_A d_{A'} - \sum_{x\in D_{AA'}} d_{f_{AA'}(x)}$ belongs to $\mathrm{Ker}(\Phi_n)$ and $f_{AA'}(x) \equiv_{A'} {}^{x^{-1}}A \cap A'$, then we get

$$e_\lambda(d_A d_{A'}) = \sum_{x\in D_{AA'}} e_\lambda(d_{f_{AA'}(x)}).$$

Because of $f_{AA'}(x) \equiv_n f_{A'A}(x^{-1})$ for every $x \in D_{AA'}$, we obtain that $e_\lambda(d_{f_{AA'}(x)}) = e_\lambda(d_{f_{A'A}(x^{-1})})$. Hence it is immediately seen the equality $e_\lambda(d_A d_{A'}) = e_\lambda(d_{A'} d_A)$. □

Lemma 3.3. *The commutator subspace of Mantaci-Reutenauer algebra $\mathscr{MR}(W_n)$ is $[\mathscr{MR}(W_n),\mathscr{MR}(W_n)] = Ker\Phi_n$.*

Proof. Since every $x \in \mathscr{MR}(W_n)$ can be uniquely written as an expression of basis elements $d_A, A \in \mathscr{SC}(n)$, for every $x,y \in \mathscr{MR}(W_n)$ we immediately obtain from the proof of Lemma 3.2 that

$$xy - yx \in \mathrm{Ker}\Phi_n.$$

Therefore, we get the inclusion $[\mathscr{MR}(W_n),\mathscr{MR}(W_n)] \subset \mathrm{Ker}\Phi_n$. If we take into account the fact that $\mathscr{MR}(W_n)/\mathrm{Ker}\Phi_n \cong \mathbb{Q}\mathrm{Irr}W_n$ (since Φ_n is a surjective algebra morphism) and Proposition 3.1, then we obtain

$$[\mathscr{MR}(W_n),\mathscr{MR}(W_n)] = \mathrm{Ker}\Phi_n.$$

□

It is clear that the quotient space $\mathscr{MR}(W_n)/[\mathscr{MR}(W_n),\mathscr{MR}(W_n)] = \mathscr{MR}(W_n)/\mathrm{Ker}\Phi_n$ is also a free $\mathbb{Q}$-module.

Theorem 3.1. *(Main Theorem) The space of trace functions on $\mathscr{MR}(W_n)$ is $\mathbb{Q}IrrW_n$.*

Proof. For each $\lambda \in \mathscr{DP}(n)$, the characteristic class function e_λ is also a trace function on $\mathscr{MR}(W_n)$ from Lemma 3.2. Moreover, by Proposition 3.1 the space of trace functions on $\mathscr{MR}(W_n)$ and $\mathbb{Q}\mathrm{Irr}W_n$ have the same dimension, that is $|\mathscr{DP}(n)|$. So we have obtained that the trace function space of $\mathscr{MR}(W_n)$ exactly coincides with the algebra $\mathbb{Q}\mathrm{Irr}W_n$. □

As Φ_n is an algebra morphism, the π_λ corresponding to $\lambda \in \mathscr{DP}(n)$ is another trace function on $\mathscr{MR}(W_n)$.

Example 3.1. We consider the Coxeter group W_2. For all $\lambda,\mu \in \mathscr{DP}(2) = \{(2;\emptyset),\ (1,1;\emptyset),\ (1;1),\ (\emptyset;2),\ (\emptyset;1,1)\}$, the values $\varphi_\lambda(\mathrm{cox}_\mu)$ are given in the following way:

$$(\varphi_\lambda(\mathrm{cox}_\mu))_{\lambda,\mu} = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 2 & 2 & 0 & 2 \\ 0 & 0 & 2 & 0 & 4 \\ 0 & 0 & 0 & 2 & 4 \\ 0 & 0 & 0 & 0 & 8 \end{pmatrix},\ (u_{\lambda\mu})_{\lambda,\mu} = \begin{pmatrix} 1 & \frac{-1}{2} & 0 & \frac{-1}{2} & \frac{1}{4} \\ 0 & \frac{1}{2} & \frac{-1}{2} & 0 & \frac{1}{8} \\ 0 & 0 & \frac{1}{2} & 0 & \frac{-1}{4} \\ 0 & 0 & 0 & \frac{1}{2} & \frac{-1}{4} \\ 0 & 0 & 0 & 0 & \frac{1}{8} \end{pmatrix}$$

Thus, a basis of the space of trace functions on $\mathscr{MR}(W_n)$ is stated as follows:

$$\begin{aligned} e_{(2;\emptyset)} &= \phi_{(2;\emptyset)} - \frac{1}{2}\phi_{(1,1;\emptyset)} - \frac{1}{2}\phi_{(\emptyset;2)} + \frac{1}{4}\phi_{(\emptyset;1,1)} \\ e_{(1,1;\emptyset)} &= \frac{1}{2}\phi_{(1,1;\emptyset)} - \frac{1}{2}\phi_{(1;1)} + \frac{1}{8}\phi_{(\emptyset;1,1)} \\ e_{(1;1)} &= \frac{1}{2}\phi_{(1;1)} - \frac{1}{4}\phi_{(\emptyset;1,1)} \\ e_{(\emptyset;2)} &= \frac{1}{2}\phi_{(\emptyset;2)} - \frac{1}{4}\phi_{(\emptyset;1,1)} \\ e_{(\emptyset;1,1)} &= \frac{1}{8}\phi_{(\emptyset;1,1)}. \end{aligned}$$

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