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# Evaluating Additional Observations of the Same Units to Estimate Misclassification Probabilities in Measurement System Analysis

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**Abstract**— Measurement System Analysis evaluates the accuracy and precision of measurement processes; in the literature, part variability and measurement error are typically assumed to follow normal distributions, and we adopt this convention. We derive closed-form formulas for Types I and II misclassification probabilities using univariate and bivariate normal cumulative distribution functions, avoiding numerical integration and enabling efficient computation (e.g., in R). Building on these results, we derive explicit maximum likelihood estimates of misclassification probabilities for both the classical approach based on measurements from different parts and the repeated-measurement approach using multiple measurements on the same part at different times. A Monte Carlo study shows that incorporating repeated measurements reduces bias and mean squared error. A brief numerical example with simulated data demonstrates practical implementation.

***Keywords*** — *Monte Carlo simulation, maximum likelihood estimation, measurement system analysis, misclassification probabilities, quality control*

## 1. Introduction

Statistical Quality Control (SQC) is essential for ensuring product consistency by monitoring process variations and minimizing defects. A fundamental component of SQC is Measurement System Analysis (MSA), which evaluates the accuracy and reliability of measurement processes. One of the key tools in MSA is Gage Repeatability and Reproducibility (Gage R&R), which quantifies measurement variability by analyzing repeatability and reproducibility. To assess the capability of a production process, statistical indices such as the process capability index and the process performance index are utilized. These indices ensure that process variation remains within defined specification limits. For a comprehensive discussion on Gage R&R analysis, we refer to several key studies, including [1–10]. Similarly, for process capability indices, we refer to recent contributions from [11–18].

MSA also aims to provide critical insights into process variation and measurement reliability by distinguishing the variation between the measured items from the errors inherent in the measurement system. Two types of probabilities are associated with the misclassification of an item, which are analogous to Type-I and Type-II errors in hypothesis testing. According to [19], the $\alpha$ probability represents the combined probability of incorrectly failing an item that meets specifications (referred to

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as a false failure or producer's risk), while the $\beta$ probability refers to the probability of incorrectly accepting an item that does not meet specifications (also called a missed fault or consumer's risk). These risks guide manufacturers in determining whether the measurement system is adequate or needs improvement.

Under the assumption that both parts and measurement errors follow a normal distribution, several studies have evaluated the misclassification probabilities $\alpha$ and $\beta$. Notable contributions in this context include the following: [20] discussed the probabilities associated with the misclassification of items due to measurement variability. [21] used a case study to assess the probabilities of misclassifying items due to measurement variability in the case of multiple sources of product variability. [22] presented methods for expanding the univariate MSA to the multivariate cases, and evaluated the performances of the production test process through misclassification probabilities. [23] presented the generalized inference method for constructing confidence intervals for misclassification probabilities in a Gage R&R study. [24] proposed a bootstrap method to construct confidence intervals for misclassification probabilities in MSA and compared its performance with the generalized inference method. [25] developed a robust method to evaluate misclassification probabilities under the two-component measurement error model. Recently, [26] presented a novel statistical methodology to improve the estimation process for misclassification probabilities, while also constructing uncorrected likelihood ratio confidence intervals for misclassification probabilities in MSA.

In this study, closed-form expressions for misclassification probabilities are derived under a normal distribution framework, providing a more computationally efficient approach than integral-based methods. The maximum likelihood estimation of these misclassification probabilities is also discussed using the methodology presented by [26]. Furthermore, a comprehensive simulation study is conducted to evaluate the performance of this methodology, assessing its accuracy in different scenarios. The remainder of this paper is organized as follows: Section 2 introduces both the classical model and the closed-form expressions for misclassification probabilities. The Maximum Likelihood Estimators (MLEs) of model parameters and misclassification probabilities are discussed in Section 3. A comprehensive simulation study is conducted, considering in-control stages, with the results presented in Section 4. To illustrate the proposed methodologies, a numerical example is provided in Section 5. Finally, Section 6 presents the conclusion of the study.

## 2. Model Setup

In this paper, we study a model of the form

$$Y = X + M \tag{2.1}$$

where $X$ and $M$ are independent random variables following $\mathcal{N}(\mu_{\text{part}}, \sigma^2_{\text{part}})$ and $\mathcal{N}(0, \sigma^2_{\text{measure}})$, respectively. Then, the random variable $Y = X + M$ following $Y \sim \mathcal{N}\left(\mu_{\text{part}}, \sigma^2_{\text{part}} + \sigma^2_{\text{measure}}\right)$. Moreover, the misclassification probabilities, denoted as $\alpha$ and $\beta$, for lower and upper specification limits $L$ and $U$ is defined by [21]:

$$\begin{aligned} \alpha &= P\left(L < X < U, \, (Y < L \vee Y > U)\right) \\ &= \int_{-\infty}^{L}\int_{L}^{U} f(x,y)\,dx\,dy + \int_{U}^{\infty}\int_{L}^{U} f(x,y)\,dx\,dy \end{aligned} \tag{2.2}$$

and

$$\begin{aligned}\beta &= P\left(L<Y<U,(X<L \vee X>U)\right)\\ &= \int_{-\infty}^{L}\int_{L}^{U} f(x,y)dydx+\int_{U}^{\infty}\int_{L}^{U} f(x,y)dydx \end{aligned} \tag{2.3}$$

where the joint pdf of $X$ and $Y$ is given in (2.4).

$$f_{X,Y}(x,y)=\frac{1}{2\pi\sigma_{\text{part}}\sigma_{\text{measure}}}\exp\left(-\frac{(x-\mu_{\text{part}})^2}{2\,\sigma_{\text{part}}^2}-\frac{(y-x)^2}{2\,\sigma_{\text{measure}}^2}\right) \tag{2.4}$$

[26] did not provide closed-form expressions for the integrals in (2.2) and (2.3). In this paper, closed-form solutions for these integrals are derived based on one-dimensional and two-dimensional normal cumulative distribution functions. Consider the following facts [27], which can be used to obtain explicit expressions for misclassification probabilities:

$$\int_{-\infty}^{z}\Phi\left(c-d\,u\right)\phi(u)\,du=\Phi_2\left(z,\frac{c}{\sqrt{1+d^2}};\frac{d}{\sqrt{1+d^2}}\right)$$

and

$$\int_{-\infty}^{z}\Phi\left(c+du\right)\phi(u)\,du=\Phi_2\left(z,\frac{c}{\sqrt{1+d^2}};-\frac{d}{\sqrt{1+d^2}}\right)$$

where, $\phi(\cdot)$ and $\Phi(\cdot)$ denote the probability density function (PDF) and cumulative distribution function (CDF) of the standard normal distribution, respectively. The function $\Phi_2(\cdot,\cdot;\rho)$ represents the CDF of a bivariate normal distribution with mean vector $\mu=\begin{bmatrix}0\\0\end{bmatrix}$ and covariance matrix $\Sigma=\begin{bmatrix}1&\rho\\ \rho&1\end{bmatrix}$ where $\rho$ is the correlation coefficient. These functions are readily available in the `stats` and `mvtnorm` packages in R. The standard normal PDF can be computed using `dnorm(x)`, while the CDF is given by `pnorm(x)`. For the bivariate normal CDF, the `pmvnorm(lower, upper, mean, sigma)` function from the `mvtnorm` package can be used. These functions allow for efficient numerical evaluation of probability distributions, making it easy to verify theoretical results.

Let $z=\frac{x-\mu_{part}}{\sigma_{part}}$. By substituting (2.4) into (2.2), we obtain the explicit form of the misclassification probability $\alpha$ as

$$\begin{aligned}\alpha &= \Phi_2\left(b,\frac{\gamma_1}{\sqrt{1+d^2}};\frac{d}{\sqrt{1+d^2}}\right)-\Phi_2\left(a,\frac{\gamma_1}{\sqrt{1+d^2}};\frac{d}{\sqrt{1+d^2}}\right)\\ &\quad+\Phi_2\left(b,\frac{\gamma_2}{\sqrt{1+d^2}};-\frac{d}{\sqrt{1+d^2}}\right)-\Phi_2\left(a,\frac{\gamma_2}{\sqrt{1+d^2}};-\frac{d}{\sqrt{1+d^2}}\right)\end{aligned}$$

where $a=\frac{L-\mu_{\text{part}}}{\sigma_{\text{part}}}, b=\frac{U-\mu_{\text{part}}}{\sigma_{\text{part}}}, d=\frac{\sigma_{\text{part}}}{\sigma_{\text{measure}}}, \gamma_1=\frac{L-\mu_{\text{part}}}{\sigma_{\text{measure}}}$, and $\gamma_2=\frac{\mu_{\text{part}}-U}{\sigma_{\text{measure}}}$.

Similarly, by substituting (2.4) into (2.3), the explicit form of another misclassification probability, $\beta$, is also obtained as

$$\begin{aligned}\beta &= \Phi_2\left(a,\frac{\gamma_3}{\sqrt{1+d^2}};\frac{d}{\sqrt{1+d^2}}\right)-\Phi_2\left(a,\frac{\gamma_4}{\sqrt{1+d^2}};\frac{d}{\sqrt{1+d^2}}\right)+\Phi\left(\frac{\gamma_3}{\sqrt{1+d^2}}\right)\\ &\quad-\Phi_2\left(b,\frac{\gamma_3}{\sqrt{1+d^2}};\frac{d}{\sqrt{1+d^2}}\right)-\Phi\left(\frac{\gamma_4}{\sqrt{1+d^2}}\right)+\Phi_2\left(b,\frac{\gamma_4}{\sqrt{1+d^2}};\frac{d}{\sqrt{1+d^2}}\right)\end{aligned}$$

where $\gamma_3=\frac{U-\mu_{\text{part}}}{\sigma_{\text{measure}}}$ and $\gamma_4=\frac{L-\mu_{\text{part}}}{\sigma_{\text{measure}}}$.

## 3. Statistical Inference

Let the measurements $Y_1, Y_2, \ldots, Y_n$ represent daily routine measurements collected on the product, where $Y_i$ (for $i = 1, \ldots, n$) follow $\mathcal{N}\left(\mu_{part},\, \sigma^2_{part} + \sigma^2_{measure}\right)$. Hence, the log-likelihood expression based on sample $Y_1, Y_2, \ldots, Y_n$ is

$$\ell_Y(\boldsymbol{\delta_1}) \propto -\frac{n}{2}\log\left(\sigma^2_{part} + \sigma^2_{measure}\right) - \frac{1}{2(\sigma^2_{part} + \sigma^2_{measure})}\sum_{i=1}^{n}(y_i - \mu_{part})^2 \tag{3.1}$$

where $\boldsymbol{\delta_1} = (\mu_{part}, \sigma_{part}, \sigma_{measure})$, and $\widehat{\boldsymbol{\delta_1}} = (\widehat{\mu}_{part}, \widehat{\sigma}_{part}, \widehat{\sigma}_{measure})$. Then, the associated gradients are found to be:

$$\frac{\partial \ell_Y(\boldsymbol{\delta_1})}{\partial \mu_{part}} = \frac{1}{\left(\sigma^2_{part} + \sigma^2_{measure}\right)}\sum_{i=1}^{n}(y_i - \mu_{part}) = 0 \tag{3.2}$$

$$\frac{\partial \ell_Y(\boldsymbol{\delta_1})}{\partial \sigma_{part}} = \frac{\sigma_{part}}{(\sigma^2_{part} + \sigma^2_{measure})^2}\left(\sum_{i=1}^{n}(y_i - \mu_{part})^2 - n\left(\sigma^2_{part} + \sigma^2_{measure}\right)\right) = 0 \tag{3.3}$$

$$\frac{\partial \ell_Y(\boldsymbol{\delta_1})}{\partial \sigma_{measure}} = \frac{\sigma_{measure}}{(\sigma^2_{part} + \sigma^2_{measure})^2}\left(\sum_{i=1}^{n}(y_i - \mu_{part})^2 - n\left(\sigma^2_{part} + \sigma^2_{measure}\right)\right) = 0 \tag{3.4}$$

The MLEs of parameters can be obtained by solving these equations simultaneously, but only the solution for $\widehat{\mu}_{part}$ can be obtained analytically. Since (3.3) and (3.4) do not provide explicit solutions, they can be solved by using an iterative algorithm to obtain the MLEs of $\boldsymbol{\delta_1}$.

$$\widehat{\mu}_{part} = \frac{\sum_{i=1}^{n} Y_i}{n} \tag{3.5}$$

The solutions of $\sigma_{part}$ and $\sigma_{measure}$ in (3.3) and (3.4) cannot be obtained in closed form because the total variance of the measurement system, $\sigma^2$, involves both $\sigma^2_{part}$ and $\sigma^2_{measure}$, i.e., $\sigma^2 = \sigma^2_{part} + \sigma^2_{measure}$. Consequently, using data from a single source alone makes it impossible to separate $\sigma_{part}$ and $\sigma_{measure}$ uniquely, and therefore, there is no single solution. An analytical solution to these equations is only possible if either $\sigma_{part}$ or $\sigma_{measure}$ is known in advance. Although the `optim` function in R may appear to obtain MLE estimates, changing the optimization method and initial values can lead to different results. For example, we generate data from the $\mathcal{N}\left(\mu_{part},\, \sigma^2_{part} + \sigma^2_{measure}\right)$, where $\mu_{part} = 5$, $\sigma_{part} = 0.2$, and $\sigma_{measure} = 0.05$. The generated data is $4.7409, 4.9212, 5.0696, 4.6897, 5.0527, 5.0081, 5.0229, 5.3006, 4.6718, 5.3412$, and the results are illustrated with Table 1.

**Table 1.** An illustrative example using different estimation methods in `optim` function

| $n = 10$ | Initial Value | Method | $\widehat{\mu}_{part}$ | $\widehat{\sigma}_{part}$ | $\widehat{\sigma}_{measure}$ | $\ell\left(\widehat{\boldsymbol{\delta_1}}\right)$ |
|---|---|---|---|---|---|---|
| | (5,0.2,0.05) | BFGS | 4.9818 | 0.2211 | 0.0007 | −0.900035 |
| | | Nelder-Mead | 4.9818 | 0.1976 | 0.0990 | −0.900035 |
| | (5,1,1) | BFGS | 4.9818 | 0.1563 | 0.1563 | −0.900035 |
| | | Nelder-Mead | 4.9818 | 0.1831 | 0.1238 | −0.900035 |

Table 1 shows that although the value of the log-likelihood function remains identical between different optimization methods, the estimates are different. This demonstrates a classical case of non-identifiability: while the total variance $\sigma^2 = \sigma^2_{part} + \sigma^2_{measure}$ is estimable from the data, its individual components cannot be uniquely separated without additional assumptions or external information. As seen in Table 1, multiple combinations of $(\sigma_{part}, \sigma_{measure})$ can yield the same log-likelihood value, leading to different MLEs depending on the optimization method. This flat likelihood surface, often referred to as a *ridge*, indicates that the parameter space contains infinitely many equally likely solutions. Therefore, the model is overparameterized and non-identifiable in its current form.

In practice, although the random variable $Y$ in (2.1) can be observed, its components $X$ and $M$ cannot be directly measured. However, following the methodology in [26], by repeatedly measuring a specific product at different times, denoted as $Z_1, Z_2, \ldots, Z_r$, independently and identically distributed (iid) normal random variables with a nuisance mean $\mu_{nui}$ and variance $\sigma^2_{measure}$ can be observed. Here, $\mu_{nui}$ represents the uninterested mean of the repeatedly measured product, and $\sigma^2_{measure}$ is the variance of $M$. Thus, additional information about the variance $\sigma^2_{measure}$ of the measurement system is collected, which is expected to improve the estimate of $\sigma^2_{measure}$. It is important to emphasize that the results are derived assuming that the repeated measurements taken by the operator on the same unit are independent. Several strategies can be implemented to ensure that repeated measurements on the same sample remain as independent as possible. First, randomizing the order of measurements and including filler samples can help prevent the operator from realizing that the same item is being measured repeatedly. Second, keeping the product's identity hidden from the operator (i.e., blinding) can reduce the risk of conscious or subconscious bias. Third, allowing sufficient time between measurements can disrupt short-term memory and reduce correlations that might arise from immediate recall. Finally, maintaining a consistent measurement protocol while incorporating minor random variations can minimize residual patterns or habits, strengthening the independence assumption between repeated measurements.

The measurements $Z_1, Z_2, \ldots, Z_r$ can be observed as follows: A previously manufactured product is sent to the laboratory daily along with newly produced products, disguised as a new product. However, the operator will not know it is a previously measured product. Let $Z$ denote the daily measurement for this product. Parameter estimates can be obtained using the following methodology, assuming that the variance of the $Z$ measurements equals the variance of the measurement error.

Let $Y_1, Y_2, \ldots, Y_n$ be iid production measurements and $Z_1, Z_2, \ldots, Z_r$ are iid measurements from daily monitoring of the measurement system. Then, the log-likelihood function is given by

$$\begin{aligned} \ell_{Y,Z}(\boldsymbol{\delta_2}) \;\; \propto \;\; & -\frac{n}{2}\log\left(\sigma^2_{part}+\sigma^2_{measure}\right) - \frac{1}{2(\sigma^2_{part}+\sigma^2_{measure})}\sum_{i=1}^{n}(y_i-\mu_{part})^2 \\ & - \;\; \frac{r}{2}\log\left(\sigma^2_{measure}\right) - \frac{1}{2\sigma^2_{measure}}\sum_{j=1}^{r}\left(z_j-\mu_{nui}\right)^2 \end{aligned} \tag{3.6}$$

where $\boldsymbol{\delta_2} = (\mu_{part}, \sigma_{part}, \sigma_{measure}, \mu_{nui})$, $\widetilde{\boldsymbol{\delta_2}} = (\widetilde{\mu}_{part}, \widetilde{\sigma}_{part}, \widetilde{\sigma}_{measure}, \widetilde{\mu}_{nui})$, and $\mu_{nui}$ is a nuisance parameter. It can be considered a truly measured value of the previously manufactured product. However, it is not used to compute the misclassification probabilities, our goal. The gradients for $\mu_{part}$, $\sigma_{part}$, $\sigma_{measure}$, and $\mu_{nui}$ are found to be

$$\frac{\partial \ell_{Y,Z}\left(\boldsymbol{\delta_2}\right)}{\partial \mu_{part}} = \frac{1}{\left(\sigma^2_{part}+\sigma^2_{measure}\right)}\sum_{i=1}^{n}(y_i-\mu_{part}) = 0$$

$$\frac{\partial \ell_{Y,Z}\left(\boldsymbol{\delta_2}\right)}{\partial \sigma_{part}} = \frac{\sigma_{part}}{(\sigma^2_{part}+\sigma^2_{measure})^2}\left(-\,n\,(\sigma^2_{part}+\sigma^2_{measure}) \;+\; \sum_{i=1}^{n}(y_i-\mu_{part})^2\right) = 0$$

$$\frac{\partial \ell_{Y,Z}(\boldsymbol{\delta_2})}{\partial \sigma_{measure}} = -\frac{n\,\sigma_{measure}}{\sigma_{part}^2+\sigma_{measure}^2} + \frac{\sigma_{measure}}{(\sigma_{part}^2+\sigma_{measure}^2)^2}\sum_{i=1}^{n}(y_i-\mu_{part})^2 - \frac{r}{\sigma_{measure}} + \frac{1}{\sigma_{measure}^3}\sum_{j=1}^{r}(z_j-\mu_{nui})^2 = 0$$

and

$$\frac{\partial \ell_{Y,Z}(\boldsymbol{\delta_2})}{\partial \mu_{nui}} = -\frac{r}{2}\log(2\pi\sigma_{measure}^2) - \frac{1}{2\sigma_{measure}^2}\sum_{j=1}^{r}(z_j-\mu_{nui})^2 = 0$$

respectively. The maximum likelihood estimators are obtained by solving the above equations, and can be explicitly derived as follows:

$$\widetilde{\mu}_{part} = \frac{\sum_{i=1}^{n} Y_i}{n} \tag{3.7}$$

$$\widetilde{\sigma}_{part} = \sqrt{\frac{1}{n}\sum_{i=1}^{n}(Y_i-\widetilde{\mu}_{part})^2 - \frac{1}{r}\sum_{j=1}^{r}(Z_j-\widetilde{\mu}_{nui})^2} \tag{3.8}$$

$$\widetilde{\sigma}_{measure} = \sqrt{\frac{1}{r}\sum_{j=1}^{r}(Z_j-\widetilde{\mu}_{nui})^2} \tag{3.9}$$

and

$$\widetilde{\mu}_{nui} = \frac{\sum_{j=1}^{r} Z_j}{r}$$

respectively. Then, the MLEs of $\alpha$ and $\beta$ probabilities can be easily obtained by the invariance property of MLEs.

**Remark 3.1.** Incorporating the $Z$ observations alongside the $Y$ data permits closed-form expressions for the MLEs (see (3.7)–(3.9)). This approach fully resolves the identifiability problem inherent in the likelihood function of (3.1), which relies exclusively on the $Y$ data.

**Remark 3.2.** Let $r$ denote the number of independent observations $Z_1, Z_2, \ldots, Z_r \sim \mathcal{N}(\mu_\nu, \sigma_{measure}^2)$. Then, as $r \to \infty$, MLE of $\sigma_{measure}$, denoted by $\widetilde{\sigma}_{measure}$, converges in probability to the true value $\sigma_{measure}$; that is,

$$\widetilde{\sigma}_{measure} \xrightarrow{p} \sigma_{measure}$$

Consequently, as $r \to \infty$, the full maximum likelihood estimation procedure that uses both $Y$ and $Z$ converges to the estimation procedure in which $\sigma_{measure}$ is treated as known and fixed. In particular, the estimator $\widetilde{\sigma}_{part}$ derived from the full dataset (including $Z$) asymptotically behaves like the estimator obtained without access to $Z$, under the assumption that $\sigma_{measure}$ is known.

## 4. Simulation Study

In this section, we perform a simulation study to evaluate the performance of the estimation method considered in Section 3. Specifically, the parts are assumed to follow a normal distribution with $\mu_{part} = 13$ and $\sigma_{part} = 3$. Additionally, measurement errors are assumed to be normally distributed with a mean of 0 and a standard deviation of $\sigma_{measure} = 1.5$. Also, $\mu_{nui} = 15$ is pre-determined. The simulation study is designed for different combinations of sample sizes $(n, r)$ for $n = 50, 100, 250$ and $r = 5, 10, 25, 50, 100, 250, 500, 1000, 2000$ by considering in control stages. In this regard, $L$ and $U$ are determined such that 90% of the true values of the products $(X)$ fall between $L$ and $U$. In this case, for $\mu_{part} = 15$ and $\sigma_{part} = 3$ it is predetermined that 90% of the products fall within this range when

$L = 11.15$ and $U = 27.82$. 50000 trials are used in the simulation, and the performance of the MLEs for the model parameters and misclassification probabilities are evaluated using mean squared error (MSE) and bias criteria. The results for $\widetilde{\boldsymbol{\delta_2}}$, $\widetilde{\alpha}$ and $\widetilde{\beta}$ are presented in Figures 1-10.

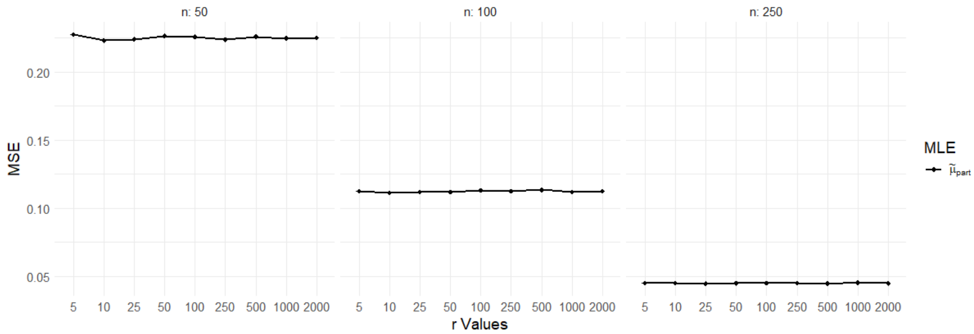


**Figure 1.** MSE values in $\widetilde{\mu}_{part}$ for different $n$ and $r$ values

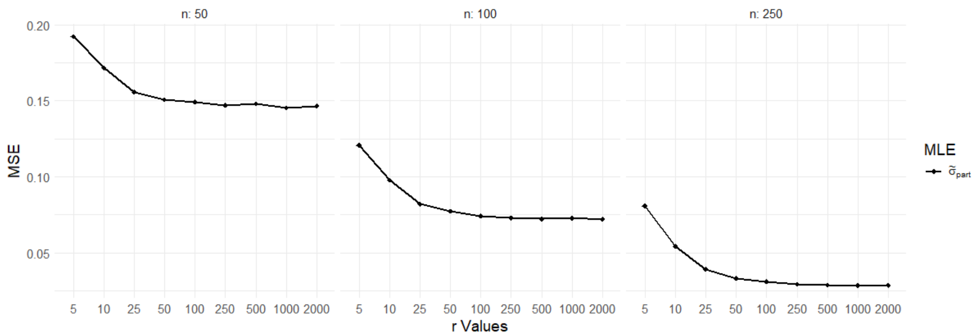


**Figure 2.** MSE values in $\widetilde{\sigma}_{part}$ for different $n$ and $r$ values

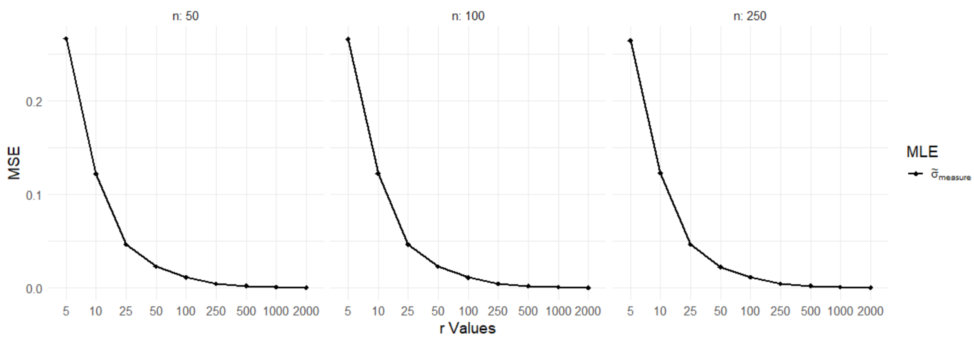


**Figure 3.** MSE values in $\widetilde{\sigma}_{measure}$ for different $n$ and $r$ values

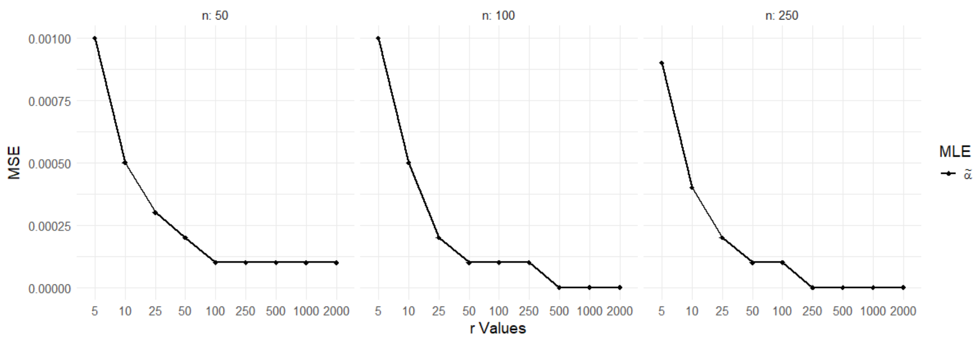


**Figure 4.** MSE values in $\widetilde{\alpha}$ for different $n$ and $r$ values

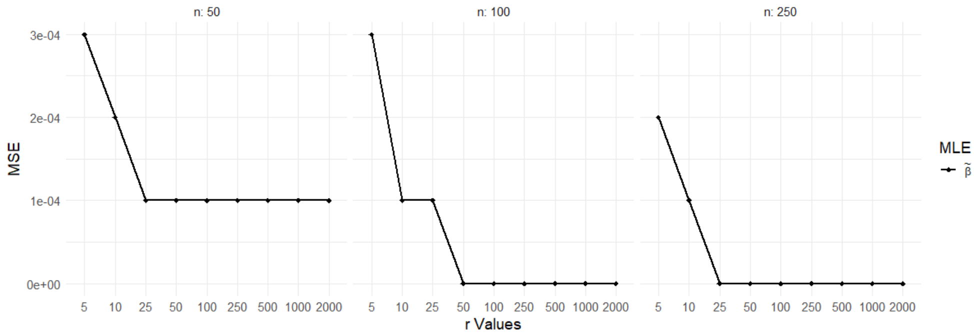


**Figure 5.** MSE values in $\widetilde{\beta}$ for different $n$ and $r$ values

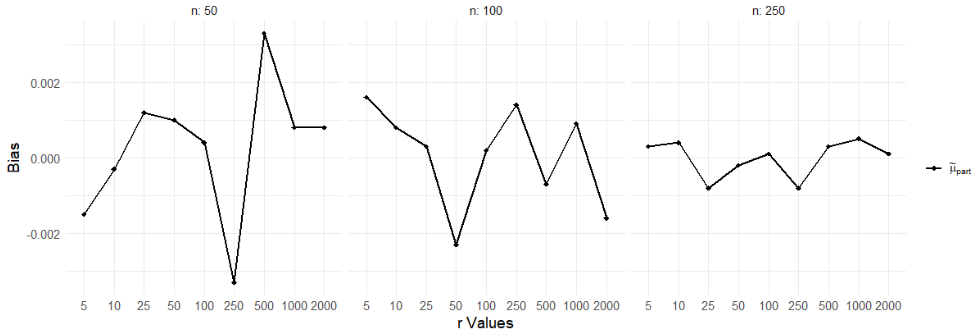


**Figure 6.** Bias values in $\widetilde{\mu}_{part}$ for different $n$ and $r$ values

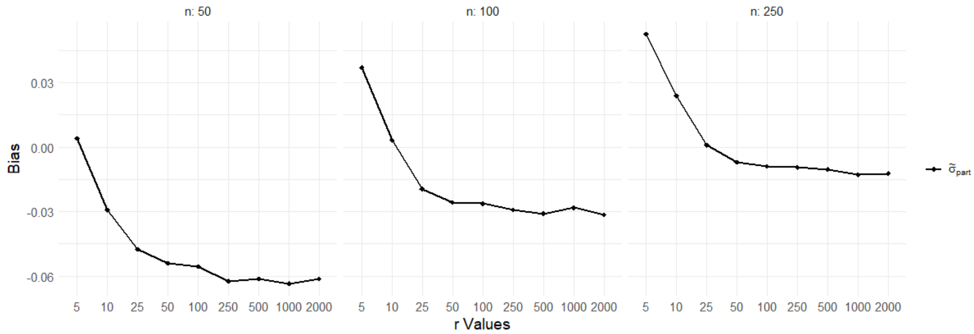


**Figure 7.** Bias values in $\widetilde{\sigma}_{part}$ for different $n$ and $r$ values

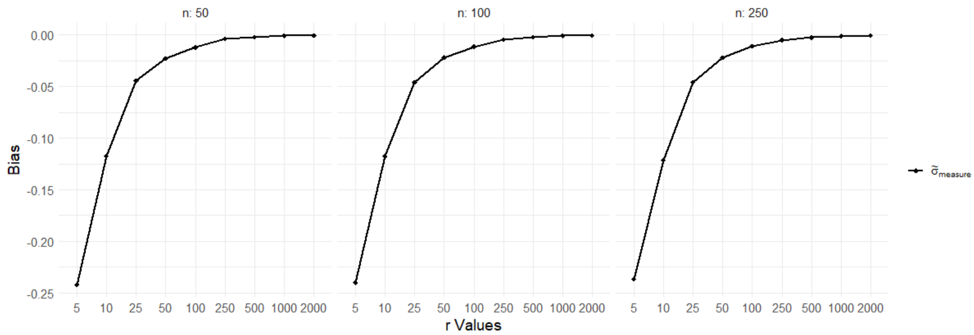


**Figure 8.** Bias values in $\widetilde{\sigma}_{measure}$ for different $n$ and $r$ values

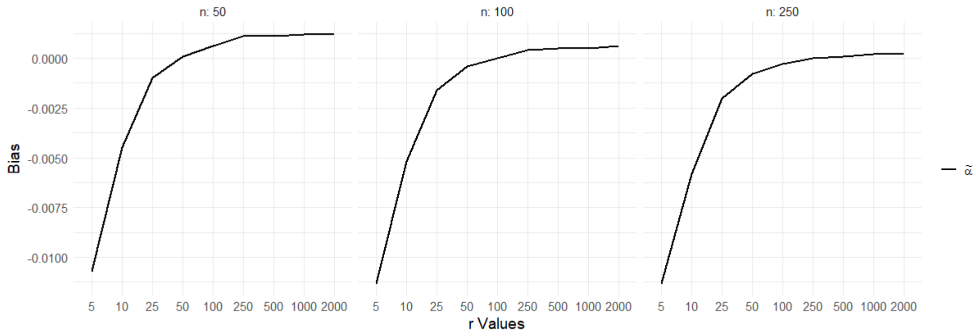


**Figure 9.** Bias values in $\widetilde{\alpha}$ for different $n$ and $r$ values

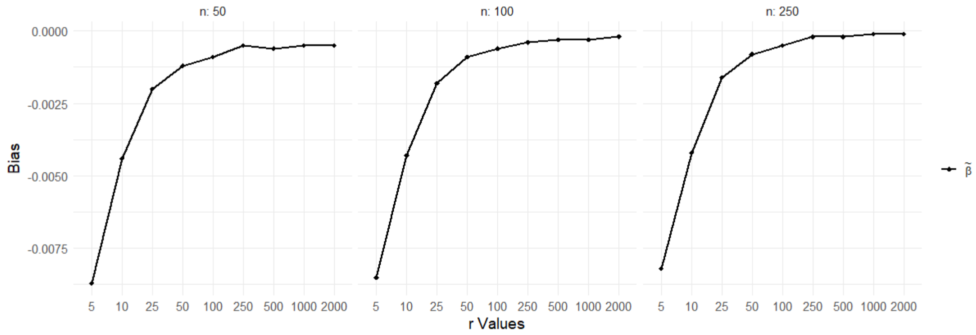


**Figure 10.** Bias values in $\widetilde{\beta}$ for different $n$ and $r$ values

According to Figures 1–10, several key observations can be made. Firstly, as expected, since $\mu_{part}$ is primarily identified from the $Y$ data, the inclusion of additional $Z$ observations does not significantly affect its estimation. As expected, the MSE and bias of all MLEs for each model parameter, as well as the MLEs of the misclassification probabilities $\alpha$ and $\beta$, tend to zero as both the sample size $n$ and the number of repeated measurements $r$ increase. Moreover, as $r$ increases (i.e., as the number of repeated measurements for the same subject increases), there is a positive effect on the MSEs of all estimators.

## 5. Illustrative example

In this section, a simulated data set is used for illustration purposes. This data set is generated assuming parts are distributed normally with parameters $\mu_{part} = 20$ and $\sigma_{part} = 0.3$, and measurement errors are assumed to be distributed normally with mean zero and standard deviation 0.05. Then, the measurements are generated using (2.1) and by fixing `set.seed(1)`. 20 parts are measured, with an extra $r = 10$ observations collected. The measurements ($Y$) and the additional observations ($Z$) are obtained as follows:

**Measurements ($Y$):**
19.8094, 20.0558, 19.7458, 20.4851, 20.1002, 19.7504, 20.1482, 20.2245, 20.1751, 19.9071, 20.4597, 20.1185, 19.8110, 19.3264, 20.3421, 19.9863, 19.9950, 20.2870, 20.2497, 20.1806.
**Extra Observations ($Z$):**

20.6479, 20.5748, 20.5993, 20.5773, 20.5111, 20.5592, 20.5602, 20.5770, 20.6350, 20.6181,

where $\mu_{nui} = 20.58$ is pre-determined. It is also assumed that the manufacturer has specified the specification limits as $L = 19.7$ and $U = 20.4$. Measurements including measurement errors are displayed in Figure 11, where the dashed lines indicate the lower and upper specification limits.

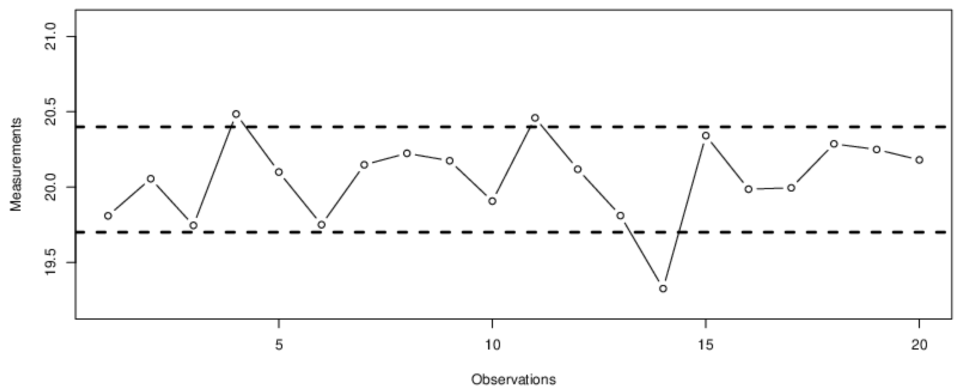


**Figure 11.** Measurements

The MLEs of model parameters and misclassification probabilities based on the above data are given in Table 2.

**Table 2.** MLEs of model parameters and misclassification probabilities for the numerical example

| Parameter | $n$ | $r$ | $\mu_{part}$ | $\sigma_{part}$ | $\sigma_{measure}$ | $\mu_{nui}$ | $\alpha$ | $\beta$ |
|---|---|---|---|---|---|---|---|---|
| True Values | 20 | - | 20 | 0.3 | 0.05 | 20.58 | 0.0302 | 0.0239 |
| $\widehat{\boldsymbol{\delta}_1}$ | 20 | - | 20.0579 | 0.1727 | 0.2085 | - | 0.1699 | 0.0165 |
| $\widetilde{\boldsymbol{\delta}_2}$ | 20 | 10 | 20.0579 | 0.2669 | 0.0453 | 20.5733 | 0.0262 | 0.0199 |

Table 2 illustrates that incorporating additional observations into the measurement system enhances the accuracy of estimating $\sigma_{part}$ and $\sigma_{measure}$, even in cases with small sample sizes, and consequently improves the estimation of the misclassification probabilities $\alpha$ and $\beta$.

## 6. Conclusion

MSA plays a critical role across various fields including manufacturing, engineering, and reliability of measurement systems. One of the main problems in MSA is the presence of measurement errors, which may lead to misclassification, specifically Type I and Type II errors. This study provides closed-form expressions for misclassification probabilities, which are derived based on one-dimensional and two-dimensional normal CDFs. The proposed analytical approach offers a computationally efficient alternative, especially in platforms like `R`. In addition to the classical sample, the study investigates the impact of repeated measurements by the same operator on the same part, highlighting the positive effects of these additional observations on the MLEs of misclassification probabilities in MSA. Overall, the findings indicate that incorporating repeated measurements via $Z$ leads to notable improvements in both the estimation of model parameters—by reducing MSE and bias—and the estimation of misclassification probabilities $\alpha$ and $\beta$, thereby enhancing the overall accuracy of the measurement system. Finally, a numerical illustration using simulated data confirms the practical applicability and effectiveness of the proposed approach. The primary limitation of this study is the assumption of normally distributed part variation and measurement errors, as well as the use of a single-operator measurement structure. In future studies, new estimators can be proposed, and their properties can be analyzed under the assumption that more than one operator is involved and that their measurements are dependent rather than independent.

## Author Contributions

All the authors equally contributed to this work. This paper is derived from the first author's doctoral dissertation supervised by the second author. They all read and approved the final version of the paper.

## Conflicts of Interest

All the authors declare no conflict of interest.

## Ethical Review and Approval

No approval from the Board of Ethics is required.

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