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On the Continuous Generalized Solutions of a Lossless Transmission Line System with Josephson Junction

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Abstract

The present paper is devoted to the investigation of lossless transmission lines with Josephson junction. Such lines are described by first order nonlinear hyperbolic system partial differential equations. We consider the mixed problem for this system with boundary conditions generated by a circuit corresponding to Josephson junction formulated by V. Angelov. We present the mixed problem in an operator form and obtain a suitable sequence converging to a continuous solution.

Keywords: transmission lines, Josephson junction, nonlinear hyperbolic system, mixed problem.
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1. Introduction

The present paper deals with the problem of analysis of lossless transmission lines with Josephson junction. The superconductivity problem has been treated in many papers [1]-[4]. The describing of the problem leads to the following nonlinear hyperbolic system

$$\begin{cases} \dfrac{\partial u(x,t)}{\partial x} = -L\dfrac{\partial i(x,t)}{\partial t}, \\ \dfrac{\partial i(x,t)}{\partial x} = -C\dfrac{\partial u(x,t)}{\partial t} - j_0 \sin\dfrac{2\pi\Phi(x,t)}{\Phi_0}, \ (x,t)\in\Pi=\{(x,t):(x,t)\in[0,\Lambda]\times[0,T])\} \\ \dfrac{\partial \Phi(x,t)}{\partial t} = u(x,t), \end{cases} \tag{1.1}$$

Here $u(x,t)$, $i(x,t)$ and $\Phi(x,t)$ are unknown functions – voltage, current and Josephson flux, L and C are prescribed specific parameters of the line, $\Lambda > 0$ is its length; j_0 is maximal Josephson current per unit length and $\Phi_0 = \dfrac{h}{2e}$ is flux induction quantum, h is Planck constant and $\dfrac{1}{\Phi_0} = J_K$ is Josephson constant.

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In [5], [6] (cf. also [7], [8]) V. Angelov showed that it is better to consider (1.1) instead of the well-known Sin-Gordon equation. As in [5] we present the flux function as an integral of the voltage

$$\Phi(x,t) = \int_0^t u(x,s)ds, \tag{1.2}$$

and then formulate the following initial-boundary value (mixed) problem: to find the unknown functions $u(x,t)$ and $i(x,t)$ in Π satisfying the system

$$\begin{cases} \dfrac{\partial u(x,t)}{\partial t} + \dfrac{1}{C}\dfrac{\partial i(x,t)}{\partial x} = -\dfrac{j_0}{C}\sin\left(\dfrac{2\pi}{\Phi_0}\int_0^t u(x,s)ds\right) \\ \dfrac{\partial i(x,t)}{\partial t} + \dfrac{1}{L}\dfrac{\partial u(x,t)}{\partial x} = 0, \end{cases} , (x,t)\in\Pi, \tag{1.3}$$

with initial conditions

$$u(x,0) = u_0(x),\ i(x,0) = i_0(x),\ x\in[0,\Lambda] \tag{1.4}$$

and boundary conditions ([5], [6])

$$E(t) - u(0,t) - R_0 i(0,t) = 0,\ t\in(0,T], \tag{1.5}$$

$$C_0\frac{du(\Lambda,t)}{dt} = i(\Lambda,t) - \frac{1}{R_1}u(\Lambda,t),\ t\in[0,T], \tag{1.6}$$

where $i_0(x)$, $u_0(x)$ are prescribed functions – the current and voltage at the initial instant, $E(t)$ is a prescribed source function, R_0, R_1, C_0 are specific parameters of the elements of the circuits (cf. [5]).

Here we are based on the operator form of (1.3) – (1.6) used in [5], [6] and obtain a solution by the method of successive approximations. Although we cannot overcome the cutting of the domain Π we choose a better initial approximation such that the sequence obtained tends to the continuous solution of the corresponding operator equation.

2. Diagonalization of the hyperbolic system

Following [8] we multiply by $\sqrt{\dfrac{L}{C}}$ the second equation of the system (1.3)

$$\begin{cases} \dfrac{\partial u(x,t)}{\partial t} + \dfrac{1}{C}\dfrac{\partial i(x,t)}{\partial x} = -\dfrac{j_0}{C}\sin\left(\dfrac{2\pi}{\Phi_0}\int_0^t u(x,s)ds\right) \\ \sqrt{\dfrac{L}{C}}\dfrac{\partial i(x,t)}{\partial t} + \dfrac{1}{\sqrt{LC}}\dfrac{\partial u(x,t)}{\partial x} = 0. \end{cases} \tag{2.1}$$

Adding the above equations we get

$$\frac{\partial}{\partial t}\left(u(x,t) + \sqrt{\frac{L}{C}}\ i(x,t)\right) + \frac{1}{\sqrt{LC}}\frac{\partial}{\partial x}\left(u(x,t) + \sqrt{\frac{L}{C}}\ i(x,t)\right) = -\frac{j_0}{C}\sin\left(\frac{2\pi}{\Phi_0}\int_0^t u(x,s)ds\right).$$

Subtracting the equations of (2.1) we get:

$$\frac{\partial}{\partial t}\left(u(x,t) - \sqrt{\frac{L}{C}}\ i(x,t)\right) - \frac{1}{\sqrt{LC}}\frac{\partial}{\partial x}\left(u(x,t) - \sqrt{\frac{L}{C}}\ i(x,t)\right) = -\frac{j_0}{C}\sin\left(\frac{2\pi}{\Phi_0}\int_0^t u(x,s)ds\right).$$

So we obtain a new system:

$$\left\{\begin{array}{l} \dfrac{\partial}{\partial t}\big(u(x,t)+\sqrt{\dfrac{L}{C}}i(x,t)\big)+\dfrac{1}{\sqrt{LC}}\dfrac{\partial}{\partial x}\big(u(x,t)+\sqrt{\dfrac{L}{C}}i(x,t)\big)=-\dfrac{j_0}{C}\sin\left(\dfrac{2\pi}{\Phi_0}\int\limits_0^t u(x,s)ds\right)\\ \dfrac{\partial}{\partial t}\big(u(x,t)-\sqrt{\dfrac{L}{C}}i(x,t)\big)-\dfrac{1}{\sqrt{LC}}\dfrac{\partial}{\partial x}\big(u(x,t)-\sqrt{\dfrac{L}{C}}i(x,t)\big)=-\dfrac{j_0}{C}\sin\left(\dfrac{2\pi}{\Phi_0}\int\limits_0^t u(x,s)ds\right). \end{array}\right. \tag{2.2}$$

As usually we put $v=\dfrac{1}{\sqrt{LC}}$, $Z_0=\sqrt{\dfrac{L}{C}}$ and define two functions of new real variables $(z_1,z_2): z_1=\dfrac{x}{\Lambda}\in[0,1],\ z_2=\dfrac{tv}{\Lambda}\in\left[0,\dfrac{Tv}{\Lambda}\right]$; or $x=\Lambda z_1,\ t=\dfrac{\Lambda}{v}z_2$:

$$\begin{aligned} f_1(z_1,z_2)&=\frac{1}{j_0Z_0\Lambda}\left[u\left(\Lambda z_1,\frac{\Lambda}{v}z_2\right)+Z_0 i\left(\Lambda z_1,\frac{\Lambda}{v}z_2\right)\right];\\ f_2(z_1,z_2)&=\frac{1}{j_0Z_0\Lambda}\left[u\left(\Lambda z_1,\frac{\Lambda}{v}z_2\right)-Z_0 i\left(\Lambda z_1,\frac{\Lambda}{v}z_2\right)\right], \end{aligned} \tag{2.3}$$

for $z=(z_1,z_2)\in P\subset\mathbb{R}^2$, where $\mathbb{R}^2=\mathbb{R}\times\mathbb{R}$ $(\mathbb{R}=(-\infty,\infty))$; $P=[0,1]\times\left[0,\dfrac{Tv}{\Lambda}\right]$.

The inverse transformation is

$$\begin{aligned} u(x,t)&=\frac{j_0Z_0\Lambda}{2}\left[f_1\left(\frac{x}{\Lambda},\frac{tv}{\Lambda}\right)+f_2\left(\frac{x}{\Lambda},\frac{tv}{\Lambda}\right)\right];\\ i(x,t)&=\frac{j_0\Lambda}{2}\left[f_1\left(\frac{x}{\Lambda},\frac{tv}{\Lambda}\right)-f_2\left(\frac{x}{\Lambda},\frac{tv}{\Lambda}\right)\right], \end{aligned}\qquad (x,t)\in\Pi. \tag{2.4}$$

For the partial derivatives of f_k $(k=1,2)$ we obtain as follows:

$$\frac{\partial f_k(z_1,z_2)}{\partial z_1}=\frac{1}{j_0Z_0}\frac{\partial}{\partial x}\left[u\left(\Lambda z_1,\frac{\Lambda}{v}z_2\right)+(-1)^{k-1}Z_0 i\left(\Lambda z_1,\frac{\Lambda}{v}z_2\right)\right];$$

$$\frac{\partial f_k(z_1,z_2)}{\partial z_2}=\frac{1}{j_0Z_0v}\frac{\partial}{\partial t}\left[u\left(\Lambda z_1,\frac{\Lambda}{v}z_2\right)+(-1)^{k-1}Z_0 i\left(\Lambda z_1,\frac{\Lambda}{v}z_2\right)\right].$$

Consequently, by substituting into the equations of the system (2.2) we obtain as follows:

$$\frac{\partial f_1(z_1,z_2)}{\partial z_2}+\frac{\partial f_1(z_1,z_2)}{\partial z_1}=\frac{C}{j_0}\left(\frac{\partial}{\partial t}+\frac{1}{\sqrt{LC}}\frac{\partial}{\partial x}\right)\left[u\left(\Lambda z_1,\Lambda\sqrt{LC}z_2\right)+\sqrt{\frac{L}{C}}i\left(\Lambda z_1,\Lambda\sqrt{LC}z_2\right)\right]=$$

$$=-\sin\left(\frac{2\pi}{\Phi_0}\int\limits_0^{\Lambda\sqrt{LC}z_2}u\left(\Lambda z_1,s\right)ds\right)=-\sin\left(\frac{2\pi}{\Phi_0}\cdot\Lambda\sqrt{LC}\cdot\frac{j_0Z_0\Lambda}{2}\int\limits_0^{z_2}\left(f_1(z_1,r)+f_2(z_1,r)\right)dr\right)=$$

$$=-\sin\left(\frac{\pi\Lambda^2Lj_0}{\Phi_0}\int\limits_0^{z_2}\left(f_1(z_1,r)+f_2(z_1,r)\right)dr\right);$$

$$\frac{\partial f_1(z_1\,,z_2)}{\partial z_2}-\frac{\partial f_1(z_1,z_2)}{\partial z_1}=\frac{C}{j_0}\left(\frac{\partial}{\partial t}-\frac{1}{\sqrt{LC}}\frac{\partial}{\partial x}\right)\left[u\left(\Lambda z_1,\Lambda\sqrt{LC}z_2\right)-\sqrt{\frac{L}{C}}i\left(\Lambda z_1,\Lambda\sqrt{LC}z_2\right)\right]=$$

$$=-\sin\left(\frac{2\pi}{\Phi_0}\int\limits_0^{\Lambda\sqrt{LC}z_2}u\left(\Lambda z_1\,,s\right)ds\right)=-\sin\left(\frac{\pi\Lambda^2Lj_0}{\Phi_0}\int\limits_0^{z_2}\left(f_1(z_1,r)+f_2(z_1,r)\right)dr\right).$$

Thus we reduce the system (2.2) to the following one:

$$\left\{\begin{array}{l} \dfrac{\partial f_1}{\partial z_2}+\dfrac{\partial f_1}{\partial z_1}=-\sin\left(\dfrac{\pi\Lambda^2Lj_0}{\Phi_0}\int\limits_0^{z_2}\left(f_1(z_1,r)+f_2(z_1,r)\right)dr\right)\\ \dfrac{\partial f_2}{\partial z_2}-\dfrac{\partial f_2}{\partial z_1}=-\sin\left(\dfrac{\pi\Lambda^2Lj_0}{\Phi_0}\int\limits_0^{z_2}\left(f_1(z_1,r)+f_2(z_1,r)\right)dr\right) \end{array}\right.,(z_1,z_2)\in P. \tag{2.5}$$

The new initial conditions become:

$$\left.\begin{aligned} f_1(z_1,0) &= \frac{u_0(\Lambda z_1)}{j_0 Z_0 \Lambda} + \frac{i_0(\Lambda z_1)}{j_0\Lambda} = \frac{u_0(\Lambda z_1)\sqrt{C} + i_0(\Lambda z_1)\sqrt{L}}{j_0\Lambda\sqrt{L}}; \\ f_2(z_1,0) &= \frac{u_0(\Lambda z_1)}{j_0 Z_0 \Lambda} - \frac{i_0(\Lambda z_1)}{j_0\Lambda} = \frac{u_0(\Lambda z_1)\sqrt{C} - i_0(\Lambda z_1)\sqrt{L}}{j_0\Lambda\sqrt{L}} \end{aligned}\right\} ,\ z_1 \in [0,\,1]\,. \tag{2.6}$$

We obtain the new boundary conditions substituting u and i from (2.4) into (1.5) and (1.6) respectively:

$E(t) - \frac{j_0 Z_0 \Lambda}{2}\left[f_1\left(0,\,\frac{tv}{\Lambda}\right) + f_2\left(0,\,\frac{tv}{\Lambda}\right)\right] - \frac{j_0 R_0 \Lambda}{2}\left[f_1\left(0,\,\frac{tv}{\Lambda}\right) - f_2\left(0,\,\frac{tv}{\Lambda}\right)\right] = 0\ (t\in(0,T])$ and

with $t = \frac{\Lambda}{v}z_2$ we get $f_1\left(0,z_2\right) = \frac{2}{(R_0+Z_0)j_0\Lambda}E\left(\frac{\Lambda z_2}{v}\right) + \frac{R_0 - Z_0}{R_0+Z_0}f_2(0,z_2)$ for $z_2 \in \left(0,\,\frac{Tv}{\Lambda}\right]$;

$C_0\frac{d}{dt}\left[f_1\Big(1,\frac{tv}{\Lambda}\Big) + f_2\Big(1,\frac{tv}{\Lambda}\Big)\right] = \frac{1}{Z_0}\left[\Big(1,\frac{tv}{\Lambda}\Big) - f_2\Big(1,\frac{tv}{\Lambda}\Big)\right] - \frac{1}{R_1}\left[f_1\Big(1,\frac{tv}{\Lambda}\Big) + f_2\Big(1,\frac{tv}{\Lambda}\Big)\right] =$

$= \frac{R_1 - Z_0}{R_1 Z_0}f_1\Big(1,\frac{tv}{\Lambda}\Big) - \frac{R_1+Z_0}{R_1 Z_0}f_2\Big(1,\frac{tv}{\Lambda}\Big) = \frac{2}{Z_0}f_1\Big(1,\frac{tv}{\Lambda}\Big) - \frac{R_1+Z_0}{R_1Z_0}\left[f_1\Big(1,\frac{tv}{\Lambda}\Big) + f_2\Big(1,\frac{tv}{\Lambda}\Big)\right],$

consequently, with $t = \frac{\Lambda}{v}r,\ r\in\left(0,\,\frac{Tv}{\Lambda}\right]$ we get

$\frac{d}{dr}\left[f_1\left(1,r\right) + f_2\left(1,r\right)\right] + \frac{\Lambda\left(R_1+Z_0\right)}{vZ_0C_0R_1}\left[f_1\left(1,r\right) + f_2\left(1,r\right)\right] = \frac{2\Lambda}{vZ_0C_0}f_1\left(1,r\right)$, or $\left(vZ_0 = \frac{1}{C}\right)$

$\frac{d}{dr}\left\{\left[f_1\left(1,r\right) + f_2\left(1,r\right)\right]e^{\frac{R_1+Z_0}{R_1}\frac{\Lambda C}{C_0}r}\right\} = \frac{2\Lambda C}{C_0}f_1(1,r).e^{\frac{R_1+Z_0}{R_1}\frac{\Lambda C}{C_0}r}$

and integrating the last equation from 0 to z_2 (with a reasonable assumption that there exist the limits $\lim\limits_{r\to+0}f_1(1,r) = f_1(1,0);\ \lim\limits_{r\to+0}f_2(1,r)) = f_2(1,0))$, we obtain for each $z_2\in\left(0,\,\frac{Tv}{\Lambda}\right]$:

$f_2(1,z_2) = e^{-\frac{R_1+Z_0}{R_1}\frac{\Lambda C}{C_0}z_2}(f_1(1,0)+f_2(1,0)) - f_1(1,z_2) + \frac{2\Lambda C}{C_0}\int\limits_0^{z_2}f_1(1,s)e^{-\frac{R_1+Z_0}{R_1}\frac{\Lambda C}{C_0}(z_2-r)}dr.$

Finally, the new boundary conditions become:

$$f_1\left(0,z_2\right) = \frac{2}{(R_0+Z_0)j_0\Lambda}E\Big(\frac{\Lambda z_2}{v}\Big) + \frac{R_0-Z_0}{R_0+Z_0}f_2(0,z_2),\ z_2\in\left(0,\,\frac{Tv}{\Lambda}\right]; \tag{2.7}$$

$$\begin{aligned} f_2(1,z_2) = {}& e^{-\frac{R_1+Z_0}{R_1}\frac{\Lambda C}{C_0}z_2}(f_1(1,0)+f_2(1,0)) - f_1(1,z_2) + \\ & + \frac{2\Lambda C}{C_0}\int\limits_0^{z_2}f_1(1,s)e^{-\frac{R_1+Z_0}{R_1}\frac{\Lambda C}{C_0}(z_2-r)}dr, \qquad z_2\in\left[0,\,\frac{Tv}{\Lambda}\right]. \end{aligned} \tag{2.8}$$

We introduce the following real constants: $\sigma = \frac{Tv}{\Lambda}$, $\mu = \frac{\pi\Lambda^2 L j_0}{\Phi_0}$ $(\sigma\in(0,+\infty), \mu\in(0,+\infty))$ and $\alpha = \frac{R_0-Z_0}{R_0+Z_0}$, $\beta = \frac{R_1+Z_0}{R_1}$, $\gamma = \frac{\Lambda C}{C_0} = \frac{\Lambda}{vZ_0C_0} = \frac{\Lambda L}{C_0 Z_0^2}$ $(\alpha\in(-1,\,1),\ \beta\in(1,+\infty), \gamma\in(0,+\infty))$, as well as the real-valued functions $h_1, h_2 : [0,1]\to\mathbb{R}$ and $\mathrm{K} : [0,\sigma]\to\mathbb{R}$:

$$h_1(z_1) = \frac{u_0(\Lambda z_1)}{j_0Z_0\Lambda} + \frac{i_0(\Lambda z_1)}{j_0\Lambda},\ \ h_2(z_1) = \frac{u_0(\Lambda z_1)}{j_0Z_0\Lambda} - \frac{i_0(\Lambda z_1)}{j_0\Lambda}; \tag{2.9}$$

$$\mathrm{K}(z_2) = \frac{1}{j_0Z_0\Lambda}E\Big(\frac{\Lambda z_2}{v}\Big)\ \Big(\mathrm{K}(0) = \frac{1}{j_0Z_0\Lambda}\lim_{r\to+0}E\Big(\frac{\Lambda r}{v}\Big) = \mathrm{K}(+0)\Big) \tag{2.10}$$

(depending on the original constants $\Lambda, T;\ L, C, R_0, R_1, C_0$ and functions i_0, u_o, E).

So the system (2.5) can be rewritten as:

$$\begin{cases} \dfrac{\partial f_1}{\partial z_2}+\dfrac{\partial f_1}{\partial z_1}=-\sin\left(\mu\int\limits_0^{z_2}\left(f_1(z_1,r)+f_2(z_1,r)\right)dr\right) \\ \\ \dfrac{\partial f_2}{\partial z_2}-\dfrac{\partial f_2}{\partial z_1}=-\sin\left(\mu\int\limits_0^{z_2}\left(f_1(z_1,r)+f_2(z_1,r)\right)dr\right) \end{cases},\ z=(z_1,z_2)\in P. \tag{2.11}$$

The transformed mixed problem is: to find a solution $f=(f_1,f_2)$ of the system (2.11), with initial conditions

$$f_1(z_1,0)=h_1(z_1),\ \ f_2(z_1,0)=h_2(z_1), z_1\in[0,\,1]\,; \tag{2.12}$$

and boundary conditions

$$f_1(0,z_2)=(1-\alpha)\mathrm{K}(z_2)+\Phi_{1,0}\left[(f_1,f_2)\right]\,(z_2),\ \ z_2\in(0,\,\sigma]\,, \tag{2.13}$$

$$f_2(1,z_2)=\left(f_1(1,0)+f_2(1,0)\right)\,e^{-\,\beta\gamma z_2}+\Phi_{2,1}\left[(f_1,f_2)\right]\,(z_2),\ z_2\in[0,\,\sigma]\,, \tag{2.14}$$

where the operators $\Phi_{1,0}$ and $\Phi_{2,1}$ are defined as follows:

$$\Phi_{1,0}\left[(f_1,f_2)\right]\,(z_2)=\alpha\,f_2(0,z_2),\ \ z_2\in(0,\,\sigma]\,; \tag{2.15}$$

$$\Phi_{2,1}\left[(f_1,f_2)\right]\,(z_2)=-f_1(1,z_2)+2\gamma\int\limits_0^{z_2}f_1(1,r)e^{-\beta\gamma\left(z_2-r\right)}dr,\ \ z_2\in[0,\,\sigma]\,. \tag{2.16}$$

3. An operator formulation of the mixed problem.

Our aim in this section is to reduce the initial-boundary value problem (2.11) – (2.14) to an operator equation for a corresponding mapping, acting on a suitable space of vector-functions.

Throughout this section (and the next), $C(I)$ will be the set of all continuous functions from the interval $I\subset\mathbb{R}$ into $\mathbb{R}$ $(\mathbb{R}=(-\infty,\infty))$, $C\left(K\right)=\left(C(K),d_{C(K)}\right)$ will be the complete metric space of all continuous functions from the compact set $K\subset\mathbb{R}^2$ into $\mathbb{R}$ with the usual uniform metric $d_{C(K)}$ induced by the max-norm $\|\cdot\|_{C(K)}$ and $C\left(K;\mathbb{R}^2\right)=\{g=(g_1,g_2):g_k\in C\left(K\right)\ \forall k=1,2\}$.

Consider Cauchy problem for the characteristics of (2.11) – (2.14): $\dfrac{d\xi}{d\eta}=\lambda_k(z_1,z_2),\xi(z_2)=z_1$ for any fixed $(z_1,z_2)\in P$ (cf. [5], [6]), where $\lambda_1(z_1,z_2)=1$ and $\lambda_2(z_1,z_2)=-1$ for all $z=(z_1,z_2)\in P$, which guarantee uniqueness to the left from z_2 of the solution $\xi=\varphi_k(\eta;z_1,z_2)$. Then we obtain two families $(c^k)=\{c^k_z:z=(z_1,z_2)\in P\}$, corresponding to $\lambda_k=\lambda_k(z_1,z_2)\equiv(-1)^{k-1}(k=1,2)$:

$$\frac{d\xi}{d\eta}=1,\ \xi(z_2)=z_1\text{ for each }(z_1,z_2)\in P\ \Rightarrow\varphi_1(\eta;z_1,z_2)=\eta+z_1-z_2,\eta_1\le\eta\le z_2\ (\eta_1\ge0)\,; \tag{3.1}$$

$$\frac{d\xi}{d\eta}=-1,\ \xi(z_2)=z_1\text{ for each }(z_1,z_2)\in P\ \Rightarrow\varphi_2(\eta;z_1,z_2)=-\eta+z_1+z_2,\eta_2\le\eta\le z_2\ (\eta_2\ge0)\,. \tag{3.2}$$

Denote by $\chi_k(z_1,z_2)\,(k=1,2)$ the smallest value of η such that $(\varphi_k(\eta;z_1,z_2),\eta)\in P=[0,1]\times[0,\sigma]$, i.e. $0\le\chi_k(z_1,z_2)\le z_2$ and if $\chi_k(z_1,z_2)>0$ then $\varphi_k(\chi_k(z_1,z_2);z_1,z_2)=0$ or $\varphi_k(\chi_k(z_1,z_2);z_1,z_2)=1$.

In fact, for the solution of (3.1) we have: if $\chi_1(z_1,z_2)>0$, then $\varphi_1(\chi_1(z_1,z_2);z_1,z_2)=0$ as well as for the solution of (3.2) – if $\chi_2(z_1,z_2)>0$, then $\varphi_2(\chi_2(z_1,z_2);z_1,z_2)=1$. Consequently

$$\chi_1(z_1,z_2)=\begin{cases} z_2-z_1, \text{ if } z_2-z_1>0\\ 0, \qquad\quad \text{if } z_2-z_1\le0\end{cases};\chi_2(z_1,z_2)=\begin{cases} z_2+z_1-1, \text{ if } z_2+z_1-1>0\\ 0, \qquad\qquad\quad \text{if } z_2+z_1-1\le0\end{cases}\ ((z_1,z_2)\in P).$$

Remark 3.1. It is easy to see that $\varphi_1(0;z_1,z_2)=z_1-z_2$ and $\varphi_2(0;z_1,z_2)=z_1+z_2$.

Introduce the sets
$\Pi_{in,1} = \{(z_1,z_2) \in P : \chi_1(z_1,z_2) = 0\} = \{(z_1,z_2) \in [0,1]\times[0,\sigma] : 0 \le z_1 \le 1,\ 0 \le z_2 \le z_1\}$,
$\Pi_{0,1} = \{(z_1,z_2) \in P : \chi_1(z_1,z_2) > 0,\ \ \varphi_1(\chi_1(z_1,z_2);(z_1,z_2)) = 0\} = P\backslash\Pi_{in,1}$,
$\Pi_{in,2} = \{(z_1,z_2) \in P : \chi_2(z_1,z_2) = 0\} = \{(z_1,z_2) \in [0,1]\times[0,\sigma] : 0 \le z_1 \le 1,\ 0 \le z_2 \le 1-z_1\}$,
$\Pi_{1,2} = \{(z_1,z_2) \in P : \chi_2(z_1,z_2) > 0,\ \varphi_2(\chi_2(z_1,z_2);z_1,z_2) = 1\} = P\backslash\Pi_{in,2}$.

Prior to present problem (2.11) – (2.14) in an operator form we introduce the pair of operators $\Phi = (\Phi_1,\Phi_2)$, defined on a suitable function space $M \subset \{f = (f_1,f_2) : f_k : P \to \mathbb{R}, k = 1,2\}$, such that for any fixed $f = (f_1,f_2) \in M$ the functions $\Phi_1[f]\,,\, \Phi_2[f] : P \to \mathbb{R}$ are well defined as follows:

$$\Phi_1\,[f]\,(z_1,z_2) = \begin{cases} h_1(z_1-z_2), & (z_1,z_2) \in \Pi_{in,1} \\ (1-\alpha)\mathrm{K}(z_2-z_1) + \Phi_{1,0}\,[f]\ (z_2-z_1), & (z_1,z_2) \in \Pi_{0,1}\,; \end{cases} \tag{3.3}$$

$$\Phi_2\,[f]\,(z_1,z_2) = \begin{cases} h_2(z_1+z_2), & (z_1,z_2) \in \Pi_{in,2} \\ (f_1(1,0)+f_2(1,0))\ e^{-\beta\gamma(z_1+z_2-1)} + \Phi_{2,1}\,[f]\ (z_1+z_2-1), & (z_1,z_2) \in \Pi_{1,2}\,, \end{cases} \tag{3.4}$$

where $h_{1,2}$, K, $\Phi_{1,0}[f] = \Phi_{1,0}\,[(f_1,f_2)]$ and $\Phi_{2,1}[f] = \Phi_{2,1}\,[(f_1,f_2)]$ are introduced in the previous section, in (2.9), (2.10) , (2.15) and (2.16) respectively.

Remark 3.2. If $f = (f_1,f_2)$ is a solution of (2.11) – (2.14), it is clear that for every $z = (z_1,z_2) \in P$
$\Phi_k[f](z_1,z_2) = f_k(\varphi_k(\chi_k(z_1,z_2);z_1,z_2),\chi_k(z_1,z_2))\,(k=1,2)$, or
$\Phi_1[f](z_1,z_2) = \begin{cases} h_1(\varphi_1(0;z_1,z_2)), & (z_1,z_2) \in \Pi_{in,1} \\ f_1(0,\chi_1(z_1,z_2)), & (z_1,z_2) \in \Pi_{0,1} \end{cases}$; $\Phi_2[f](z_1,z_2) = \begin{cases} h_2(\varphi_2(0;z_1,z_2)), & (z_1,z_2) \in \Pi_{in,2} \\ f_2(1,\chi_2(z_1,z_2)), & (z_1,z_2) \in \Pi_{1,2}\,. \end{cases}$

Now then, let us integrate equations of (2.11) along characteristics from (c^1) and (c^2) respectively. For any fixed $z = (z_1,z_2) \in P\ (z_2 > 0)$ there exist two characteristics, $c^1_z \in (c^1)$ and $c^2_z \in (c^2)$:
$c^k_z = \{(\xi,\eta) \in P : \xi = \varphi_k(\eta;z_1,z_2), \chi_k(z_1,z_2) \le \eta \le z_2\}\,,\ k = 1,2\ (c^1_{(0,\sigma)} = \{(0,\sigma)\}; c^2_{(1,\sigma)} = \{(1,\sigma)\})$.

We rewrite definitions of $c^k_z(k=1,2)$, substituting η for a function of ξ: $\eta = \psi_k(\xi;z_1,z_2)$, where ξ belongs to the closed interval between z_1 and $\theta_k(z) = \theta_k(z_1,z_2)$ $(\theta_k(z) = \varphi_k(\chi_k(z);z) : (\xi,\eta) \in c^k_z \subset P)$ such that $\varphi_k(\psi_k(\xi;z);z) = \xi;\ \psi_k(\varphi_k(\eta;z);z) = \eta$. In particular:

$$\psi_1(\xi;z_1,z_2) = z_2 - z_1 + \xi,\ \theta_1(z_1,z_2) \le \xi \le z_1\,;\ \ \psi_2(\xi;z_1,z_2) = z_2 + z_1 - \xi,\ z_1 \le \xi \le \theta_2(z_1,z_2) \tag{3.5}$$

and we get the following representations for the characteristics:

$$c^1_z = \{(\xi,\eta) \in P : \eta = \psi_1(\xi;z_1,z_2),\ \theta_1(z_1,z_2) \le \xi \le z_1\}\,;$$

$$c^2_z = \{(\xi,\eta) \in P : \eta = \psi_2(\xi;z_1,z_2),\ z_1 \le \xi \le \theta_2(z_1,z_2)\}\,,$$

where

$$\theta_1(z) = \varphi_1(0;z) = z_1 - z_2,\ \forall z \in \Pi_{in,1};\ \theta_1(z) = \varphi_1(\chi_1(z);z) = 0,\ \forall z = (z_1,z_2) \in \Pi_{0,1};$$

$$\theta_2(z) = \varphi_2(0;z) = z_1 + z_2,\ \forall z \in \Pi_{in,2};\ \theta_2(z) = \varphi_2(\chi_2(z);z) = 1,\ \forall z = (z_1,z_2) \in \Pi_{1,2}.$$

Therefore, integrating the first and the second of equations in (2.11) along c^1_z and c^2_z, respectively, we reduce the initial-boundary value problem (2.11) – (2.14) to the following system:

$$\begin{cases} f_1(z_1,z_2) = \Phi_1[f](z_1,z_2) - \int\limits_{\theta_1(z_1,z_2)}^{z_1} \sin\Big[\mu \int\limits_0^{\psi_1(\xi;z_1,z_2)} (f_1(\xi,r)+f_2(\xi,r))\,dr\Big]d\xi \\ f_2(z_1,z_2) = \Phi_2[f](z_1,z_2) - \int\limits_{z_1}^{\theta_2(z_1,z_2)} \sin\Big[\mu \int\limits_0^{\psi_2(\xi;z_1,z_2)} (f_1(\xi,r)+f_2(\xi,r))\,dr\Big]d\xi \end{cases}, \tag{3.6}$$

where $(z_1,z_2) \in P$, or in an operator form:

$$f = B(f), \tag{3.7}$$

where $f = (f_1, f_2)\,, B(f) = (B_1(f), B_2(f)) = (B_1(f_1, f_2), B_2(f_1, f_2))$ and for $z = (z_1, z_2) \in P$:

$$B_k(f)(z) = \Phi_k[f](z) - G_k(f)(z)\ (k = 1, 2); \tag{3.8}$$

$$G_1(f)(z) = \int_{\theta_1(z_1,z_2)}^{z_1} F[f](\xi\,, \psi_1(\xi; z))d\xi\,;\ \ G_2(f)(z) = \int_{z_1}^{\theta_2(z_1,z_2)} F[f](\xi\,, \psi_2(\xi; z))d\xi; \tag{3.9}$$

$$F[f](\xi\,, \eta) = \sin\left[\mu \int_0^{\eta} (f_1(\xi, r) + f_2(\xi, r))\, dr\right],\ (\xi\,, \eta) \in P. \tag{3.10}$$

In particular:

$B_1(f)(z_1, z_2) = h_1(z_1 - z_2) - \int\limits_{z_1-z_2}^{z_1} \sin\Big[\mu \int\limits_0^{z_2-z_1+\xi} (f_1(\xi, r) + f_2(\xi, r))\ dr\Big]d\xi$ for $(z_1, z_2) \in \Pi_{in,1}$;

$B_1(f)(z_1, z_2) = (1 - \alpha)\mathrm{K}(z_2 - z_1) + \alpha\, f_2(0, z_2 - z_1) -$
$\quad - \int\limits_0^{z_1} \sin\Big[\mu \int\limits_0^{z_2-z_1+\xi} (f_1(\xi, r) + f_2(\xi, r))\, dr\Big]d\xi$ for $(z_1, z_2) \in \Pi_{0,1}$;

$B_2(f)(z_1, z_2) = h_2(z_2 + z_1) - \int\limits_{z_1}^{z_2+z_1} \sin\Big[\mu \int\limits_0^{z_2+z_1-\xi} (f_1(\xi, r) + f_2(\xi, r))\, dr\Big]d\xi$ for $(z_1, z_2) \in \Pi_{in,2}$

and for $(z_1, z_2) \in \Pi_{1,2}$:

$B_2(f)(z_1, z_2) = (f_1(1, 0) + f_2(1, 0))\, e^{-\beta\gamma(z_2+z_1-1)} - f_1(1, z_2 + z_1 - 1)+$
$\quad +2\gamma \int\limits_0^{z_2+z_1-1} f_1(1, r)e^{-\beta\gamma(z_2+z_1-1-r)}dr - \int\limits_{z_1}^{1} \sin\Big[\mu \int\limits_0^{z_2+z_1-\xi} (f_1(\xi, r) + f_2(\xi, r))\, dr\Big]d\xi.$

Remark 3.3. Let $f = (f_1, f_2) \in C\left(P; \mathbb{R}^2\right)$ be a fixed point of the operator B (i.e. f is a continuous solution of the equation (3.7)). Then by necessity $h_1, h_2 \in C([0,\, 1])$; $\mathrm{K} \in C([0,\, \sigma])$ (where the functions h_1, h_2 and K are defined by (2.9) and (2.10), respectively) and the following conformity conditions have to be fulfilled:

$h_1(0) = f_1(0, 0) = \lim\limits_{\delta\to+0} f_1(0,\delta) = (1-\alpha)\mathrm{K}(0) + \alpha \lim\limits_{\delta\to+0} f_2(0,\delta) = (1-\alpha)\mathrm{K}(0) + \alpha h_2(0);$

$h_2(1) = f_2(1,0) = \lim\limits_{\delta\to+0} f_2(1,\delta) = h_1(1) + h_2(1) + \lim\limits_{\delta\to+0}[-f_1(1, \delta) + 2\gamma e^{-\beta\gamma\delta} \int\limits_0^{\delta} f_1(1, r)e^{\beta\gamma r}dr].$

Note that the last condition is identically satisfied for any continuous solution f of (3.7).

4. Main result

Everywhere in this section we assume that for the initial current $i_0 = i_0(x)$, voltage $u_0 = u_0(x)$ and source function $E = E(t)$ (by (1.4), (1.5)) the following Conformity Condition **(CC)** is fulfilled:

(CC) The functions i_0 and u_0 are defined and continuous everywhere on $[0, \Lambda]$, the function $E = E(t)$ is defined and continuous at each $t \in [0, T]$, such that $E(0) := u_0(0) + R_0 i_0(0)$ and $\lim\limits_{t\to+0} E(t) = E(0)$.

In terms of initial-boundary conditions (2.12), (2.13), **(CC)** is equivalent to the following conformity condition for the functions, defined by (2.9), (2.10):

$$h_1, h_2 \in C([0,\, 1]); \mathrm{K} \in C([0,\, \sigma]) \text{ and } h_1(0) = (1 - \alpha)\mathrm{K}(0) + \alpha h_2(0). \tag{4.1}$$

We define the metric $d : C(P; \mathbb{R}^2) \times C(P; \mathbb{R}^2) \to [0, \infty)$ as follows

$d(u, w) = \|u - w\| := \max\limits_{k=1,2} \|u_k - w_k\|_{C(P)} = \max\limits_{k=1,2} \left\{ \max\limits_{(\omega,\zeta)\in P} |u_k(\omega, \zeta) - w_k(\omega, \zeta)| \right\}$, for every two pairs $u = (u_1, u_2), w = (w_1, w_2) \in C(P; \mathbb{R}^2)$. Thus $(C(P; \mathbb{R}^2), d)$ becomes a complete metric space.

We introduce the set $M = \{f = (f_1, f_2) \in C\left(P; \mathbb{R}^2\right) : f(\xi, 0) = (h_1(\xi), h_2(\xi))\, \forall\xi \in [0, 1]\}$.
(M, d) is also a complete metric space, as M is a closed subset of $C\left(P; \mathbb{R}^2\right)$.

Lemma 4.1. *The operator B, defined by* (3.7) – (3.10)*, maps the set M into itself, whenever the conformity condition* (4.1) *is fulfilled.*

Proof. Let $f = (f_1, f_2) \in C\left(P; \mathbb{R}^2\right)$ be a fixed pair of continuous functions.

Then the function $f_1 + f_2$ belongs to $C(P)$ (as a sum of continuous functions).

Therefore for any two points $(\xi, \eta), (\bar{\xi}, \bar{\eta}) \in P$ the following inequalities are fulfilled:

$$\left|F[f](\xi\,,\eta) - F[f](\bar{\xi}\,,\bar{\eta})\right| \le 2\left|\sin\left[\frac{\mu}{2}\int\limits_0^{\eta}(f_1(\xi,r)+f_2(\xi,r))\,dr - \frac{\mu}{2}\int\limits_0^{\bar{\eta}}\left(f_1(\bar{\xi},r)+f_2(\bar{\xi},r)\right)dr\right]\right| \le$$

$$\le \mu\left|\int\limits_0^{\eta}(f_1(\xi,r)+f_2(\xi,r))\,dr - \int\limits_0^{\bar{\eta}}\left(f_1(\bar{\xi},r)+f_2(\bar{\xi},r)\right)dr\right| \le$$

$$\le \mu\left[\max_{0\le r\le\sigma}\left|(f_1+f_2)(\xi,r) - (f_1+f_2)(\bar{\xi},r)\right|.\min\{\eta,\bar{\eta}\} + |\eta-\bar{\eta}|\max_{(s,r)\in P}|(f_1+f_2)(s,r)|\right].$$

It follows from the uniform continuity of $f_1 + f_2$ on the compact set P that $F[f] \in C(P)$.

Let $k \in \{1, 2\}$ be fixed. The function $\widetilde{\psi}_k : \mathbb{R}^3 \to \mathbb{R} :\ \widetilde{\psi}_k(\xi, z_1, z_2) = z_2 - (-1)^{k-1}(z_1 - \xi)$ is a smooth function such that $\dfrac{\partial\widetilde{\psi}_k}{\partial\xi} \equiv (-1)^{k-1}$, $\dfrac{\partial\widetilde{\psi}_k}{\partial z_1} \equiv (-1)^k$, $\dfrac{\partial\widetilde{\psi}_k}{\partial z_2} \equiv 1$, $\forall(\xi, z_1, z_2) \in \mathbb{R}^3$ and its restrictions to the sets of characteristics are at least continuous functions.

Thus, as a consequence of the continuity of integrands in (3.9) it follows, at first, the continuity of $G_{\mathrm{k}}(f)(z)$ at each point $z \in \Pi_{in,\,k}$ or $z \in P\backslash\Pi_{in,\,k}$. Then, by the continuity of θ_k, we get $G_k(f) \in C(P)$.

The continuity on the set $\Pi_{in,\,k}$ of $\varphi_k(0;\cdot)$ together with $h_k \in C([0,\,1])$ implies continuity of $\Phi_k[f](z)$ at each point $z \in \Pi_{in,\,k}$.

On the other hand, by the continuity of $\varphi_k(\chi_k(z); z)$ and the requirements for h_1,$h_2 \in C([0,\,1])$, $\mathrm{K} \in C([0,\,\sigma])$, $f_1, f_2 \in C\,(P)$, it follows that $\Phi_k[f](z)$ is continuous at every point $z \in P\backslash\Pi_{in,\,k}$.

Finally, if $f \in M$ and z belongs to the common boundary of the sets $\Pi_{in,\,k}$ and $P\backslash\Pi_{in,\,k}$, then by the conformity condition (4.1) we obtain as follows:

for $k = 1$: $\displaystyle\lim_{\bar{z}\overset{\bar{z}\in\Pi_{0,1}}{\longrightarrow} z}\Phi_1[f](\bar{z}) = (1-\alpha)\mathrm{K}(0) + \alpha\lim_{\delta\to 0} f_2(0,\delta) = (1-\alpha)\mathrm{K}(0) + \alpha h_2(0) = h_1(0) = \Phi_1[f](z);$

for $k = 2$: $\displaystyle\lim_{\bar{z}\overset{\bar{z}\in\Pi_{1,2}}{\longrightarrow} z}\Phi_2[f](\bar{z}) = h_1(1) + h_2(1) - \lim_{\delta\to+0} f_1(1,\delta) = h_2(1) = \Phi_2[f](z).$

Hence $\Phi_k[f] \in C(P)$, which implies $B_k(f) = \Phi_k[f] - G_k(f) \in C(P)$.

Moreover, $G_k\,(f)\,(\xi, 0) = 0$, consequently, $B_k\,(f)\,(\xi, 0) = \Phi_k[f](\xi, 0) = h_k(\xi)\,\forall\xi \in [0, 1]$ $(k = 1, 2)$. Thus we have proved that $B(f) \in M$, $\forall f \in M$.

Our next aim in this section is to define a suitable sequence of successive approximations, which is convergent in (M, d), for to find a continuous solution of the operator equation (3.7).

Let $f^0 = \left(f^0_1, f^0_2\right) \in C\left(P; \mathbb{R}^2\right)$ be a fixed pair of continuous functions such that $f^0_1(\xi, 0) = h_1(\xi)\,;\ f^0_2(\xi, 0) = h_2(\xi)\,\forall\xi \in [0, 1]$ and $\max\limits_{(\xi,\eta)\in P}\left|f^0_k(\xi,\eta)\right| \le \max\limits_{0\le\xi\le1}|h_k(\xi)|$ $(k = 1, 2)$ (for example, we can choose as an initial approximation the pair of continuous functions $f^0 = \left(f^0_1, f^0_2\right) \in M$ for which: $f^0_1(\xi,\eta) = h_1(\xi)\,;\ f^0_2(\xi,\eta) = h_2(\xi)\ \forall(\xi,\eta) \in P = [0,1]\times[0,\sigma]$).

Define the sequences $\{g^n = (g^n_1, g^n_2)\}^{\infty}_{n=0}, \{f^{n+1} = (f^{n+1}_1, f^{n+1}_2)\}^{\infty}_{n=0}$ as follows:

$$\begin{array}{l} f^{n+1}_1(z) = B_1(f^n)(z) = g^n_1(z) - G_1(f^n)(z); \\ f^{n+1}_2(z) = B_2(f^n)(z) = g^n_2(z) - G_2(f^n)(z) \end{array}\ (z \in P), \tag{4.2}$$

where:

$G_1(f^n)(z) = \displaystyle\int\limits_{\theta_1(z_1,z_2)}^{z_1} F[f^n](\xi,\psi_1(\xi;z))d\xi;\ G_2(f^n)(z) = \int\limits_{z_1}^{\theta_2(z_1,z_2)} F[f^n](\xi,\psi_2(\xi;z))d\xi\ (z = (z_1,z_2) \in P)$, or for every $n = 0, 1, 2, ...$

$$G_1(f^n)\,(z) = \begin{cases} \int\limits_{z_1-z_2}^{z_1} \sin\left[\mu \int\limits_0^{z_2-z_1+\xi} \left(f_1^n(\xi,r) + f_2^n(\xi,r)\right) dr\right] d\xi, \text{ if } z = (z_1,z_2) \in \Pi_{in,\,1} \\ \int\limits_0^{z_1} \sin\left[\mu \int\limits_0^{z_2-z_1+\xi} \left(f_1^n(\xi,r) + f_2^n(\xi,r)\right) dr\right] d\xi\,, \quad \text{if } z = (z_1,z_2) \in \Pi_{0,\,1}; \end{cases}$$
$$G_2(f^n)\,(z) = \begin{cases} \int\limits_{z_1}^{z_2+z_1} \sin\left[\mu \int\limits_0^{z_2+z_1-\xi} \left(f_1^n(\xi,r) + f_2^n(\xi,r)\right) dr\right] d\xi\,, \text{ if } z = (z_1,z_2) \in \Pi_{in,\,2} \\ \int\limits_{z_1}^{1} \sin\left[\mu \int\limits_0^{z_2+z_1-\xi} \left(f_1^n(\xi,r) + f_2^n(\xi,r)\right) dr\right] d\xi\,, \quad \text{if } z = (z_1,z_2) \in \Pi_{1,\,2}; \end{cases} \tag{4.3}$$

$$g_1^n(z) = \begin{cases} h_1(z_1 - z_2), & \text{if } z = (z_1,z_2) \in \Pi_{in,\,1} \\ (1-\alpha)\mathrm{K}(z_2 - z_1) + \alpha f_2^n(0, z_2 - z_1), & \text{if } z = (z_1,z_2) \in \Pi_{0,\,1}; \end{cases}$$

$$g_2^n(z) = \begin{cases} h_2(z_1 + z_2), & \text{if } z = (z_1,z_2) \in \Pi_{in,\,2} \\ (h_1(1) + h_2(1))e^{-\beta\gamma(z_2+z_1-1)} - f_1^n(1, z_2 + z_1 - 1) + \\ \quad +2\gamma \int\limits_0^{z_2+z_1-1} f_1^n(1,r)e^{-\beta\gamma(z_2+z_1-1-r)}dr, & \text{if } z = (z_1,z_2) \in \Pi_{1,\,2}. \end{cases} \tag{4.4}$$

First, we show that $g^0 \in M$ and, by induction, that $g^{n+1} \in M, f^{n+1} \in M (n = 0, 1, 2, ...)$.

Indeed, by (4.1) and $f^0 = \left(f_1^0, f_2^0\right) \in M \subset C\left(P; \mathbb{R}^2\right)$ it follows:

$(1-\alpha)\mathrm{K}(0) + \lim\limits_{\delta\to+0} f_2^0(0,\delta) = (1-\alpha)\mathrm{K}(0) + f_2^0(0,0) = (1-\alpha)\mathrm{K}(0) + h_2(0) = h_1(0) \Rightarrow g_1^0 \in C\,(P)$;

$\lim\limits_{\delta\to+0}\left[(h_1(1)+h_2(1))e^{-\beta\gamma\delta} - f_1^0(1,\delta) + 2\gamma e^{-\beta\gamma\delta}\int\limits_0^{\delta} f_1^0(1,r)e^{\beta\gamma r}dr\right] = h_2(1) \Rightarrow g_2^0 \in C\,(P)$.

Therefore $\left(g_1^0, g_2^0\right) = g^0 \in C\left(P; \mathbb{R}^2\right)$, and in view of $g^0(\xi, 0) = (h_1(\xi), h_2(\xi))\,\forall\xi \in [0, 1]$ (by definition), $g^0 \in M$, which guarantees $f^1 = B(f^0) = (g_1^0 - G_1(f^0), g_2^0 - G_2(f^0))$ belongs to M.

If, by assumption, for some natural number n we suppose $g^m \in M$, $f^m \in M$ $\forall m \le n$, then for the next number $n+1$, we have $f^{n+1} = B(f^n) \in M$, from which (and the conformity condition (4.1)) it follows the continuity of g_k^{n+1} on the common boundary of the sets $\Pi_{in,\,k}$ and $\Pi_{k-1,\,k}$ $(k = 1, 2)$:

$(1-\alpha)\mathrm{K}(0) + \lim\limits_{\delta\to+0} f_2^{n+1}(0,\delta) = (1-\alpha)\mathrm{K}(0) + f_2^{n+1}(0,0) = (1-\alpha)\mathrm{K}(0) + h_2(0) = h_1(0) \Rightarrow g_1^{n+1} \in C\,(P)$;

$\lim\limits_{\delta\to+0}\left[(h_1(1)+h_2(1))e^{-\beta\gamma\delta} - f_1^{n+1}(1,\delta) + 2\gamma e^{-\beta\gamma\delta}\int\limits_0^{\delta} f_1^{n+1}(1,r)e^{\beta\gamma r}dr\right] = h_2(1) \Rightarrow g_2^{n+1} \in C\,(P)$.

This completes the proof that $g^n \in M$, $f^{n+1} \in M$ $(\forall n = 0, 1, 2, ...)$. In particular,

$$g^n(z) = \Phi(f^n)(z) \text{ and } f^{n+1}(\xi, 0) = g^n(\xi, 0) = (h_1(\xi), h_2(\xi))\ \forall\xi \in [0, 1].$$

Next, for any fixed $\theta \in [0, \sigma]$ we denote by P_θ the set $P_\theta = \{(\omega, \zeta) \in P : 0 \le \omega \le 1; 0 \le \zeta \le \theta\}$ $(P_\sigma \equiv P)$, define (for $u = (u_1, u_2), w = (w_1, w_2) \in C(P; \mathbb{R}^2)$) the distance function

$$d_\theta(u, w) = \max_{k=1,2} \|u_k - w_k\|_{C(P_\theta)} = \max_{k=1,2}\left\{\max_{(\omega,\zeta)\in P_\theta} |u_k(\omega,\zeta) - w_k(\omega,\zeta)|\right\}$$

and introduce the functions $p_n, q_n : [0, \sigma] \to [0, \infty)$ as follows:

$$p_n(\theta) = d_\theta(f^{n+1}, f^n)\,; \qquad q_n(\theta) = d_\theta(g^{n+1}, g^n)\,(n = 0, 1, 2, ...). \tag{4.5}$$

Thus, by definitions of the sequences we get: $p_{n+1}(\theta) \le q_n(\theta) + \max\limits_{k=1,2}\left\|G_k(f^{n+1}) - G_k(f^n)\right\|_{C(P_\theta)}$.

On the other hand, $\forall k \in \{1, 2\}$ for any fixed pairs $f = (f_1, f_2), \widetilde{f} = (\widetilde{f}_1, \widetilde{f}_2) \in C(P; \mathbb{R}^2)$ and for every $z = (z_1, z_2) \in P$ it follows: $\left|G_k(f)(z) - G_k(\widetilde{f})(z)\right| =$

$$= \left|(-1)^{k-1} \int\limits_{\theta_k(z)}^{z_1} \left[\sin\left(\mu \int\limits_0^{\psi_k(\xi;z_1,z_2)} (f_1(\xi,r)+f_2(\xi,r))dr\right) - \sin\left(\mu \int\limits_0^{\psi_k(\xi;z_1,z_2)} (\widetilde{f}_1(\xi,r)+\widetilde{f}_2(\xi,r))dr\right)\right] d\xi\right| =$$

$$= \left|(-1)^{k-1} \int\limits_{\theta_k(z)}^{z_1} 2\sin\left[\frac{\mu}{2}\int\limits_0^{\psi_k(\xi;z_1,z_2)} [(f_1+f_2)-(\widetilde{f}_1+\widetilde{f}_2)](\xi,r)dr\right] \cos\left[\frac{\mu}{2}\int\limits_0^{\psi_k(\xi;z_1,z_2)} [(f_1+f_2)+(\widetilde{f}_1+\widetilde{f}_2)](\xi,r)dr\right] d\xi\right|.$$

Consequently,

$\left|G_1(f)(z) - G_1(\widetilde{f})(z)\right| \le 2\,(z_1 - \theta_1(z_1,z_2)) \le 2z_1 \le 2;$

$\left|G_2(f)(z) - G_2(\widetilde{f})(z)\right| \le 2\,(\theta_2(z_1,z_2) - z_1) \le 2(1-z_1) \le 2.$

Moreover,

$$\left|G_k(f)(z) - G_k(\widetilde{f})(z)\right| \le (-1)^{k-1} \int\limits_{\theta_k(z)}^{z_1} \left|\mu \int\limits_0^{\psi_k(\xi;z_1,z_2)} [(f_1+f_2)-(\widetilde{f}_1+\widetilde{f}_2)](\xi,r)dr\right| d\xi \le$$

$$\le \mu.(-1)^{k-1} \int\limits_{\theta_k(z)}^{z_1} \int\limits_0^{\psi_k(\xi;z_1,z_2)} \max_{\substack{0\le\xi\le1 \\ 0\le\eta\le r}} \left[|f_1(\xi,r)-\widetilde{f}_1(\xi,r)| + |f_2(\xi,r)-\widetilde{f}_2(\xi,r)|\right] dr d\xi \le$$

$$\le 2\mu \int\limits_0^{z_2} \max_{l=1,2} \left\|f_l - \widetilde{f}_l\right\|_{C(P_r)} dr\,.\,(-1)^{k-1} \int\limits_{\theta_k(z)}^{z_1} d\xi = 2\mu\,.\,|z_1 - \theta_k(z)|\,.\int\limits_0^{z_2} d_r(f-\widetilde{f})dr.$$ Consequently,

$$\begin{aligned} \left|G_1(f)(z) - G_1(\widetilde{f})(z)\right| &\le 2\mu z_1 \int\limits_0^{z_2} d_r(f-\widetilde{f})dr \le 2\mu \int\limits_0^{z_2} d_r(f-\widetilde{f})dr \\ \left|G_2(f)(z) - G_2(\widetilde{f})(z)\right| &\le 2\mu(1-z_1) \int\limits_0^{z_2} d_r(f-\widetilde{f})dr \le 2\mu \int\limits_0^{z_2} d_r(f-\widetilde{f})dr \end{aligned} \quad \forall z=(z_1,z_2)\in P. \tag{4.6}$$

Thus we obtain: $\max\limits_{k=1,2} \left\|G_k(f^{n+1}) - G_k(f^n)\right\|_{C(P_\theta)} \le 2\mu \int\limits_0^\theta p_n(\tau)d\tau$, which implies the following inequalities are valid for every $n = 0,1,2,...$:

$$p_{n+1}(\theta) \le q_n(\theta) + 2\mu \int\limits_0^\theta p_n(\tau)d\tau \;\; \forall\theta\in[0,\sigma]. \tag{4.7}$$

We will prove the following statements:

Proposition 4.2. *Let $u=(u_1,u_2)$, $w=(w_1,w_2)\in M$ be two arbitrary chosen pairs. Then the following a priory estimates are valid, whenever the conformity condition (4.1) is fulfilled:*
if $\sigma > 1$, then:

$$\begin{aligned} &d_\theta(\Phi(B(u)),\Phi(B(w))) \le 2\mu \int\limits_0^\theta d_\tau(u,w)d\tau + 2\gamma \int\limits_0^\theta d_\tau(B(u),B(w))d\tau\,,\ \forall\theta\in[0,1] \;\; \textit{and} \\ &d_\theta(\Phi(B(u)),\Phi(B(w))) \le |\alpha|\, d_{\theta-1}(u,w) + 2(\gamma+\mu) \int\limits_0^\theta d_\tau(u,w)d\tau + \\ &\qquad\qquad + 2\gamma \int\limits_0^\theta d_\tau(B(u),B(w))d\tau\,,\ \ \forall\theta\in(1,\sigma]; \end{aligned} \tag{4.8}$$

if $\sigma \le 1$, then:

$$d_\theta(\Phi(B(u)),\Phi(B(w))) \le 2\mu \int\limits_0^\theta d_\tau(u,w)d\tau + 2\gamma \int\limits_0^\theta d_\tau(B(u),B(w))d\tau,\ \forall\theta\in[0,\sigma], \tag{4.9}$$

where $\Phi=(\Phi_1,\Phi_2): M\to M$ and $B=(B_1,B_2): M\to M$ are the pairs of mappings, defined in Section 3.

Remark 4.3. Recall that the real constants above were introduced in Section 2. In particular, by definition: $|\alpha| = \left|\dfrac{R_0 - Z_0}{R_0 + Z_0}\right| \le 1$; $\sigma = \dfrac{Tv}{\Lambda}$, that is $\sigma \le 1 \Leftrightarrow Tv \le \Lambda$.

Proof of Proposition 4.2. Let $\sigma > 1$. Let θ be any fixed point, belonging to the interval $[0, \sigma]$. Then there exist, by continuity, an index $l \in \{1, 2\}$ and a point $(\omega_0, \zeta_0) \in P_\theta$, such that:

$$\max_{k=1,2}\left\{\max_{(\omega,\zeta)\in P_\theta} |\Phi_k[B(u)](\omega,\zeta) - \Phi_k[B(w)](\omega,\zeta)|\right\} = |\Phi_l[B(u)](\omega_0,\zeta_0) - \Phi_l[B(w)](\omega_0,\zeta_0)|. \tag{4.10}$$

Without restrictions, we assume $(\omega_0, \zeta_0) \in P\backslash\Pi_{in,\,l}$ (if, on the contrary, $(\omega_0, \zeta_0) \in \Pi_{in,\,l}$, then the right-hand side of (4.10) is equal to zero). Let us consider both possible cases: $l = 1$ or $l = 2$:

Case 1: $l = 1$. In this case $(\omega_0, \zeta_0) \in P_\theta \cap \Pi_{0,\,1} \Rightarrow 0 \le \omega_0 \le 1,\ 0 \le \zeta_0 \le \theta$; $0 < \zeta_0 - \omega_0 \le \theta - \omega_0$ and the right-hand side of (4.10) becomes

$|\Phi_1[B(u)](\omega_0,\zeta_0) - \Phi_1[B(w)](\omega_0,\zeta_0)| = |\alpha|\,|B_2(u)(0,\zeta_0-\omega_0) - B_2(w)(0,\zeta_0-\omega_0)| =$
$= |\alpha|\,|\{\Phi_2[u](0,\zeta_0-\omega_0) - \Phi_2[w](0,\zeta_0-\omega_0)\} - \{G_2(u)(0,\zeta_0-\omega_0) - G_2(w)(0,\zeta_0-\omega_0)\}| \le$
$\le |\alpha|\,|\Phi_2[u](0,\zeta_0-\omega_0) - \Phi_2[w](0,\zeta_0-\omega_0)| + 2\,|\alpha|\,\mu \int\limits_0^{\zeta_0-\omega_0} d_r(u,w)dr$ (in view of (4.6)).

If $\zeta_0 - \omega_0 \le 1$ (which, in particular, is ever fulfilled whenever $\theta \in [0, 1]$), then the first addend in the last sum vanishes, since $(0, \zeta_0 - \omega_0) \in \Pi_{in,\,2}$, and the second one does not be greater than

$$2\,|\alpha|\,\mu \int\limits_0^\theta d_\tau(u,w)d\tau \le 2\mu \int\limits_0^\theta d_\tau(u,w)d\tau.$$

Let the inequalities $1 < \zeta_0 - \omega_0 \le \theta - \omega_0$ be fulfilled (for $\theta \in (1, \sigma]$, by necessity).

Consequently the point $(0, \zeta_0 - \omega_0)$ belongs to $\Pi_{1,\,2}$, which implies:

$\Phi_2[u](0,\zeta_0-\omega_0) - \Phi_2[w](0,\zeta_0-\omega_0) = [(u_1(1,0) + u_2(1,0)) - (w_1(1,0) + w_2(1,0))]e^{-\beta\gamma(\zeta_0-\omega_0-1)} +$
$+[w_1(1,\zeta_0-\omega_0-1) - u_1(1,\zeta_0-\omega_0-1)] + 2\gamma \int\limits_0^{\zeta_0-\omega_0-1} (u_1(1,r) - w_1(1,r))\, e^{-\beta\gamma(\zeta_0-\omega_0-1-r)}dr =$
$= [w_1(1,\zeta_0-\omega_0-1) - u_1(1,\zeta_0-\omega_0-1)] + 2\gamma \int\limits_0^{\zeta_0-\omega_0-1} (u_1(1,r) - w_1(1,r))\, e^{-\beta\gamma(\zeta_0-\omega_0-1-r)}dr,$

since $u,\ w \in M$.

Therefore $d_\theta(\Phi(B(u)), \Phi(B(w))) = |\Phi_1[B(u)](\omega_0,\zeta_0) - \Phi_1[B(w)](\omega_0,\zeta_0)| \le$
$\le |\alpha| \max\{|u_1(\omega,\zeta) - w_1(\omega,\zeta)| : 0 \le \omega \le 1, 0 \le \zeta \le \zeta_0 - \omega_0 - 1\} +$
$+2\,|\alpha|\,\gamma \int\limits_0^{\zeta_0-\omega_0-1} |u_1(1,r) - w_1(1,r)|dr + 2\,|\alpha|\,\mu \int\limits_0^{\zeta_0-\omega_0} d_r(u,w)dr \le$
$\le |\alpha|\, d_{\zeta_0-1}(u,w) + 2\,|\alpha|\,(\gamma+\mu) \int\limits_0^{\zeta_0} d_r(u,w)dr \le |\alpha|\, d_{\theta-1}(u,w) + 2\,|\alpha|\,(\gamma+\mu) \int\limits_0^{\theta} d_\tau(u,w)d\tau.$

In view of $\gamma > 0$; $|\alpha|\,(\gamma + \mu) \le \gamma + \mu$, we have just obtained the estimates (4.8), which completes the proof of Proposition 4.2 in Case 1.

Case 2: $l = 2$. In this case $(\omega_0, \zeta_0) \in P_\theta \cap \Pi_{1,\,2} \Rightarrow 0 \le \omega_0 \le 1,\ 0 \le \zeta_0 \le \theta$; $1 < \zeta_0 + \omega_0 \le \theta + \omega_0$ and we obtain as follows:

$d_\theta(\Phi(B(u)), \Phi(B(w))) = |\Phi_2[B(u)](\omega_0,\zeta_0) - \Phi_2[B(w)](\omega_0,\zeta_0)| =$
$= |(B_1(w)(1,\zeta_0+\omega_0-1) - B_1(u)(1,\zeta_0+\omega_0-1)) + 2\gamma \int\limits_0^{\zeta_0+\omega_0-1} (B_1(u)(1,r) - B_1(w)(1,r))\, e^{-\beta\gamma(\zeta_0+\omega_0-1-r)}dr| \le$
$\le |\Phi_1[u](1,\zeta_0+\omega_0-1) - \Phi_1[w](1,\zeta_0+\omega_0-1)| + |G_1(u)(1,\zeta_0+\omega_0-1) - G_1(w)(1,\zeta_0+\omega_0-1)| +$
$+2\gamma \int\limits_0^{\zeta_0+\omega_0-1} |B_1(u)(1,r) - B_1(w)(1,r)|\, dr \le |\Phi_1[u](1,\zeta_0+\omega_0-1) - \Phi_1[w](1,\zeta_0+\omega_0-1)| +$
$+2\mu \int\limits_0^{\zeta_0+\omega_0-1} |u(1,r) - w(1,r)|\, dr + 2\gamma \int\limits_0^{\zeta_0+\omega_0-1} |B_1(u)(1,r) - B_1(w)(1,r)|\, dr.$

If $\zeta_0+\omega_0-1\le 1$ (in particular, such is the situation provided $\theta\in[0,1]$), then $\Phi_1[u](1,\zeta_0+\omega_0-1)-\Phi_1[w](1,\zeta_0+\omega_0-1)=h_1(2-\zeta_0-\omega_0)-h_1(2-\zeta_0-\omega_0)\equiv 0$.

Consequently, $|\Phi_2[B(u)](\omega_0,\zeta_0)-\Phi_2[B(w)](\omega_0,\zeta_0)|\le$

$$\le 2\mu\int_0^{\zeta_0+\omega_0-1}|u(1,r)-w(1,r)|\,dr+2\gamma\int_0^{\zeta_0+\omega_0-1}|B_1(u)(1,r)-B_1(w)(1,r)|\,dr\le$$

$$\le 2\mu\int_0^{\zeta_0}d_r(u,w)dr+2\gamma\int_0^{\zeta_0}d_r(B(u),B(w))dr\le 2\mu\int_0^{\theta}d_\tau(u,w)d\tau+2\gamma\int_0^{\theta}d_\tau(B(u),B(w))d\tau.$$

If $\zeta_0+\omega_0-1>1$, then $0<\zeta_0+\omega_0-1-1\le\zeta_0-1\le\theta-1$ (wherever, by necessity $1<\theta\le\sigma$) and $|\Phi_1[u](1,\zeta_0+\omega_0-1)-\Phi_1[w](1,\zeta_0+\omega_0-1)|=|\alpha|\,|u_2(0,\zeta_0+\omega_0-1-1)-w_2(0,\zeta_0+\omega_0-1-1)|\le$ $\le|\alpha|\max\{|u_2(\omega,\zeta)-w_2(\omega,\zeta)|:0\le\omega\le 1,0\le\zeta\le\zeta_0-1\}$.

Therefore,

$$|\Phi_2[B(u)](\omega_0,\zeta_0)-\Phi_2[B(w)](\omega_0,\zeta_0)|\le|\alpha|\,d_{\zeta_0-1}(u,w)+2\mu\int_0^{\zeta_0}d_r(u,w)dr+2\gamma\int_0^{\zeta_0}d_r(B(u),B(w))dr\le$$

$$\le|\alpha|\,d_{\theta-1}(u,w)+2\mu\int_0^{\theta}d_\tau(u,w)d\tau+2\gamma\int_0^{\theta}d_\tau(B(u),B(w))d\tau.$$

In view of $0<\mu<\gamma+\mu$, it follows (4.8), which completes the proof of Proposition 4.2 in Case 2.

Finally, if $\sigma\le 1$, then, by repeating the proof of the first of estimates from (4.8), we get (4.9), whenever $\theta\in[0,\sigma]\subset[0,1]$. The proof of Proposition 4.2 is completed.

As a consequence of Proposition 4.2 we obtain the following corollaries.

Corollary 4.4. *Under the terms and conditions of Proposition 4.2 the following inequalities hold: if $\sigma>1$, then:*

$$\begin{aligned}&d_\theta(B(B(u)),B(B(w)))\le 2\mu\int_0^{\theta}d_\tau(u,w)d\tau+2(\gamma+\mu)\int_0^{\theta}d_\tau(B(u),B(w))d\tau,\quad\forall\theta\in[0,1];\\&d_\theta(B(B(u)),B(B(w)))\le|\alpha|\,d_{\theta-1}(u,w)+\\&\qquad\qquad+2(\gamma+\mu)\int_0^{\theta}[d_\tau(u,w)+d_\tau(B(u),B(w))]\,d\tau,\qquad\forall\theta\in(1,\sigma];\end{aligned}\tag{4.11}$$

if $\sigma\le 1$, then:

$$d_\theta(B(B(u)),B(B(w)))\le 2\mu\int_0^{\theta}d_\tau(u,w)d\tau+2(\gamma+\mu)\int_0^{\theta}d_\tau(B(u),B(w))d\tau,\ \forall\theta\in[0,\sigma].\tag{4.12}$$

Corollary 4.5. *If $f=(f_1,f_2)\in C(P;\mathbb{R}^2)$ and $\widetilde{f}=(\widetilde{f}_1,\widetilde{f}_2)\in C(P;\mathbb{R}^2)$ are solutions of the equation (3.7), then $f\equiv\widetilde{f}$ (uniqueness of the fixed point).*

Corollary 4.6. *For the sequences of functions $p_n,q_n(n=0,1,2,...)$, defined by (4.5), the following estimates are valid:*

If $\sigma>1$, then for all $n=1,2,...$:

$$\begin{aligned}&q_n(\theta)\le 2\mu\int_0^{\theta}p_{n-1}(\tau)d\tau+2\gamma\int_0^{\theta}p_n(\tau)d\tau,\ \forall\theta\in[0,1]\ \text{and}\\&q_n(\theta)\le|\alpha|\,p_{n-1}(\theta-1)+2(\gamma+\mu)\int_0^{\theta}p_{n-1}(\tau)d\tau+2\gamma\int_0^{\theta}p_n(\tau)d\tau\,,\ \forall\theta\in(1,\sigma].\end{aligned}\tag{4.13}$$

If $\sigma\le 1$, then for all $n=1,2,...$:

$$q_n(\theta)\le 2\mu\int_0^{\theta}p_{n-1}(\tau)d\tau+2\gamma\int_0^{\theta}p_n(\tau)d\tau,\ \forall\theta\in[0,\sigma].\tag{4.14}$$

Proof of Corollary 4.4. The statement of this corollary is immediate from Proposition 4.2 together with (4.6) and the fact that:

$$d_\theta(B(B(u)), B(B(w))) \leq d_\theta(\Phi(B(u)), \Phi(B(w))) + \max_{k=1,2} \|G_k(B(u)) - G_k(B(w))\|_{C(P_\theta)}.$$

Proof of Corollary 4.5. Let $f = (f_1, f_2) \in C(P;\mathbb{R}^2)$ and $\widetilde{f} = (\widetilde{f}_1, \widetilde{f}_2) \in C(P;\mathbb{R}^2)$ be two fixed points of the operator B. Then $f = B(f) = B(B(f)) \in M$, $\widetilde{f} = B(\widetilde{f}) = B(B(\widetilde{f})) \in M$ and applying Corollary 4.4 to the continuous function : $\theta \to p(\theta) := d_\theta\left(f, \widetilde{f}\right)$ $(0 \leq \theta \leq \sigma)$ we obtain as follows:

If $\sigma \leq 1$, then:

$$0 \leq p(\theta) = d_\theta\left(f, \widetilde{f}\right) = d_\theta\left(B(B(f)), B(B(\widetilde{f}))\right) \leq 2\mu \int_0^\theta d_\tau(f, \widetilde{f}) d\tau + 2(\gamma+\mu) \int_0^\theta d_\tau(B(f), B(\widetilde{f})) d\tau, \forall\theta \in [0, \sigma].$$

Therefore $0 \leq p(\theta) \leq (2\gamma + 4\mu) \int_0^\theta p(\tau) d\tau$ for all $\theta \in [0, \sigma]$.

Then the classical Gronwall's inequality implies that $d_\theta\left(f, \widetilde{f}\right) = p(\theta) = 0,\ \forall\theta \in [0, \sigma]$.

Thus $\left\|f - \widetilde{f}\right\| = \max_{k=1,2} \left\|f_k - \widetilde{f}_k\right\|_{C(P)} = d_\sigma\left(f, \widetilde{f}\right) = 0 \Leftrightarrow f \equiv \widetilde{f}$.

Let $\sigma > 1$.

Then there exists a unique natural number $m \in \{1, 2, 3, ...\}$ such that $\sigma \in (m, m+1]$.

At first, it is easy to see, by induction, that $p(\tau) = d_\tau\left(f, \widetilde{f}\right) = 0$ for every $\tau \in [0, m]$.

Indeed, for $\tau \in [0, 1]$ the claim was, in fact, just proved.

By assumption that it is true for some $k \in \{2, ..., m-1\}$, we have $p(\tau) = d_\tau\left(f, \widetilde{f}\right) = 0, \forall\tau \in [0, k]$ and applying (4.11) on $[0, k+1]$ for any $\theta \in (k, k+1] \subset (1, k+1]$ it follows

$$p(\theta) = d_\theta(f, \widetilde{f}) = d_\theta(B(B(f)), B(B(\widetilde{f}))) \leq |\alpha|\, p(\theta - 1) + 4(\gamma + \mu) \int_0^\theta p(\tau) d\tau.$$

But $p(\theta - 1) = d_{\theta-1}(f, \widetilde{f}) = 0$, in view of $\theta - 1 \in (k-1, k] \subset (0, k]$.

Consequently, $p(\theta) \leq 4(\gamma+\mu) \int_0^\theta p(\tau) d\tau = 4(\gamma+\mu) \int_k^\theta p(\tau) d\tau$, this, via Gronwall's inequality, completes the induction.

Thus we proved $p(\tau) = d_\tau\left(f, \widetilde{f}\right) = 0$ for every $\tau \in [0, m]$.

Finally, (4.11) applied once again, but to the interval $(m, \sigma]$, implies that for any $\theta \in (m, \sigma]$:

$$p(\theta) = d_\theta(f, \widetilde{f}) = d_\theta(B(B(f)), B(B(\widetilde{f}))) \leq |\alpha|\, p(\theta - 1) + 4(\gamma+\mu) \int_0^\theta p(\tau) d\tau = 4(\gamma+\mu) \int_m^\theta p(\tau) d\tau,$$

since $\theta - 1 \in (m-1, \sigma-1] \subset (m-1, m]$.

Then the Gronwall's inequality applied to the interval $[0, \sigma]$ implies $p(\theta) = d_\theta\left(f, \widetilde{f}\right) = 0$ for every $\theta \in (m, \sigma]$ too.

Therefore the function $p(\theta) = d_\theta(f, \widetilde{f})$ is equal to zero everywhere onto the interval $[0, \sigma]$, that is $\|f - \widetilde{f}\| = \max_{k=1,2} \|f_k - \widetilde{f}_k\|_{C(P)} = d_\sigma(f, \widetilde{f}) = 0 \Leftrightarrow f \equiv \widetilde{f}$, which completes the proof of Corollary 4.5 and thus we have proven that the equation (3.7) has at most one solution, belonging to $C(P;\mathbb{R}^2)$.

Proof of Corollary 4.6. In order to prove that result, it is enough to recall that (by definition) $f^{n+1} = B(f^n) = B(B(f^{n-1}))$ and to apply Proposition 4.2 to the pairs of functions $u = f^n$, $w = f^{n-1}$, with the corresponding distances

$$q_n(\theta) = d_\theta(g^{n+1}, g^n) = d_\theta(\Phi(B(f^n)), \Phi(B(f^{n-1}))), \quad p_{n-1}(\theta) = d_\theta(f^n, f^{n-1}),$$

$$p_n(\theta) = d_\theta(f^{n+1}, f^n) = d_\theta(B(f^n), B(f^{n-1}))\, (n = 1, 2, ...),$$

for to complete the proof.

Thus we have almost proven the following Lemma:

Lemma 4.7. *There exists $A \in (0, \infty)$ such that for all $n = 1, 2, ...$ the following inequalities are satisfied:*
if $\sigma > 1$:

$$\begin{aligned} &p_{n+1}(\theta) \leq A \int_0^\theta (p_n(\tau) + p_{n-1}(\tau))\, d\tau, \ \forall \theta \in [0,1] \ \text{and} \\ &p_{n+1}(\theta) \leq A[p_{n-1}(\theta - 1) + \int_0^\theta (p_n(\tau) + p_{n-1}(\tau))\, d\tau]\,, \ \forall \theta \in (1, \sigma]\,; \end{aligned} \tag{4.15}$$

if $\sigma \leq 1$:

$$p_{n+1}(\theta) \leq A \int_0^\theta (p_n(\tau) + p_{n-1}(\tau))\, d\tau\,, \ \forall \theta \in [0, \sigma]. \tag{4.16}$$

Proof. In order to complete the proof of Lemma 4.7, it is only necessary to combine the Corollary 4.6 with the estimates in (4.7) above and to show that inequalities (4.15), (4.16) are satisfied, for example, with the following real constant: $A = \max\{|\alpha|,\ 2(\gamma + \mu)\} > 0$ (in view of $0 < \mu < \gamma + \mu$).

Lemma 4.7 is thus proved.

Remark 4.8. In terms of the original given physical constants, we have by definition:

$$\max\{|\alpha|,\ 2(\gamma + \mu)\} = \max\left\{\left|\frac{R_0 - Z_0}{R_0 + Z_0}\right|,\ 2\left(\frac{\Lambda C}{C_0} + \frac{\pi \Lambda^2 L j_0}{\Phi_0}\right)\right\}.$$

Now, we state our main result:

Theorem 4.9. *Let the condition* (4.1) *(equivalently,* **(CC)***) be fulfilled. Then the operator equation* (3.7) *has a unique solution $f = (f_1, f_2) \in C\left(P; \mathbb{R}^2\right)$. That solution is a fixed point of the operator $B : M \to M$, where $M = \{f \in C\left(P; \mathbb{R}^2\right) : f(\xi, 0) = (h_1(\xi), h_2(\xi))\, \forall \xi \in [0, 1]\}$, and it can be obtained as the uniform limit of the sequence of successive approximations $\{f^n\}_{n=0}^\infty$, defined by $f^{n+1} = B\left(f^n\right) \ \ (n = 0, 1, 2, ...)$ as in* (4.2), *where $f^0 = \left(f_1^0, f_2^0\right) \in M$ is a suitable chosen pair.*

Proof. We choose as an initial approximation the pair $f^0 = \left(f_1^0, f_2^0\right)$, defined as follows:
$f_1^0(\xi, \eta) = h_1(\xi)$, $f_2^0(\xi, \eta) = h_2(\xi)\ \forall (\xi, \eta) \in P = [0, 1] \times [0, \sigma]$.

The functions f_1^0 and f_2^0 are continuous on the rectangle P, since, by the conformity condition (4.1), $h_1, h_2 \in C([0, 1])$. Therefore $f^0 \in M$ and for every $n = 0, 1, 2, ...$ the function $f^{n+1} = B\left(f^n\right)$ belongs to M, as we have already shown above.

Let a_0, a_1 be such positive numbers that: $p_0(\theta) \leq a_0,\ p_1(\theta) \leq a_1\, \forall \theta \in [0, \sigma]$
(such constants exist by the continuity of $f_0,\ f_1, f_2$ on the rectangle P).

Define the sequence of real numbers $\{a_n\}_{n=0}^\infty : a_{n+1} = \dfrac{a_n + a_{n-1}}{3} + Ae^{-3A}a_{n-1},\ n = 1, 2, ...$, where $A > 0$ is the constant from Lemma 4.7 ($A = \max\{|\alpha|,\ 2(\gamma + \mu)\} > 0$).

We will prove, by induction, the following estimates are valid for $n = 0, 1, 2, ...$:

$$p_n(\theta) \leq a_n e^{3A\theta}\,,\ \forall \theta \in [0, \sigma]; \qquad a_n \leq a^* q^{\left\lfloor \frac{n}{2} \right\rfloor}, \tag{4.17}$$

where $a^* = \max\{a_0, a_1\}$, $\left\lfloor \frac{n}{2} \right\rfloor = \begin{cases} k, n = 2k \\ k, n = 2k+1 \end{cases}$ $(k = 0, 1, 2, ...)$ is the integer part of $\frac{n}{2}$ and
$q = \frac{2}{3} + Ae^{-3A} : \frac{2}{3} < q = \frac{2}{3} + Ae^{-3A} \leq \frac{2}{3} + \frac{1}{3} \cdot \max\limits_{s \geq 0}\{se^{-s}\} = \frac{2}{3} + \frac{1}{3} \cdot e^{-1} = \frac{2 + e^{-1}}{3} < \frac{4}{5} < 1$.

Indeed, (4.17) are fulfilled by definition for $n = 0,\ 1$ ($e^{3A\theta} \geq 1,\ \forall \theta \in [0, \sigma]$). Next, if for a fixed $n \geq 2$ we suppose that the inequalities (4.17) are satisfied for every natural $m \leq n$, then for $m = n + 1$, by Lemma 4.7 and the inductive assumption, it follows (in the more general case: $\sigma > 1$ and $\theta \in (1, \sigma]$):

$$p_{n+1}(\theta) \leq A\Big[p_{n-1}(\theta - 1) + \int_0^\theta \Big(p_n(\tau) + p_{n-1}(\tau)\Big)d\tau\Big] \leq Aa_{n-1}e^{3A(\theta - 1)} + \tfrac{a_n + a_{n-1}}{3} \int_0^\theta e^{3A\tau} d(3A\tau) =$$

$$= e^{3A\theta}\left[Ae^{-3A}a_{n-1} + \tfrac{a_n + a_{n-1}}{3}(1 - e^{-3A\theta})\right] \leq e^{3A\theta}\left[Ae^{-3A}a_{n-1} + \tfrac{a_n + a_{n-1}}{3}\right] = e^{3A\theta}a_{n+1};$$

$$a_{n+1} = \tfrac{a_n + a_{n-1}}{3} + Ae^{-3A}a_{n-1} \leq a^* q^{\left\lfloor \frac{n-1}{2} \right\rfloor}\left[\tfrac{1}{3}\left(q^{\left\lfloor \frac{n}{2} \right\rfloor - \left\lfloor \frac{n-1}{2} \right\rfloor} + 1\right) + Ae^{-3A}\right] \leq a^* q^{\left\lfloor \frac{n-1}{2} \right\rfloor + 1} = a^* q^{\left\lfloor \frac{n+1}{2} \right\rfloor},$$

which completes the proof of (4.17).

Thus we are ready to prove the convergence of the series $\sum_{n=0}^{\infty} \left\| f^{n+1} - f^n \right\|$, where
$\left\| f^{n+1} - f^n \right\| = d(f^{n+1}, f^n) = d_\sigma(f^{n+1}, f^n) = p_n(\sigma) = \max_{0 \le \theta \le \sigma} p_n(\theta)$ $(n = 0, 1, 2, ...)$.

Appling (4.17), we first get that there exists the sum of the series $\sum_{n=0}^{\infty} a_n$:

$$\sum_{n=0}^{\infty} a_n \le a^* \sum_{n=0}^{\infty} q^{\left\lfloor \frac{n}{2} \right\rfloor} = a^* \cdot \frac{2}{1-q} = \frac{6a^*}{1 - 3Ae^{-3A}} \le \frac{6a^*}{1 - e^{-1}} < 10a^* \qquad \left(\frac{2}{3} < q < \frac{4}{5} < 1 \right).$$

Next, the inequalities $e^{-3A\sigma} \left\| f^{n+1} - f^n \right\| \le \max_{0 \le \theta \le \sigma} \{ e^{-3A\theta} p_n(\theta) \} \le a_n$ $(\forall n = 0, 1, 2, ...)$ imply:

$$e^{-3A\sigma} \sum_{n=0}^{\infty} \left\| f^{n+1} - f^n \right\| = e^{-3A\sigma} \sum_{n=0}^{\infty} p_n(\sigma) \le \sum_{n=0}^{\infty} \max_{0 \le \theta \le \sigma} \{ e^{-3A\theta} p_n(\theta) \} \le \sum_{n=0}^{\infty} a_n;$$

$$\sum_{n=0}^{\infty} \left\| f^{n+1} - f^n \right\| = \sum_{n=0}^{\infty} p_n(\sigma) \le e^{3A\sigma} \sum_{n=0}^{\infty} a_n \le \frac{6a^* e^{3A\sigma}}{1 - 3Ae^{-3A}}.$$

Therefore the sequence $\{f^n\}_{n=0}^{\infty}$ converges uniformly on P to some pair $f \in C(P; \mathbb{R}^2)$ and, in view of Corollary 4.6, there exists $g \in C(P; \mathbb{R}^2)$ such that $\{g^n\}_{n=0}^{\infty}$ converges uniformly on P to g.

Moreover, for any fixed $\xi \in [0, 1]$: $\lim_{n \to \infty} f^{n+1}(\xi, 0) = \lim_{n \to \infty} g^n(\xi, 0) = (h_1(\xi), h_2(\xi))$, consequently, $f(\xi, 0) = (h_1(\xi), h_2(\xi)) = g(\xi, 0)$ $\forall \xi \in [0, 1]$, that is, $f, g \in M$.

Finally, by taking the limits in (4.2) – (4.4) above, we conclude that the uniform limit $f \in M$ is a fixed point of the operator B, that is, f is a continuous solution of the equation (3.7), which completes the proof of the "existence" part of Theorem 4.9. The "uniqueness" part has already been proven (in Corollary 4.5).

Theorem 4.9 is thus proved.

Conclusion.

1) We have the following estimates:

$$\left\| f^1 - f^0 \right\| \le a_0 \le a^*; \ \left\| f^2 - f^0 \right\| \le \left\| f^2 - f^1 \right\| + \left\| f^1 - f^0 \right\| \le a_1 + a_0 \le 2a^*;$$

$$\left\| f^{m+1} - f^0 \right\| = \left\| \sum_{k=0}^{m} \left(f^{k+1} - f^k \right) \right\| \le e^{3A\sigma} a^* \sum_{k=0}^{m} q^{\left\lfloor \frac{k}{2} \right\rfloor} \le e^{3A\sigma} a^* \cdot \frac{2}{1-q} \left(1 - q^{\left\lfloor \frac{m}{2} \right\rfloor + 1} \right) \ (\forall m = 2, 3, ...);$$

$$\left\| f^{m+1} - f^n \right\| = \left\| \sum_{k=n}^{m} \left(f^{k+1} - f^k \right) \right\| \le \sum_{k=n}^{m} p_k(\sigma) \le e^{3A\sigma} a^* \sum_{k=n}^{m} q^{\left\lfloor \frac{k}{2} \right\rfloor} \le e^{3A\sigma} a^* q^{\left\lfloor \frac{n}{2} \right\rfloor} \cdot \frac{2}{1-q},$$

for any fixed natural n and for every $m \in \{n, n+1, n+2, ...\}$.

So, by taking the limit as $m \to \infty$ we obtain as follows:

$$\left\| f - f^n \right\| \le \frac{2a^* e^{3A\sigma}}{1-q} q^{\left\lfloor \frac{n}{2} \right\rfloor} = \frac{6a^* e^{3A\sigma}}{1 - 3Ae^{-3A}} \cdot \left(\frac{2}{3} + Ae^{-3A} \right)^{\left\lfloor \frac{n}{2} \right\rfloor} \ (n = 0, 1, 2, ...).$$

2) We can choose a_0, a_1 as follows: $a_0 := 4 \left\| \mathrm{K} \right\|_{C([0,\sigma])} + 3 \max_{k=1,2} \left\| h_k \right\|_{C([0,1])} + 1$, $a_1 := 3a_0 + 2$.

Indeed, we have as follows:

$$\left\| f^0 \right\| = \max\{ \| f_1^0 \|_{C(P)}, \| f_2^0 \|_{C(P)} \} = \max_{k=1,2} \| h_k \|_{C([0,1])};$$

$$\left| G_1[f^0](z_1, z_2) \right| = \left| \int_{\theta_1(z_1,z_2)}^{z_1} \sin \left(\mu \int_0^{\psi_1(\xi; z_1, z_2)} (h_1(\xi) + h_1(\xi)) dr \right) d\xi \right| \le z_1 - \theta_1(z_1, z_2) \le z_1 \le 1;$$

$$\left| G_2[f^0](z_1, z_2) \right| = \left| \int_{z_1}^{\theta_2(z_1,z_2)} \sin \left(\mu \int_0^{\psi_2(\xi; z_1, z_2)} (h_1(\xi) + h_1(\xi)) dr \right) d\xi \right| \le \theta_2(z_1, z_2) - z_1 \le 1 - z_1 \le 1;$$

$\left|g_1^0(z_1,z_2)-f_1^0(z_1,z_2)\right|=|h_1(z_1-z_2)-h_1(z_1)|\ ,\forall(z_1,z_2)\in\Pi_{in,1};$

$\left|g_1^0(z_1,z_2)-f_1^0(z_1,z_2)\right|=$
$=|(1-\alpha)\,\mathrm{K}(z_2-z_1)+\alpha h_2(0)-h_1(z_1)|\le(1-\alpha)\ |\mathrm{K}(z_2-z_1)-\mathrm{K}(0)|+|h_1(0)-h_1(z_1)|\,,\forall(z_1,z_2)\in\Pi_{0,1}.$

Consequently, $\left\|f_1^1-f_1^0\right\|_{C(P)}\le 2(1-\alpha)\left\|\mathrm{K}\right\|_{C([0,\sigma])}+2\left\|h_1\right\|_{C([0,1])}+1.$

$\left|g_2^0(z_1,z_2)-f_2^0(z_1,z_2)\right|=|h_2(z_1+z_2)-h_2(z_1)|\,,\forall(z_1,z_2)\in\Pi_{in,2};$

$\left|g_2^0(z_1,z_2)-f_2^0(z_1,z_2)\right|=$
$=|(h_1(1)+h_2(1))e^{-\beta\gamma(z_2+z_1-1)}-h_1(1)+\frac{2}{\beta}h_1(1)\int\limits_0^{z_2+z_1-1}e^{-\beta\gamma(z_2+z_1-1-r)}d(\beta\gamma r)-h_2(z_1)|=$
$=\left|h_2(1)e^{-\beta\gamma(z_2+z_1-1)}-h_2(z_1)+h_1(1)\left(\frac{2}{\beta}-1\right)(1-e^{-\beta\gamma(z_2+z_1-1)})\right|\le$
$\le|h_2(1)|+|h_2(z_1)|+|h_1(1)|,\forall(z_1,z_2)\in\Pi_{1,2}$ (in view of $\left|\dfrac{2}{\beta}-1\right|=\left|\dfrac{R_1-Z_0}{R_1+Z_0}\right|\le 1$).

Therefore, $\left\|f_2^1-f_2^0\right\|_{C(P)}\le 2\left\|h_2\right\|_{C([0,1])}+\left\|h_1\right\|_{C([0,1])}+1\le 3\max\limits_{k=1,2}\left\|h_k\right\|_{C([0,1])}+1.$

Thus, for every point $\theta\in[0,\sigma]$ we obtain as follows
$p_0(\theta)=d_\theta(f^1,f^0)=\max\limits_{k=1,2}\left\|f_k^1-f_k^0\right\|_{C(P_\theta)}\le\max\limits_{k=1,2}\left\|f_k^1-f_k^0\right\|_{C(P)}\le$
$\le\max\{2(1-\alpha)\left\|\mathrm{K}\right\|_{C([0,\sigma])}+2\max\limits_{k=1,2}\left\|h_k\right\|_{C([0,1])}+1,\ 3\max\limits_{k=1,2}\left\|h_k\right\|_{C([0,1])}+1\}\le a_0$
(in view of $1-\alpha=\dfrac{2Z_0}{R_0+Z_0}\in(0,2)$).

Moreover,
$\left\|g_1^1-g_1^0\right\|_{C(P)}\le|\alpha|a_0,\ \left\|g_2^1-g_2^0\right\|_{C(P)}\le\left(1+\dfrac{2}{\beta}\right)a_0\le 3a_0$, which implies
$p_1(\theta)=d_\theta(f^2,f^1)=\max\limits_{k=1,2}\left\|f_k^2-f_k^1\right\|_{C(P_\theta)}\le\max\limits_{k=1,2}\left\|f_k^2-f_k^1\right\|_{C(P)}\le$
$\le\max\limits_{k=1,2}\left\|g_k^1-g_k^0+G_k[f^0]-G_k[f^1]\right\|_{C(P)}\le\max\limits_{k=1,2}\left\|g_k^1-g_k^0\right\|_{C(P)}+2\le a_1$, where $a_1=3a_0+2$.

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