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Research Article

Exploring the Novel Wave Structures of the Kairat-X Equation via Two Analytical Methods

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Abstract: This paper aims to investigate the Kairat-X equation in the context of the ferromagnetic materials, optical fibers, differential geometry of curves, and equivalence aspects. Two efficient techniques are used to obtain new solutions: the modified extended tanh expansion method and the $\left(\frac{G'}{G^2}\right)$-expansion function method. By applying these methods, the nonlinear ordinary differential form of the analyzed equation is obtained using the appropriate wave transform. The effective application of the proposed approaches has yielded a substantial number of analytical solutions for the model, including hyperbolic, bright-dark soliton, W-shaped soliton, and mixed-type trigonometric, rational, and trigonometric solutions. These methods are advantageous in deriving a wide variety of exact solutions; however, they can also present limitations in terms of computational complexity and the scope of applicable equations. Various graphical representations are given to enhance the understanding of the obtained solutions. To the best of our knowledge, all derived solutions are novel. Furthermore, the correctness of each solution has been verified using Maple software.

Keywords: Kairat-X equation, the modified extended tanh expansion method, the $\left(\frac{G'}{G^2}\right)$-expansion function method.

1. Introduction

Nonlinear partial differential equations (NLPDEs) are used to model complex physical phenomena in physics, mechanics, biology, chemistry, and engineering [8, 10, 26, 30]. The study of nonlinear wave phenomena has attracted significant attention in recent years, including breathing waves, rogue waves, and solitons. The derivation of soliton solutions for NLPDEs has become an extremely fascinating and active field of research for many scientists working in engineering and applied sciences. Solitons, which are widely used in science and engineering, play a crucial technological role in enabling the transmission of digital information through optical fibers. In electromagnetics,

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solitons are also studied, such as transverse electromagnetic waves between two superconducting metal strips. Optical solitons are an important area of study in nonlinear optics, covering a wide range of topics such as metasurfaces, crystals, birefringence, magneto-optics, optical fibers, and optical couplers [27]. Optical solitons, also known as soliton wave packets, are characterized by their stability over long propagation distances. High-speed data transmission over optical fibers and the operation of technologies such as all-optical switches depend on this property. For modern telecommunications to be reliable and efficient, optical solitons and their stability are crucial [31]. Researchers are increasingly recognizing the significant contributions that mathematical approaches and computational technologies make to science, especially in areas where technological advancements and real-world applications are involved [28].

In recent years, finding exact solutions to NLPDEs has become crucial. Many direct and effective approaches have been developed to help engineers and physicists better understand the mechanisms governing these physical models, as well as the associated challenges and potential applications. Several efficient methods have been proposed for determining the implicit soliton solutions of nonlinear equations, including the tanh-function method [19], the modified simplest equation approach [18], the sine-cosine method [5, 29], the complete discriminant system method [3], the Jacobi elliptic function method [12, 14, 15], the first integral method [6], the modified sub-equation method [11, 24], the modified expansion rational function method [7] and the sub-equation method [1, 22].

This paper focuses on the Kairat-X equation (K-XE), a NLPDE that emerges in contexts such as nonlinear optics, ferromagnetic media, and optical fiber systems. The K-XE is given by [13, 23]:

$$\rho_{tt} + \rho_{xxxt} - 3\left(\rho_x \rho_t\right)_x = 0, \tag{1}$$

where $\rho = \rho(x,t)$ denotes the real wave function, with the nonlinear interaction and dispersion effects represented by the terms $(\rho_x \rho_t)_x$ and ρ_{xxxt}, respectively. The K-XE was initially formulated by Myrzakulova, who studied its Lax pair representation in order to demonstrate its integrable properties [23]. Faridi et al. employed the new auxiliary equation method to derive new soliton solutions for the same equation [13]. In their study, numerous soliton solutions with diverse characteristics such as complex waves, plane waves, shock waves, and exponential wave forms were obtained. Iqbal et al. investigated the fractional form of this equation and applied the extended simple equation method to obtain various solutions in trigonometric, exponential, and rational forms [17]. Ahmad et al. utilized the unified method to derive several exact analytical solutions for the same model [2]. Samina et al. used the generalized auxiliary equation method to obtain soliton solutions of the (1) and also performed a detailed analysis of its bifurcation structure, chaotic dynamics, and sensitivity characteristics [25].

The aim of this work is to improve wave behavior through study and enhance its practical applications, particularly in the field of telecommunications [9]. It takes an interdisciplinary approach by combining physics, computer science, and mathematics, emphasizing the role of active scientific research in solving real-world problems and advancing technology.

The existence of a Lax pair implies that the model possesses infinitely many conservation laws and can admit soliton-type solutions. This feature justifies the use of powerful analytical techniques such as the modified extended tanh expansion method (METEM) and the $\left(\frac{G'}{G^2}\right)$-expansion function method, as applied in this study. By utilizing different solution prototypes for the considered model, new approaches are presented to improve data transmission rates, optimize optical systems, and advance nonlinear optics toward more reliable and efficient communication technologies.

2. Methodology

Suppose that the presence of a NLPDE of the form:

$$N(\rho, \rho_x, \rho_t, \rho_{xx}, \rho_{xt}, \rho_{tt}, \cdots) = 0, \tag{2}$$

in which $\rho = \rho(x,t)$ is an arbitrary function of x and t with its partial derivatives.

Applying the next wave transformation

$$\rho(x,t) = V(\xi), \quad \xi = (\kappa x - \eta t), \tag{3}$$

then (2) reduces to the following form:

$$O(V, V', V'', V''', \cdots) = 0. \tag{4}$$

Here, η and κ are real constants different from zero.

2.1. Basic Steps of the METEM

This section presents the fundamental steps of the METEM approach [20].

Step 1: Consider the general solution of (4) in the form:

$$V(\xi) = M_0 + \sum_{s=1}^{R} \left(M_s \Phi^s(\xi) + L_s \Phi^{-s}(\xi)\right) \quad (M_R \neq 0 \text{ or } L_R \neq 0), \tag{5}$$

where $\Phi(\xi)$ defined as follows:

$$\frac{d\Phi(\xi)}{d\xi} = \Theta + \left(\Phi(\xi)\right)^2, \tag{6}$$

in which Θ is arbitrary constant. The following expressions represent the general solutions of (6):

Case 1: When $\Theta < 0$, the corresponding **hyperbolic solutions** can be written as follows:

$$\Phi_1(\xi) = -\sqrt{-\Theta}\tanh\left(\sqrt{-\Theta}\,(\xi+\xi_0)\right), \tag{7}$$

$$\Phi_2(\xi) = -\sqrt{-\Theta}\coth\left(\sqrt{-\Theta}\,(\xi+\xi_0)\right), \tag{8}$$

$$\Phi_3(\xi) = -\sqrt{-\Theta}\left(\tanh\left(2\sqrt{-\Theta}\,(\xi+\xi_0)\right) + i\varepsilon\,\mathrm{sech}\left(2\sqrt{-\Theta}\,(\xi+\xi_0)\right)\right), \tag{9}$$

$$\Phi_4(\xi) = \frac{-\sqrt{-\Theta}\tanh\left(\sqrt{-\Theta}\,(\xi+\xi_0)\right)+\Theta}{\sqrt{-\Theta}\tanh\left(\sqrt{-\Theta}\,(\xi+\xi_0)\right)+1}, \tag{10}$$

$$\Phi_5(\xi) = \frac{\sqrt{-\Theta}\left(-4\cosh\left(2\sqrt{-\Theta}\,(\xi+\xi_0)\right)+5\right)}{4\sinh\left(2\sqrt{-\Theta}\,(\xi+\xi_0)\right)+3}, \tag{11}$$

$$\Phi_6(\xi) = \frac{\varepsilon\sqrt{-\Theta\,(c^2+d^2)} - c\sqrt{-\Theta}\cosh\left(2\sqrt{-\Theta}\,(\xi+\xi_0)\right)}{c\sinh\left(2\sqrt{-\Theta}\,(\xi+\xi_0)\right)+d}, \tag{12}$$

$$\Phi_7(\xi) = \varepsilon\sqrt{-\Theta}\left[1 - \frac{2c}{c+\cosh\left(2\sqrt{-\Theta}\,(\xi+\xi_0)\right) - \varepsilon\sinh\left(2\sqrt{-\Theta}\,(\xi+\xi_0)\right)}\right]. \tag{13}$$

Case 2: If $\Theta > 0$, the desired **trigonometric solutions** can be expressed as follows:

$$\Phi_8(\xi) = \sqrt{\Theta}\tan\left(\sqrt{\Theta}\,(\xi+\xi_0)\right), \tag{14}$$

$$\Phi_9(\xi) = -\sqrt{\Theta}\cot\left(\sqrt{\Theta}\,(\xi+\xi_0)\right), \tag{15}$$

$$\Phi_{10}(\xi) = \sqrt{\Theta}\left(\tan\left(2\sqrt{\Theta}\,(\xi+\xi_0)\right) + \varepsilon\sec\left(2\sqrt{\Theta}\,(\xi+\xi_0)\right)\right), \tag{16}$$

$$\Phi_{11}(\xi) = -\frac{\sqrt{\Theta}\left(1-\tan\left(\sqrt{\Theta}\,(\xi+\xi_0)\right)\right)}{1+\tan\left(\sqrt{\Theta}\,(\xi+\xi_0)\right)}, \tag{17}$$

$$\Phi_{12}(\xi) = \frac{\sqrt{\Theta}\left(-5\cos\left(2\sqrt{\Theta}\,(\xi+\xi_0)\right)+4\right)}{5\sin\left(2\sqrt{\Theta}\,(\xi+\xi_0)\right)+3}, \tag{18}$$

$$\Phi_{13}(\xi) = \frac{\varepsilon\sqrt{\Theta\,(c^2+d^2)} - c\sqrt{\Theta}\cos\left(2\sqrt{\Theta}\,(\xi+\xi_0)\right)}{c\sin\left(2\sqrt{\Theta}\,(\xi+\xi_0)\right)+d}, \tag{19}$$

$$\Phi_{14}(\xi) = i\varepsilon\sqrt{\Theta}\left[1 - \frac{2c}{c+\cos\left(2\sqrt{\Theta}\,(\xi+\xi_0)\right) - i\varepsilon\sin\left(2\sqrt{\Theta}\,(\xi+\xi_0)\right)}\right]. \tag{20}$$

Case 3: For $\Theta = 0$, the relevant **rational solution** can be derived as follows:

$$\Phi_{15}(\xi) = -\frac{1}{\xi + \xi_0}. \tag{21}$$

Here, $\varepsilon = \pm 1,\ c \neq 0,\ d,\ \Theta,\ \xi_0$ are real arbitrary parameters.

Step 2: By taking the homogeneous balance between the highest order derivative and the most considerable nonlinear term in (4), the value of R is obtained.

Step 3: Inserting (5) and its derivatives into (4), with respect to (6), we obtain a polynomial in terms of $V(\xi)$. By setting the coefficients of each power of $V(\xi)$ to zero, we obtain a system of equations involving the unknown parameters $\Theta,\ M_s,\ L_s\ (s = 1, 2, \ldots, R)$. By solving this system, we derive the analytical solutions of (4).

Step 4: Lastly, the application of the transformation in (3) to the solutions of (4) enables the construction of several analytical solutions for (2). Under three distinct cases, the corresponding solutions to (6) have been obtained.

2.2. Description of the $\left(\frac{G'}{G^2}\right)$ Expansion Function Method

The principal steps of the $\left(\frac{G'}{G^2}\right)$-expansion function method is specified in this subsection [21]. To solve (1), we assume a solution of the form:

$$H(\xi) = m_0 + \sum_{i=1}^{K}\left(m_i\left(\frac{G'}{G^2}\right)^{i} + n_i\left(\frac{G'}{G^2}\right)^{-i}\right) \quad (m_i \neq 0 \text{ or } n_i \neq 0), \tag{22}$$

where $G = G(\xi)$ defined as follows:

$$\left(\frac{G'}{G^2}\right)' = \tau + \varphi\left(\frac{G'}{G^2}\right)^2. \tag{23}$$

Here, the constants $\varphi \neq 0$ and $\tau \neq 1$ are assumed, and the unknown constants $m_0,\ m_i,\ n_i$ $(i = 1, 2, 3, \cdots, K)$ will be defined later. The corresponding three families of solutions to (22) are as follows:

When $\tau\varphi > 0$, we have the following **trigonometric solution**:

$$\left(\frac{G'}{G^2}\right) = \sqrt{\frac{\tau}{\varphi}}\left(\frac{A_1\cos\left(\sqrt{\tau\varphi}\xi\right) + A_2\sin\left(\sqrt{\tau\varphi}\xi\right)}{A_2\cos\left(\sqrt{\tau\varphi}\xi\right) - A_1\sin\left(\sqrt{\tau\varphi}\xi\right)}\right). \tag{24}$$

When $\tau\varphi < 0$, we derive the subsequent **hyperbolic solution**:

$$\left(\frac{G'}{G^2}\right) = -\sqrt{\frac{\tau\varphi}{\varphi}}\left(\frac{A_1\sinh\left(2\sqrt{\tau\varphi}\xi\right) + A_1\cosh\left(2\sqrt{\tau\varphi}\xi\right) + A_2}{A_1\sinh\left(2\sqrt{\tau\varphi}\xi\right) + A_1\cosh\left(2\sqrt{\tau\varphi}\xi\right) - A_2}\right). \tag{25}$$

When $\varphi \neq 0, \tau = 0$, we have the next **rational solution**:

$$\left(\frac{G'}{G^2}\right) = \left(-\frac{A_1}{\varphi\left(A_1\xi + A_2\right)}\right). \tag{26}$$

Here, A_1, A_2 are constants. Substituting (22) and (23) into (4), and equating the coefficients of like powers of $\left(\frac{G'}{G^2}\right)$ to zero, yields a system of algebraic equations solved via Maple software program.

3. Application of the Offered Methods

Consider the wave transformation given by:

$$\rho(x,t) = V(\xi), \quad \xi = \kappa x - \eta t. \tag{27}$$

Substituting (3) into (1), then we reach

$$\eta V''(\xi) - \kappa^3 V(\xi)'''' + 3\kappa^2\left(V'^2(\xi)\right)' = 0. \tag{28}$$

When integrating (28) with respect to ξ, we obtain

$$\eta V'(\xi) - \kappa^3 V(\xi)''' + 3\kappa^2\left(V'^2(\xi)\right) + c_0 = 0. \tag{29}$$

where suppose that the integration constant c_0 is zero. Assuming $V'(\xi) = S$, where $S(\xi)$ is real-valued, (1) reduces to the following ODE:

$$\eta S(\xi) - \kappa^3 S(\xi)'' + 3\kappa^2 S^2(\xi) = 0. \tag{30}$$

3.1. The Solutions to the Proposed Model Using the METEM

Applying the equilibrium principle to (30) yields $n = 2$. Therefore, the (5) turns into

$$S(\xi) = M_0 + M_1\Phi(\xi) + M_2\Phi^2(\xi) + \frac{L_1}{\Phi(\xi)} + \frac{L_2}{\Phi^2(\xi)}. \tag{31}$$

In this case, M_0, M_1, M_2, L_1 and L_2 are parameters. Adhering to the suggested method, then we reach the subsequent equation system:

$$\begin{cases} \eta M_0 - \kappa^3\left(2M_2\Theta^2 + 2L_2\right) + 3\kappa^2\left(2L_2M_2 + 2L_1M_1 + M_0^2\right) = 0, \\ \eta L_2 - 8\kappa^3 L_2\Theta + 3\kappa^2\left(L_1^2 + 2L_2M_0\right) = 0, \\ \eta L_1 - 2\kappa^3 L_1\Theta + 3\kappa^2\left(2L_1M_0 + 2L_2M_1\right) = 0, \\ 6\kappa^2 M_1M_2 - 2\kappa^3 M_1 = 0, \\ 3\kappa^2 M_2^2 - 6\kappa^3 M_2 = 0, \\ \eta M_2 - 8\kappa^3 M_2\Theta + 3\kappa^2\left(2M_0M_2 + M_1^2\right) = 0, \\ \eta M_1 - 2\kappa^3 M_1\Theta + 3\kappa^2\left(2L_1M_2 + 2M_1M_0\right) = 0, \\ -6\kappa^3 L_2\Theta^2 + 3\kappa^2 L_2^2 = 0, \\ -2\kappa^3 L_1\Theta^2 + 6\kappa^2 L_2L_1 = 0. \end{cases}$$

Solving the above system of algebraic equation, we obtain the following sets:

Set 1:

$$M_0 = -\frac{4\kappa\Theta}{3}, \quad M_1 = 0, \quad M_2 = 2\kappa, \quad L_1 = 0, \quad L_2 = 2\kappa\Theta^2, \quad \eta = 16\kappa^3\Theta. \tag{32}$$

Set 2:

$$M_0 = \frac{2\kappa\Theta}{3}, \quad M_1 = 0, \quad M_2 = 2\kappa, \quad L_1 = 0, \quad L_2 = 0, \quad \eta = 4\kappa^3\Theta. \tag{33}$$

By using **Set 1**, we get the following solutions:

Case 1: If $\Theta < 0$, then the kink type solution obtained as

$$\rho_{1,1}(x,t) = -\frac{4\kappa\Theta}{3} - 2\kappa\Theta\tanh\left(\sqrt{-\Theta}\,(\kappa x - \eta t)\right)^2 - \frac{2\kappa\Theta}{\tanh\left(\sqrt{-\Theta}\,(\kappa x - \eta t)\right)^2}. \tag{34}$$

The solitary wave solution reached as

$$\rho_{1,2}(x,t) = -\frac{4\kappa\Theta}{3} - 2\kappa\Theta\coth\left(\sqrt{-\Theta}\,(\kappa x - \eta t)\right)^2 - \frac{2\kappa\Theta}{\coth\left(\sqrt{-\Theta}\,(\kappa x - \eta t)\right)^2}. \tag{35}$$

The mixed complex bright-dark soliton solution attained as

$$\rho_{1,3}(x,t) = -\frac{4\kappa\Theta}{3} - 2\kappa\Theta\Big(\tanh\left(2\sqrt{-\Theta}\,(\kappa x - \eta t)\right) + i\,\mathrm{sech}\left(2\sqrt{-\Theta}\,(\kappa x - \eta t)\right)\Big)^2$$
$$- \frac{2\kappa\Theta}{\left(\tanh\left(2\sqrt{-\Theta}\,(\kappa x - \eta t)\right) + i\,\mathrm{sech}\left(2\sqrt{-\Theta}\,(\kappa x - \eta t)\right)\right)^2}. \tag{36}$$

The kink type solution reached as

$$\rho_{1,4}(x,t) = -\frac{4\kappa\Theta}{3} + \frac{2\kappa\left(-\Theta + \sqrt{-\Theta}\tanh\left(\sqrt{-\Theta}\,(\kappa x - \eta t)\right)\right)^2}{\left(1 + \sqrt{-\Theta}\tanh\left(\sqrt{-\Theta}\,(\kappa x - \eta t)\right)\right)^2}$$

$$+ \frac{2\kappa\Theta^2\left(1 + \sqrt{-\Theta}\tanh\left(\sqrt{-\Theta}\,(\kappa x - \eta t)\right)\right)^2}{\left(-\Theta + \sqrt{-\Theta}\tanh\left(\sqrt{-\Theta}\,(\kappa x - \eta t)\right)\right)^2}. \tag{37}$$

The solitary wave solutions obtained as

$$\rho_{1,5}(x,t) = -\frac{4\kappa\Theta}{3} - \frac{2\kappa\Theta\left(5 - 4\cosh\left(2\sqrt{-\Theta}\,(\kappa x - \eta t)\right)\right)^2}{\left(3 + 4\sinh\left(2\sqrt{-\Theta}\,(\kappa x - \eta t)\right)\right)^2}$$

$$- \frac{2\kappa\Theta\left(3 + 4\sinh\left(2\sqrt{-\Theta}\,(\kappa x - \eta t)\right)\right)^2}{\left(5 - 4\cosh\left(2\sqrt{-\Theta}\,(\kappa x - \eta t)\right)\right)^2}, \tag{38}$$

$$\rho_{1,6}(x,t) = -\frac{4\kappa\Theta}{3} + \frac{2\kappa\left(\sqrt{-\Theta\,(c^2 + d^2)} - c\sqrt{-\Theta}\cosh\left(2\sqrt{-\Theta}\,(\kappa x - \eta t)\right)\right)^2}{\left(c\sinh\left(2\sqrt{-\Theta}\,(\kappa x - \eta t)\right) + d\right)^2}$$

$$+ \frac{2\kappa\Theta^2\left(c\sinh\left(2\sqrt{-\Theta}\,(\kappa x - \eta t)\right) + d\right)^2}{\left(\sqrt{-\Theta\,(c^2 + d^2)} - c\sqrt{-\Theta}\cosh\left(2\sqrt{-\Theta}\,(\kappa x - \eta t)\right)\right)^2}, \tag{39}$$

$$\rho_{1,7}(x,t) = -\frac{4\kappa\Theta}{3} - \frac{2\kappa\Theta\left(-c + \cosh\left(2\sqrt{-\Theta}\,(\kappa x - \eta t)\right) - \sinh\left(2\sqrt{-\Theta}\,(\kappa x - \eta t)\right)\right)^2}{\left(c + \cosh\left(2\sqrt{-\Theta}\,(\kappa x - \eta t)\right) - \sinh\left(2\sqrt{-\Theta}\,(\kappa x - \eta t)\right)\right)^2}$$

$$- \frac{2\kappa\Theta\left(c + \cosh\left(2\sqrt{-\Theta}\,(\kappa x - \eta t)\right) - \sinh\left(2\sqrt{-\Theta}\,(\kappa x - \eta t)\right)\right)^2}{\left(-c + \cosh\left(2\sqrt{-\Theta}\,(\kappa x - \eta t)\right) - \sinh\left(2\sqrt{-\Theta}\,(\kappa x - \eta t)\right)\right)^2}. \tag{40}$$

Case 2: If $\Theta > 0$, then we reached the following singular periodic wave solutions:

$$\rho_{1,8}(x,t) = -\frac{4\kappa\Theta}{3} + 2\kappa\Theta\tan\left(\sqrt{\Theta}\,(\kappa x - \eta t)\right)^2 + \frac{2\kappa\Theta}{\tan\left(\sqrt{\Theta}\,(\kappa x - \eta t)\right)^2}, \tag{41}$$

$$\rho_{1,9}(x,t) = -\frac{4\kappa\Theta}{3} + 2\kappa\Theta\cot\left(\sqrt{\Theta}\,(\kappa x - \eta t)\right)^2 + \frac{2\kappa\Theta}{\cot\left(\sqrt{\Theta}\,(\kappa x - \eta t)\right)^2}. \tag{42}$$

The mixed type trigonometric soliton solutions attained as

$$\rho_{1,10}(x,t) = -\frac{4\kappa\Theta}{3} + 2\kappa\Theta\left(\tan\left(2\sqrt{\Theta}\,(\kappa x - \eta t)\right) + \sec\left(2\sqrt{\Theta}\,(\kappa x - \eta t)\right)\right)^2$$

$$+ \frac{2\kappa\Theta}{\left(\tan\left(2\sqrt{\Theta}\,(\kappa x - \eta t)\right) + \sec\left(2\sqrt{\Theta}\,(\kappa x - \eta t)\right)\right)^2}. \tag{43}$$

The explicit periodic type solution reached as

$$\rho_{1,11}(x,t) = -\frac{4\kappa\Theta}{3} + \frac{2\kappa\Theta\left(1-\tan\left(\sqrt{\Theta}\left(\kappa x-\eta t\right)\right)\right)^2}{\left(1+\tan\left(\sqrt{\Theta}\left(\kappa x-\eta t\right)\right)\right)^2} + \frac{2\kappa\Theta\left(1+\tan\left(\sqrt{\Theta}\left(\kappa x-\eta t\right)\right)\right)^2}{\left(1-\tan\left(\sqrt{\Theta}\left(\kappa x-\eta t\right)\right)\right)^2}, \tag{44}$$

$$\rho_{1,12}(x,t) = -\frac{4\kappa\Theta}{3} + \frac{2\kappa\Theta\left(4-5\cos\left(2\sqrt{\Theta}\left(\kappa x-\eta t\right)\right)\right)^2}{\left(3+5\sin\left(2\sqrt{\Theta}\left(\kappa x-\eta t\right)\right)\right)^2} + \frac{2\kappa\Theta\left(3+5\sin\left(2\sqrt{\Theta}\left(\kappa x-\eta t\right)\right)\right)^2}{\left(4-5\cos\left(2\sqrt{\Theta}\left(\kappa x-\eta t\right)\right)\right)^2}, \tag{45}$$

$$\rho_{1,13}(x,t) = -\frac{4\kappa\Theta}{3} + \frac{2\kappa\left(\sqrt{\Theta\left(c^2-d^2\right)}-c\sqrt{\Theta}\cos\left(2\sqrt{\Theta}\left(\kappa x-\eta t\right)\right)\right)^2}{\left(c\sin\left(2\sqrt{\Theta}\left(\kappa x-\eta t\right)\right)+d\right)^2} + \frac{2\kappa\Theta^2\left(c\sin\left(2\sqrt{\Theta}\left(\kappa x-\eta t\right)\right)+d\right)^2}{\left(\sqrt{\Theta\left(c^2-d^2\right)}-c\sqrt{\Theta}\cos\left(2\sqrt{\Theta}\left(\kappa x-\eta t\right)\right)\right)^2}, \tag{46}$$

$$\rho_{1,14}(x,t) = -\frac{4\kappa\Theta}{3} - 2\kappa\Theta\left(1-\frac{2c}{c+\cos\left(2\sqrt{\Theta}\left(\kappa x-\eta t\right)\right)-i\sin\left(2\sqrt{\Theta}\left(\kappa x-\eta t\right)\right)}\right)^2 - \frac{2\kappa\Theta}{\left(1-\frac{2c}{c+\cos\left(2\sqrt{\Theta}\left(\kappa x-\eta t\right)\right)-i\sin\left(2\sqrt{\Theta}\left(\kappa x-\eta t\right)\right)}\right)^2}. \tag{47}$$

Case 3: If $\Theta = 0$, then we get the rational solution as below:

$$\rho_{1,15}(x,t) = \frac{2\kappa}{\left(\kappa x-\eta t\right)^2}. \tag{48}$$

For **Set 2**, we reach the next solutions:

Case 1: If $\Theta < 0$, then the kink type solution obtained as

$$\rho_{2,1}(x,t) = -\frac{2\kappa\Theta}{3}\left(3\tanh\left(\sqrt{-\Theta}\left(\kappa x-\eta t\right)\right)^2-1\right). \tag{49}$$

The solitary wave solution obtained as

$$\rho_{2,2}(x,t) = -\frac{2\kappa\Theta}{3}\left(3\coth\left(\sqrt{-\Theta}\left(\kappa x-\eta t\right)\right)^2-1\right). \tag{50}$$

The mixed complex bright-dark soliton solution attained as

$$\rho_{2,3}(x,t) = \frac{2\kappa\Theta}{3} - 2\kappa\Theta\Big(\tanh\left(2\sqrt{-\Theta}\,(\kappa x - \eta t)\right) + i\,\mathrm{sech}\left(2\sqrt{-\Theta}\,(\kappa x - \eta t)\right)\Big)^2. \tag{51}$$

The kink type solution reached as

$$\rho_{2,4}(x,t) = \frac{2\kappa\Theta}{3} + \frac{2\kappa\left(-\Theta + \sqrt{-\Theta}\tanh\left(\sqrt{-\Theta}\,(\kappa x - \eta t)\right)\right)^2}{\left(1 + \sqrt{-\Theta}\tanh\left(\sqrt{-\Theta}\,(\kappa x - \eta t)\right)\right)^2}. \tag{52}$$

The mixed type hyperbolic solutions obtained as

$$\rho_{2,5}(x,t) = \frac{2\kappa\Theta}{3} - \frac{2\kappa\Theta\left(5 - 4\cosh\left(2\sqrt{-\Theta}\,(\kappa x - \eta t)\right)\right)^2}{\left(3 + 4\sinh\left(2\sqrt{-\Theta}\,(\kappa x - \eta t)\right)\right)^2}, \tag{53}$$

$$\rho_{2,6}(x,t) = \frac{2\kappa\Theta}{3} + \frac{2\kappa\left(\sqrt{-\Theta\,(c^2 + d^2)} - c\sqrt{-\Theta}\cosh\left(2\sqrt{-\Theta}\,(\kappa x - \eta t)\right)\right)^2}{\left(c\sinh\left(2\sqrt{-\Theta}\,(\kappa x - \eta t)\right) + d\right)^2}, \tag{54}$$

$$\rho_{2,7}(x,t) = \frac{2\kappa\Theta}{3} - \frac{2\kappa\Theta\left(-c + \cosh\left(2\sqrt{-\Theta}\,(\kappa x - \eta t)\right) - \sinh\left(2\sqrt{-\Theta}\,(\kappa x - \eta t)\right)\right)^2}{\left(c + \cosh\left(2\sqrt{-\Theta}\,(\kappa x - \eta t)\right) - \sinh\left(2\sqrt{-\Theta}\,(\kappa x - \eta t)\right)\right)^2}. \tag{55}$$

Case 2: If $\Theta > 0$, then we get as follows:

The singular periodic wave solutions obtained as

$$\rho_{2,8}(x,t) = \frac{2\kappa\Theta}{3}\left(3\tan\left(\sqrt{\Theta}\,(\kappa x - \eta t)\right)^2 + 1\right), \tag{56}$$

$$\rho_{2,9}(x,t) = \frac{2\kappa\Theta}{3}\left(3\cot\left(\sqrt{\Theta}\,(\kappa x - \eta t)\right)^2 + 1\right). \tag{57}$$

The combo trigonometric soliton solution attained as

$$\rho_{2,10}(x,t) = \frac{2\kappa\Theta}{3} + 2\kappa\Theta\Big(\tan\left(2\sqrt{\Theta}\,(\kappa x - \eta t)\right) + \sec\left(2\sqrt{\Theta}\,(\kappa x - \eta t)\right)\Big)^2. \tag{58}$$

The explicit periodic type solution reached as

$$\rho_{2,11}(x,t) = \frac{2\kappa\Theta}{3} + \frac{2\kappa\Theta\left(-1 + \tan\left(\sqrt{\Theta}\,(\kappa x - \eta t)\right)\right)^2}{\left(1 + \tan\left(\sqrt{\Theta}\,(\kappa x - \eta t)\right)\right)^2}. \tag{59}$$

The combo trigonometric soliton solution obtained as

$$\rho_{2,12}(x,t) = \frac{2\kappa\Theta}{3} + \frac{2\kappa\Theta\left(4 - 5\cos\left(2\sqrt{\Theta}\,(\kappa x - \eta t)\right)\right)^2}{\left(3 + 5\sin\left(2\sqrt{\Theta}\,(\kappa x - \eta t)\right)\right)^2}, \tag{60}$$

$$\rho_{2,13}(x,t) = \frac{2\kappa\Theta}{3} + \frac{2\kappa\left(\sqrt{\Theta\left(c^2-d^2\right)} - c\sqrt{\Theta}\cos\left(2\sqrt{\Theta}\left(\kappa x-\eta t\right)\right)\right)^2}{\left(c\sin\left(2\sqrt{\Theta}\left(\kappa x-\eta t\right)\right)+d\right)^2}. \tag{61}$$

The complex trigonometric wave solutions attained as

$$\rho_{2,14}(x,t) = \frac{2\kappa\Theta}{3} - 2\kappa\Theta\left(1 - \frac{2c}{c+\cos\left(2\sqrt{\Theta}\left(\kappa x-\eta t\right)\right) - i\sin\left(2\sqrt{\Theta}\left(\kappa x-\eta t\right)\right)}\right)^2. \tag{62}$$

Case 3: If $\Theta = 0$, then we have the following rational solution:

$$\rho_{2,15}(x,t) = \frac{2\kappa}{\left(\kappa x-\eta t\right)^2}. \tag{63}$$

3.2. Utilizing the $\left(\frac{G'}{G^2}\right)$ Expansion Function Method

Using the homogenous balance principle, (22) is as follows:

$$H\left(\xi\right) = m_0 + m_1\left(\frac{G'}{G^2}\right) + m_2\left(\frac{G'}{G^2}\right)^2 + n_1\left(\frac{G'}{G^2}\right)^{-1} + n_2\left(\frac{G'}{G^2}\right)^{-2}. \tag{64}$$

When (64) is inserted into (30) with all coefficients set to zero, we obtain

$$\begin{cases} -2\kappa^3 n_2\varphi^2 - 2\kappa^3 m_2\tau^2 + 3\kappa^2 m_0^2 + 6\kappa^2 m_1 n_1 + 6\kappa^2 m_2 n_2 + \eta m_0 = 0, \\ -8\kappa^3\varphi\tau m_2 + 6\kappa^2 m_0 m_2 + 3\kappa^2 m_1^2 + \eta m_2 = 0, \\ -2\kappa^3\varphi\tau m_1 + 6\kappa^2 m_0 m_1 + 6\kappa^2 m_2 n_1 + \eta m_1 = 0, \\ -2\kappa^3\tau\varphi n_1 + 6\kappa^2 m_0 n_1 + 6\kappa^2 m_1 n_2 + \eta n_1 = 0, \\ -8\kappa^3\varphi\tau n_2 + 6\kappa^2 m_0 n_2 + 3\kappa^2 n_1^2 + \eta n_2 = 0, \\ -2\kappa^3\varphi^2 m_1 + 6\kappa^2 m_1 m_2 = 0, \\ -2\kappa^3\tau^2 n_1 + 6\kappa^2 n_1 n_2 = 0, \\ -6\kappa^3\varphi^2 m_2 + 3\kappa^2 m_2^2 = 0, \\ -6\kappa^3\tau^2 n_2 + 3\kappa^2 n_2^2 = 0. \end{cases}$$

By solving above the algebraic system, we get the following solution sets:

Set 1:

$$\eta = 4\kappa^3\varphi\tau, \quad m_0 = \frac{2\kappa\varphi\tau}{3}, \quad m_1 = 0, \quad m_2 = 2\kappa\varphi^2, \quad n_1 = 0, \quad n_2 = 0.$$

Set 2:

$$\eta = 16\kappa^3\varphi\tau, \quad m_0 = -\frac{4\kappa\varphi\tau}{3}, \quad m_1 = 0, \quad m_2 = 2\kappa\varphi^2, \quad n_1 = 0, \quad n_2 = 2\kappa\tau^2.$$

By using **Set 1**, we have the following soliton solutions:

If $\tau\varphi > 0$, then the trigonometric solution is given by the following form:

$$\rho_1(x,t) = \frac{2\kappa\tau\varphi}{3} + \frac{2\kappa\tau\varphi\left(A_1\cos\left(\sqrt{\tau\varphi}\xi\right) + A_2\sin\left(\sqrt{\tau\varphi}\xi\right)\right)^2}{\left(A_2\cos\left(\sqrt{\tau\varphi}\xi\right) - A_1\sin\left(\sqrt{\tau\varphi}\xi\right)\right)^2}. \tag{65}$$

If $\tau\varphi < 0$, then the hyperbolic solution is found as follow:

$$\rho_2(x,t) = \frac{2\kappa\tau\varphi}{3} - \frac{2\kappa\tau\varphi\left(A_1\sinh\left(2\sqrt{-\tau\varphi}\xi\right) + A_1\cosh\left(-\sqrt{\tau\varphi}\xi\right) + A_2\right)^2}{\left(A_1\sinh\left(2\sqrt{-\tau\varphi}\xi\right) + A_1\cosh\left(2\sqrt{-\tau\varphi}\xi\right) - A_2\right)^2}. \tag{66}$$

If $\tau = 0,\ \varphi \neq 0$, then the rational solution is given by the following form:

$$\rho_3(x,t) = \frac{2\kappa A_1^2}{\left(\xi A_1 + A_2\right)^2}. \tag{67}$$

For **Set 2**, we reach the following solutions:

If $\tau\varphi > 0$, then the trigonometric solution is given by the following form:

$$\begin{aligned}\rho_{1,0}(x,t) = &-\frac{4\kappa\tau\varphi}{3} + \frac{2\kappa\tau\varphi\left(A_1\cos\left(\sqrt{\tau\varphi}\xi\right) + A_2\sin\left(\sqrt{\tau\varphi}\xi\right)\right)^2}{\left(A_2\cos\left(\sqrt{\tau\varphi}\xi\right) - A_1\sin\left(\sqrt{\tau\varphi}\xi\right)\right)^2} \\ &+ \frac{2\kappa\tau\varphi\left(A_2\cos\left(\sqrt{\tau\varphi}\xi\right) - A_1\sin\left(\sqrt{\tau\varphi}\xi\right)\right)^2}{\left(A_1\cos\left(\sqrt{\tau\varphi}\xi\right) + A_2\sin\left(\sqrt{\tau\varphi}\xi\right)\right)^2}.\end{aligned} \tag{68}$$

If $\tau\varphi < 0$, then the hyperbolic solution is found as below:

$$\begin{aligned}\rho_{2,0}(x,t) = &-\frac{2\kappa\tau\varphi}{3} - \frac{2\kappa\tau\varphi\left(A_1\sinh\left(2\sqrt{-\tau\varphi}\xi\right) + A_1\cosh\left(\sqrt{-\tau\varphi}\xi\right) + A_2\right)^2}{\left(A_1\sinh\left(2\sqrt{-\tau\varphi}\xi\right) + A_1\cosh\left(2\sqrt{-\tau\varphi}\xi\right) - A_2\right)^2} \\ &- \frac{2\kappa\tau\varphi\left(A_1\sinh\left(2\sqrt{-\tau\varphi}\xi\right) + A_1\cosh\left(\sqrt{-\tau\varphi}\xi\right) - A_2\right)^2}{\left(A_1\sinh\left(2\sqrt{-\tau\varphi}\xi\right) + A_1\cosh\left(2\sqrt{-\tau\varphi}\xi\right) + A_2\right)^2}.\end{aligned} \tag{69}$$

If $\tau = 0,\ \varphi \neq 0$, then the rational solution is given by the following form:

$$\rho_{3,0}(x,t) = \frac{2\kappa A_1^2}{\left(\xi A_1 + A_2\right)^2}. \tag{70}$$

4. Graphical Explanation

This section demonstrates the structural properties of the obtained soliton solutions using various graphical representations. Specifically, 3D, contour, and 2D graphs corresponding to analytical solutions derived in previous sections are presented. These visualizations help demonstrate the localization, amplitude, and wave propagation properties of the solutions more intuitively and comprehensively.

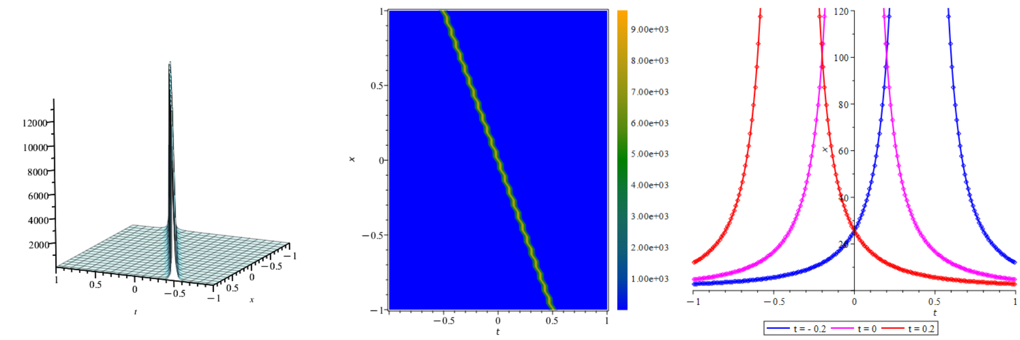
Figure 1. 3D, 2D, and contour plots of the bright soliton solution for $\rho_{1,1}(x,t)$ are presented when $\Theta = -0.5\,,\ \kappa = 0.5\,.$

Figure 2. The W-shaped soliton solution of $|\rho_{1,3}(x,t)|$ is illustrated using 3D, 2D, and contour plots for $\Theta = -0.5\,,\ \kappa = 0.5\,.$

Figure 3. The singular solitary wave structure of $\rho_{1,8}(x,t)$ is depicted through 3D, 2D, and contour plots for $\Theta = 1\,,\ \kappa = 1\,.$

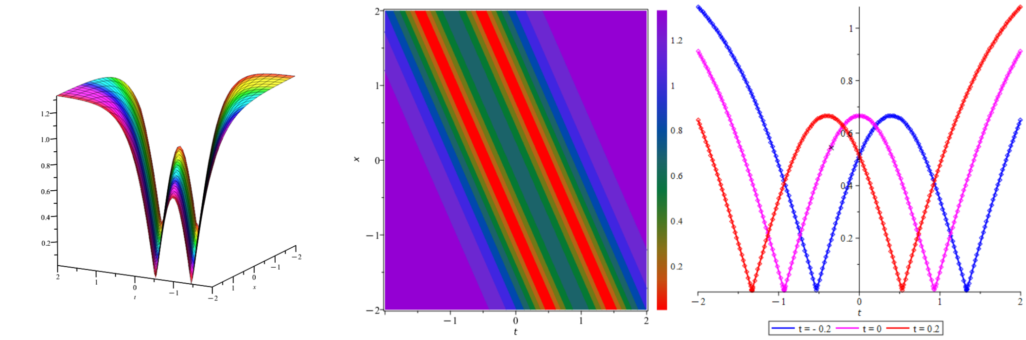
Figure 4. The dark soliton solution of $\rho_{2,1}(x,t)$ is illustrated using 3D, 2D, and contour plots for $\Theta = -0.5\,,\ \kappa = 1\,.$

The solutions in this study have characteristic graphical properties of nonlinear wave systems. These solutions are bright, dark, W-shaped soliton and singular solitary wave solutions. A bright soliton is a smooth, bell-shaped peak that does not change as it moves. In contrast, a W-shaped soliton has two peaks, meaning the wave amplitude differs between the peak and trough, forming a profile that resembles the letter "W". A singular solitary wave solution differs in that it has infinite or undefined amplitude at certain points, resulting in sharp discontinuities or singularities in the wave profile. Conversely, a dark soliton appears as a localized notch (or trough) on a smooth background wave with a phase shift and stable motion. Dark and bright solitons have smooth, localized forms at the graphical level, while W-shaped soliton and singular solitary wave introduce more complex dynamics into the wave profile. This impacts applications in fluid dynamics, optics, and Bose-Einstein condensates.



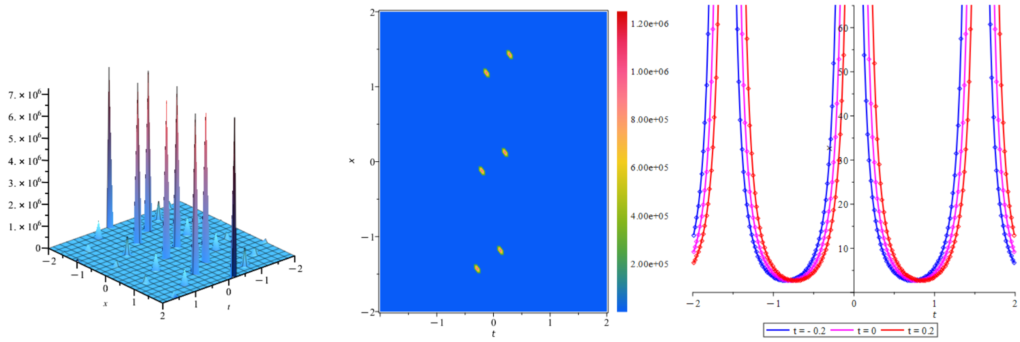
(a) 3D (b) Contour (c) 2D

Figure 1: Graphs of $\rho_{1,1}(x,t)$



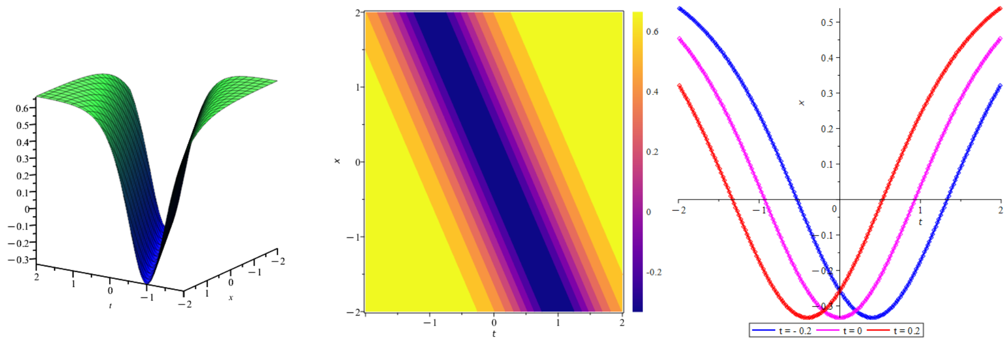
(a) 3D (b) Contour (c) 2D

Figure 2: Graphical representations of $|\rho_{1,3}(x,t)|$



(a) 3D (b) Contour (c) 2D

Figure 3: Graphical representations of $\rho_{1,8}(x,t)$



(a) 3D (b) Contour (c) 2D

Figure 4: Graphical representations of $\rho_{2,1}(x,t)$.

5. Conclusion

In this study, METEM and the $\left(\frac{G'}{G^2}\right)$-expansion function method were used to derive new analytical solutions of nonlinear K-XE. The results include a wide range of exact solutions, such as W-shaped solitons, singular solitary waves, bright and dark solitons, as well as rational, hyperbolic, and trigonometric forms. Several of these solutions were visualized using 3D, contour, 2D plots generated via Maple software. These graphical representations effectively capture the physical behavior of the solutions and validate their consistency. According to the obtained results, these two approaches provide highly accurate analytical solutions for K-XE. Another advantage of these methods is their proven ability to efficiently generate solutions. These solutions are crucial for understanding the wave dynamics of the model. All solutions have been verified using software programs. In the future, the study will be expanded to include the fractional and variable-coefficient forms of the K-XE. These efforts are expected to further enhance the model's physical interpretability and applicability in nonlinear science.

Declaration of Ethical Standards

The author declares that the materials and methods used in her study do not require ethical committee and/or legal special permission.

Conflicts of Interest

The author declares no conflict of interest.

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