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On Geodesics of Warped Sasaki Metric

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Abstract
In this paper we establish a necessary and sufficient conditions under which a curve be a geodesic respect to the warped Sasaki metric.

Keywords: *Horizontal lift; vertical lift; warped Sasaki metric; geodesic.*

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1. Introduction

The main idea in this note consists in the modification of the Sasaki metric. First we introduce a new metric called warped Sasaki metric on the tangent bundle TM. This new natural metric will lead us to interesting results. Afterward we establish a necessary and sufficient conditions under which a curve be a geodesic with respect to the warped Sasaki metric.

2. Basic Notions and Definition on TM.

Horizontal and vertical lifts on TM.

Let (M, g) be an m-dimensional Riemannian manifold and (TM, π, M) be its tangent bundle. A local chart $(U, x^i)_{i=1\dots n}$ on M induces a local chart $(\pi^{-1}(U), x^i, y^i)_{i=1\dots n}$ on TM. Denote by Γ^k_{ij} the Christoffel symbols of g and by ∇ the Levi-Civita connection of g.
We have two complementary distributions on TM, the vertical distribution $\mathcal{V}$ and the horizontal distribution $\mathcal{H}$, defined by :

$$\begin{aligned} \mathcal{V}_{(x,u)} &= Ker(d\pi_{(x,u)}) = \{a^i \frac{\partial}{\partial y^i}|_{(x,u)}; \quad a^i \in \mathbb{R}\} \\ \mathcal{H}_{(x,u)} &= \{a^i \frac{\partial}{\partial x^i}|_{(x,u)} - a^i u^j \Gamma^k_{ij} \frac{\partial}{\partial y^k}|_{(x,u)}; \quad a^i \in \mathbb{R}\}, \end{aligned}$$

where $(x, u) \in TM$, such that $T_{(x,u)}TM = \mathcal{H}_{(x,u)} \oplus \mathcal{V}_{(x,u)}$.
Let $X = X^i \frac{\partial}{\partial x^i}$ be a local vector field on M. The vertical and the horizontal lifts of X are defined by

$$X^V = X^i \frac{\partial}{\partial y^i} \tag{2.1}$$

$$X^H = X^i \frac{\delta}{\delta x^i} = X^i \{\frac{\partial}{\partial x^i} - y^j \Gamma^k_{ij} \frac{\partial}{\partial y^k}\} \tag{2.2}$$

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For consequences, we have $(\frac{\partial}{\partial x^i})^H = \frac{\delta}{\delta x^i}$ and $(\frac{\partial}{\partial x^i})^V = \frac{\partial}{\partial y^i}$, then $(\frac{\delta}{\delta x^i}, \frac{\partial}{\partial y^i})_{i=1\ldots n}$ is a local adapted frame in TTM.

Remark 2.1. .

1. if $w = w^i \frac{\partial}{\partial x^i} + \overline{w}^j \frac{\partial}{\partial y^j} \in T_{(x,u)}TM$, then its horizontal and vertical parts are defined by

$$w^h = w^i \frac{\partial}{\partial x^i} - w^i u^j \Gamma^k_{ij} \frac{\partial}{\partial y^k} \in \mathcal{H}_{(x,u)}$$

$$w^v = \{\overline{w}^k + w^i u^j \Gamma^k_{ij}\} \frac{\partial}{\partial y^k} \in \mathcal{V}_{(x,u)}$$

2. if $u = u^i \frac{\partial}{\partial x^i} \in T_x M$ then its vertical and horizontal lifts are defined by

$$\begin{aligned} u^V &= u^i \frac{\partial}{\partial y^i} \in \mathcal{V}_{(x,u)} \in \mathcal{H}_{(x,u)} \\ u^H &= u^i \{\frac{\partial}{\partial x^i} - y^j \Gamma^k_{ij} \frac{\partial}{\partial y^k}\}. \end{aligned}$$

Proposition 2.1 ([16]). *Let (M, g) be a Riemannian manifold and R its curvature tensor, then for all vector fields $X, Y \in \Gamma(TM)$ and $p \in T^2 M$ we have:*

1. $[X^H, Y^H]_p = [X, Y]^H_p - (R_x(X, Y)u)^V$,
2. $[X^H, Y^V]_p = (\nabla_X Y)^V_p$,
3. $[X^V, Y^V]_p = 0$,

where $p = (x, u)$.

3. Warped Sasaki metric.

Warped Sasaki metric.

Definition 3.1. Let (M, g) be a Riemannian manifold and $f : M \times \mathbb{R} \to]0, +\infty[$ be a smooth function. On the tangent bundle TM we define a warped Saski metric noted g^S_f by

1. $g^S_f(X^H, Y^H)_{(x,u)} = g_x(X, Y)$
2. $g^S_f(X^H, Y^V)_{(x,u)} = g^S_f(X^V, Y^H)_{(x,u)} = 0$
3. $g^S_f(X^V, Y^V)_{(x,u)} = f(x, r) g_x(X, Y)$

where $X, Y \in \Gamma(TM)$, $(x, u) \in TM$ and $r = g(u, u)$. f is called warping function.

Note that, if $f = 1$ then g^S_f is the Sasaki metric [16].

The notion of Sasaki metric and Gromol-Chegeer metric was considered in [1], [12], [13], [14], [15], [16].

Lemma 3.1. *Let (M,g) be a Riemannian manifold, then for all $x \in M$ and $u = u^i \frac{\partial}{\partial x^i} \in T_x M$, we have the following*

1. $X^H(g(u, u))_{(x,u)} = 0$
2. $X^H(g(Y, u))_{(x,u)} = g(\nabla_X Y, u)_x$

3. $X^V(g(u,u)_{(x,u)} = 2g(X,u)_x$
4. $X^V(g(Y,u)_{(x,u)} = g(X,Y)_x$

Proof. Localy, if $U : x \in M \to U_x = u^i \frac{\partial}{\partial x^i} \in TM$ be a local vector field constant on each fiber T_xM, then from formulas (2.1) and (2.2) we obtain :

$$\begin{aligned}
1.\quad X^H(g(u,u))_{(x,u)} &= [X^i \frac{\partial}{\partial x^i} g_{st} y^s y^t - \Gamma^k_{ij} X^i y^j \frac{\partial}{\partial y^k} g_{st} y^s y^t]_{(x,u)} \\
&= X(g(U,U)_x - 2(\Gamma^k_{ij} X^i y^j g_{sk} y^s)_{(x,u)} \\
&= (X(g(U,U)_x - 2g(U,\nabla_X U))_x \\
&= 0. \\
2.\quad X^H(g(Y,u))_{(x,u)} &= [X^i \frac{\partial}{\partial x^i} g_{st} Y^s y^t - \Gamma^k_{ij} X^i y^j \frac{\partial}{\partial y^k} g_{st} Y^s y^t]_{(x,u)} \\
&= X(g(Y,U)_x - (\Gamma^k_{ij} X^i y^j g_{sk} Y^s)_{(x,u)} \\
&= (X(g(Y,U)_x - g(Y,\nabla_X U))_x \\
&= g(\nabla_X Y, U))_x. \\
3.\quad X^V(g(u,u))_{(x,u)} &= [X^i \frac{\partial}{\partial y^i} g_{st} y^s y^t]_{(x,u)} = 2X^i g_{it} u^t = 2g(X,u)_x \\
4.\quad X^V(g(Y,u))_{(x,u)} &= [X^i \frac{\partial}{\partial y^i} g_{st} Y^s y^t]_{(x,u)} = X^i g_{si} Y^s = g(X,Y)_x
\end{aligned}$$

□

From Lemma 3.1, we obtain

Lemma 3.2. *Let (M,g) be a Riemannian manifold,* $F : (s,t) \in \mathbb{R}^2 \to F(s,t) \in]0,+\infty[$*,* $\alpha : M \to]0,+\infty[$ *and* $\beta : \mathbb{R} \to]0,+\infty[$ *be smooth functions. If* $f(x,r) = F(\alpha(x),\beta(r))$*, then we have the following*

1. $X^V(f)_{(x,u)} = 2\beta'(r) g_x(X,u) \frac{\partial F}{\partial t}(\alpha(x),\beta(r))$
2. $X^H(f)_{(x,u)} = g_x(grad_M \alpha, X) \frac{\partial F}{\partial s}(\alpha(x),\beta(r))$

where $(x,u) \in TM$ *and* $r = g_x(u,u)$ *.*

In the following, we consider $f(x,r) = F(\alpha(x),\beta(r))$, where $F : (s,t) \in \mathbb{R}^2 \to F(s,t) \in]0,+\infty[$, $\alpha : M \to]0,+\infty[$ and $\beta : \mathbb{R} \to]0,+\infty[$ are smooth functions.

Theorem 3.1. *Let* (M,g) *be a Riemannian manifold. If* $f(x,r) = F(\alpha(x),\beta(r))$ *and* ∇ *(resp* $\widetilde{\nabla}$*) denote the Levi-Civita connection of* (M,g) *(resp* (TM, g^S_f)*), then we have:*

$$\begin{aligned}
1.\quad (\widetilde{\nabla}_{X^H} Y^H)_p &= (\nabla_X Y)^H_p - \frac{1}{2}(R_x(X,Y)u)^V, \\
2.\quad (\widetilde{\nabla}_{X^H} Y^V)_p &= (\nabla_X Y)^V_p + \frac{f(x,r)}{2}(R_x(u,Y)X)^H \\
&\quad + \frac{1}{2f(x,r)} g_x(grad_M \alpha, X) \frac{\partial F}{\partial s}(\alpha(x),\beta(r)) Y^V_p \\
3.\quad (\widetilde{\nabla}_{X^V} Y^H)_p &= \frac{f(x,r)}{2}(R_x(u,X)Y))^H + \frac{1}{2f(x,r)} g_x(grad_M \alpha, Y) \frac{\partial F}{\partial s}(\alpha(x),\beta(r)) X^V_p \\
4.\quad (\widetilde{\nabla}_{X^V} Y^V)_p &= \frac{\beta'(r)}{f(x,r)} \frac{\partial F}{\partial t}(\alpha(x),\beta(r)) \Big[g_x(Y,U) X^V_p) + g_x(X,U) Y^V_p) - g_x(X,Y) U^V_p \Big] \\
&\quad - \frac{1}{2} g_x(X,Y) \frac{\partial F}{\partial t}(\alpha(x),\beta(r)) (grad_M \alpha)^H_p.
\end{aligned}$$

for all vector fields $X, Y \in \Gamma(TM)$ *and* $p = (x,u) \in TM$*, where* R *denote the curvature tensor of (M,g).*

The proof of Theorem 3.1 follows directly from Kozul formula, Lemma 3.1 and Lemma 3.2.

Lemma 3.3. *Let (M,g) be a Riemannian manifold . If $X,Y\in\Gamma(TM)$ are vector fields and $(x,u)\in TM$ such that $X_x=u$, then we have*

$$d_xX(Y_x)=Y^H_{(x,u)}+(\nabla_YX)^V_{(x,u)}.$$

Proof. Let (U,x^i) be a local chart on M in $x\in M$ and $(\pi^{-1}(U),x^i,y^j)$ be the induced chart on TM, if $X_x=X^i(x)\frac{\partial}{\partial x^i}|_x$ and $Y_x=Y^i(x)\frac{\partial}{\partial x^i}|_x$, then

$$d_xX(Y_x)=Y^i(x)\frac{\partial}{\partial x^i}|_{(x,X_x)}+Y^i(x)\frac{\partial X^k}{\partial x^i}(x)\frac{\partial}{\partial y^k}|_{(x,X_x)},$$

thus the horizontal part is given by

$$\begin{aligned}(d_xX(Y_x))^h &= Y^i(x)\frac{\partial}{\partial x^i}|_{(x,X_x)}-Y^i(x)X^j(x)\Gamma^k_{ij}(x)\frac{\partial}{\partial y^k}|_{(x,X_x)}\\ &= Y^H_{(x,X_x)}\end{aligned}$$

and the vertical part is given by

$$\begin{aligned}(d_xX(Y_x))^v &= \{Y^i(x)\frac{\partial X^k}{\partial x^i}(x)+Y^i(x)X^j(x)\Gamma^k_{ij}(x)\}\frac{\partial}{\partial y^k}|_{(x,X_x)}\\ &= (\nabla_YX)^V_{(x,X_x)}.\end{aligned}$$

□

4. Geodesics of warped Sasaki metric

Lemma 4.1.
Let (M,g) be a Riemannian manifold and $x:I\to M$ be a curve on M. If $C:t\in I\to C(\tau)=(x(t),y(t))\in TM$ is a curve in TM such $y(t)\in T_{x(t)}M$ (i.e. $y(t)$ is a vector field along $x(t)$), then

$$\dot{C}=\dot{x}^H+(\nabla_{\dot{x}}y)^V \tag{4.1}$$

Proof. Locally, If $Y\in\Gamma(TM)$ is a vector field such $Y(x(t))=y(t)$ then we have

$$\dot{C}(t)=dC(t)=dY(x(t))$$

Using Lemma 3.3 we obtain

$$\dot{C}(t)=dY(x(t))=\dot{x}^H+(\nabla_{\dot{x}}y)^V$$

□

Theorem 4.1.
Let (M,g) be a Riemannian manifold and (TM,g^S_f) its tangent bundle equipped with the warped Sasaki metric. If $f(x,r)=F(\alpha(x),\beta(r))$ and $C(t)=(x(t),y(t))$ is curve on TM such $y(t)$ is a vector field along $x(t)$), then

$$\begin{aligned}\widetilde{\nabla}_{\dot{C}}\dot{C} &= \Big[\nabla_{\dot{x}}\dot{x}+fR(y,\nabla_{\dot{x}}y)\dot{x}-\frac{1}{2}\|\nabla_{\dot{x}}y\|^2\frac{\partial F}{\partial s}grad_M\alpha\Big]^H\\ &+ \Big[\nabla_{\dot{x}}\nabla_{\dot{x}}y+\big[\dot{x}(\alpha)\frac{\partial\ln F}{\partial s}+2\beta'\frac{\partial\ln F}{\partial t}g(\nabla_{\dot{x}}y,y)\big]\nabla_{\dot{x}}y-\beta'\frac{\partial\ln F}{\partial t}\|\nabla_{\dot{x}}y\|^2y\Big]^V\end{aligned} \tag{4.2}$$

Proof.

We have

$$\begin{aligned}
\widetilde{\nabla}_{\dot{C}}\dot{C} &= \widetilde{\nabla}_{[\dot{x}^H + (\nabla_{\dot{x}}y)^V]}[\dot{x}^H + (\nabla_{\dot{x}}y)^V] \\
&= \widetilde{\nabla}_{\dot{x}^H}\dot{x}^H + \widetilde{\nabla}_{\dot{x}^H}(\nabla_{\dot{x}}y)^V + \widetilde{\nabla}_{(\nabla_{\dot{x}}y)^V}\dot{x}^H + \widetilde{\nabla}_{(\nabla_{\dot{x}}y)^V}(\nabla_{\dot{x}}y)^V \\
&= (\nabla_{\dot{x}}\dot{x})^H - \frac{1}{2}(R(\dot{x},\dot{x})y)^V + (\nabla_{\dot{x}}\nabla_{\dot{x}}y)^V + \frac{f}{2}(R(y,\nabla_{\dot{x}}y)\dot{x})^H + \frac{1}{2}\dot{x}(\alpha)\frac{\partial \ln F}{\partial s}(\nabla_{\dot{x}}y)^V \\
&\quad + \frac{f}{2}(R(y,\nabla_{\dot{x}}y)\dot{x})^H + \frac{1}{2}\dot{x}(\alpha)\frac{\partial \ln F}{\partial s}(\nabla_{\dot{x}}y)^V \\
&\quad + \beta'\frac{\partial \ln F}{\partial t}\Big[g(\nabla_{\dot{x}}y,y)(\nabla_{\dot{x}}y)^V + g(\nabla_{\dot{x}}y,y)(\nabla_{\dot{x}}y)^V - g(\nabla_{\dot{x}}y,\nabla_{\dot{x}}y)y^V\Big] \\
&\quad - \frac{1}{2}g(\nabla_{\dot{x}}y,\nabla_{\dot{x}}y)\frac{\partial F}{\partial s}(grad_M\alpha)^H
\end{aligned}$$

$$\begin{aligned}
\widetilde{\nabla}_{\dot{C}}\dot{C} &= \Big[\nabla_{\dot{x}}\dot{x} + fR(y,\nabla_{\dot{x}}y)\dot{x} - \frac{1}{2}\|\nabla_{\dot{x}}y\|^2\frac{\partial F}{\partial s}grad_M\alpha\Big]^H \\
&\quad + \Big[\nabla_{\dot{x}}\nabla_{\dot{x}}y + \dot{x}(\alpha)\frac{\partial \ln F}{\partial s}\nabla_{\dot{x}}y + \beta'\frac{\partial \ln F}{\partial t}\big[2g(\nabla_{\dot{x}}y,y)\nabla_{\dot{x}}y - \|\nabla_{\dot{x}}y\|^2 y\big]\Big]^V \\
&= \Big[\nabla_{\dot{x}}\dot{x} + fR(y,\nabla_{\dot{x}}y)\dot{x} - \frac{1}{2}\|\nabla_{\dot{x}}y\|^2\frac{\partial F}{\partial s}grad_M\alpha\Big]^H \\
&\quad + \Big[\nabla_{\dot{x}}\nabla_{\dot{x}}y + \big[\dot{x}(\alpha)\frac{\partial \ln F}{\partial s} + 2\beta'\frac{\partial \ln F}{\partial t}g(\nabla_{\dot{x}}y,y)\big]\nabla_{\dot{x}}y - \beta'\frac{\partial \ln F}{\partial t}\|\nabla_{\dot{x}}y\|^2 y\Big]^V
\end{aligned}$$

□

From the Theorem 4.1 we obtain

Theorem 4.2.
Let (M,g) be a Riemannian manifold and (TM, g_f^S) its tangent bundle equipped with the warped Sasaki metric. If $f(x,r) = F(\alpha(x),\beta(r))$ and $C(t) = (x(t),y(t))$ is curve on TM such $y(t)$ is a vector field along $x(t)$), then C is a geodesic on TM if and only if

$$\begin{cases}
\nabla_{\dot{x}}\dot{x} = \frac{1}{2}\|\nabla_{\dot{x}}y\|^2\frac{\partial F}{\partial s}grad_M\alpha - fR(y,\nabla_{\dot{x}}y)\dot{x} \\
\nabla_{\dot{x}}\nabla_{\dot{x}}y = \beta'\frac{\partial \ln F}{\partial t}\|\nabla_{\dot{x}}y\|^2 y - \big[\dot{x}(\alpha)\frac{\partial \ln F}{\partial s} + 2\beta'\frac{\partial \ln F}{\partial t}g(\nabla_{\dot{x}}y,y)\big]\nabla_{\dot{x}}y
\end{cases} \tag{4.3}$$

Definition 4.1 ([16])**.** Let (M,g) be a Riemannian manifold and (TM, g_f^S) its tangent bundle equipped with the warped Sasaki metric. A curve $C(t) = (x(t),y(t))$ is said to be a horizontal lift of the curve $x(t)$ if and only if $\nabla_{\dot{x}}y = 0$.

Definition 4.2 ([16] [15])**.** Let (M,g) be a Riemannian manifold and (TM, g_f^S) its tangent bundle equipped with the warped Sasaki metric. If $x(t)$ is a curve on (M,g), then the curve $C(t) = (x(t),\dot{x}(t))$ is called the natural lift of curve $x(t)$.

Using Theorem 4.2 we deduce:

Corollary 4.1. *Let (M,g) be a Riemannian manifold and (TM, g_f^S) its tangent bundle equipped with the warped Sasaki metric. The natural lift $C(t) = (x(t),\dot{x}(t))$ of any geodesic $x(t)$ on (M,g) is a geodesic on (TM, g_f^S).*

Corollary 4.2. *Let (M^m,g) be a Riemannian manifold, (TM, g_f^S) its tangent bundle equipped with the warped Sasaki metric and $C(t) = (x(t),y(t))$ be a horizontal lift of the curve $x(t)$ (i.e. $\nabla_{\dot{x}}y = 0$). Then $C(t)$ is a geodesic on (TM, g_f^S) if and only if $x(t)$ is a geodesic on (M,g).*

Remark 4.1. If $C(t) = (x(t), y(t))$ is a horizontal lift of the curve $x(t)$ then locally we have

$$\begin{aligned} \nabla_{\dot{x}} y = 0 \quad & \Leftrightarrow \quad \frac{dy^s}{dt} + \Gamma^s_{ij} y^i \frac{dx^j}{dt} = 0 \\ & \Leftrightarrow \quad y(t) = e^{-A(t)} K \end{aligned}$$

where $K \in \mathbb{R}^m$ and $A(t) = (\Gamma^s_{ij} \frac{dx^j}{dt})_{s,i}$

Remark 4.2. Using the Remark 4.1 we can construct an infinity of examples of geodesics on (TM, g^S_f).

Example 4.1. Condider the upper half-plane

$$\mathbb{R}^2_+ = \Big\{(x, y) \in \mathbb{R}^2 \quad ; \quad y > 0\Big\}\}$$

with the metric of Lobatchevski's non-euclidean geometry given by

$$g_{11} = g_{22} = \frac{1}{y} \quad , \quad g_{12} = g_{21} = 0.$$

The Christoffel symbols of the Riemannian connection are given by

$$\Gamma^1_{11} = \Gamma^1_{22} = \Gamma^2_{12} = \Gamma^2_{21} = 0, \quad \Gamma^2_{11} = \frac{1}{y}, \quad \Gamma^2_{22} = \Gamma^1_{12} = \Gamma^1_{21} = -\frac{1}{y}.$$

1. If $C(t) = (x_0, y(t), u(t), v(t))$ is horizontal lift of the curve $(x_0, y(t))$, then the matrix $A(t)$ is given by

   $$A(t) = -\frac{1}{y} \begin{pmatrix} \frac{dy}{dt} & 0 \\ 0 & \frac{dy}{dt} \end{pmatrix}$$

   and

   $$C(t) = (x_0, y(t), k_1 y(t), k_2 y(t))$$

2. If $C(t) = (x(t), y(t), u(t), v(t))$ is horizontal lift of the curve $(x(t), y(t))$ such $y(t) = ax(t) + b$ and $x \neq 0$, then the matrix $A(t)$ is given by

   $$A(t) = -\frac{dx}{(ax(t) + b)dt} \begin{pmatrix} a & -1 \\ 1 & a \end{pmatrix}$$

   and

   $$C(t) = \Big(x(t), y(t), \exp\Big[\ln(y(t)) \begin{pmatrix} a & -1 \\ 1 & a \end{pmatrix}\Big]K\Big)$$

where $K \in \mathbb{R}^2$

Example 4.2. Let $\mathbb{R}^2$ equipped with the Riemannian metric in polar coordinate defined by :

$$g = dr^2 + h(r, \theta)^2 d\theta^2$$

Relatively to the orthonormal frame

$$e_r = \frac{\partial}{\partial r}, \quad e_\theta = \frac{1}{h(r, \theta)} \frac{\partial}{\partial \theta}$$

we have

$$\nabla_{e_r} e_r = \nabla_{e_r} e_\theta = 0, \quad \nabla_{e_\theta} e_r = \frac{1}{h} \frac{\partial h}{\partial r} e_\theta, \quad \nabla_{e_\theta} e_\theta = -\frac{1}{h} \frac{\partial h}{\partial r} e_r.$$

and the matrix of Levi-Civita connection relatively to the orthonormal frame (e_r, e_θ) is given by

$$\Gamma = \begin{pmatrix} 0 & -\frac{\partial h}{\partial r} d\theta \\ \frac{\partial h}{\partial r} d\theta & 0 \end{pmatrix}$$

If $C(t)=(r(t),\theta(t),u(t),v(t))$ is horizontal lift of the curve $(r(t),\theta(t))$ then matrix A relatively to the orthonormal frame (e_r,e_θ) is given by

$$A=\begin{pmatrix} 0 & -\frac{1}{h}\frac{\partial h}{\partial r}\frac{d\theta}{dt} \\ \frac{1}{h}\frac{\partial h}{\partial r}\frac{d\theta}{dt} & 0 \end{pmatrix}$$

and

$$\begin{cases} u(t)=k_1\cos(\int\frac{1}{h}\frac{\partial h}{\partial r}\frac{d\theta}{dt})+k_2\sin(\int\frac{1}{h}\frac{\partial h}{\partial r}\frac{d\theta}{dt}), \\ v(t)=-k_1\sin(\int\frac{1}{h}\frac{\partial h}{\partial r}\frac{d\theta}{dt})+k_2\cos(\int\frac{1}{h}\frac{\partial h}{\partial r}\frac{d\theta}{dt}), \end{cases}$$

Theorem 4.3.
Let (M,g) be a Riemannian manifold and (TM,g_f^S) its tangent bundle equipped with the warped Sasaki metric and $x(t)$ be a geodesic on M. If $f(x,r)=F(\alpha(x),\beta(r))$ and $C=(x(t),y(t))$ is a geodesic on TM such $\nabla_{\dot{x}}y\neq 0$ then

$$\dot{x}(\alpha)\frac{\partial\ln F}{\partial s}\Big(\alpha(x(t)),\beta(r(t))\Big) \quad = \quad 0 \tag{4.4}$$

where $r(t)=g_{x(t)}(y(t),y(t))$.

Proof.
Let $x(t)$ be a geodesic on M then $\nabla_{\dot{x}}\dot{x}=0$. Using the first equation of formula (4.3) we obtain

$$\begin{aligned} g(\nabla_{\dot{x}}\dot{x},\dot{x})=0 \quad &\Rightarrow \quad \frac{1}{2}\|\nabla_{\dot{x}}y\|^2\frac{\partial F}{\partial s}g(grad_M\alpha,\dot{x})-fg\big(R(y,\nabla_{\dot{x}}y)\dot{x},\dot{x}\big)=0 \\ &\Rightarrow \quad \frac{1}{2}\|\nabla_{\dot{x}}y\|^2\dot{x}(\alpha)\frac{\partial F}{\partial s}\Big(\alpha(x(t)),\beta(r(t))\Big)=0 \\ &\Rightarrow \quad \dot{x}(\alpha)\frac{\partial F}{\partial s}\Big(\alpha(x(t)),\beta(r(t))\Big)=0 \end{aligned}$$

□

Corollary 4.3. *Let (M,g) be a Riemannian manifold and (TM,g_f^S) its tangent bundle equipped with the warped Sasaki metric, $f(x,r)=\alpha(x)$ and $x(t)$ be a geodesic on M. If the curve $C=(x(t),y(t))$ is a geodesic on TM such $\nabla_{\dot{x}}y\neq 0$, then f is a constant along the curve $x(t)$.*

The proof follows directly from Theorem 4.3.

Corollary 4.4.
Let (M,g) be a Riemannian manifold and (TM,g_f^S) its tangent bundle equipped with the warped Sasaki metric, $x(t)$ be a geodesic on M and f be a constant along the curve $x(t)$. If the curve $C=(x(t),y(t))$ is a geodesic on TM such $\nabla_{\dot{x}}y\neq 0$ then $\nabla_{\dot{x}}\nabla_{\dot{x}}y=0$.

The proof follows directly from Theorem 4.3 and Theorem 4.2.

Corollary 4.5. *Let (M,g) be a Riemannian manifold, (TM,g_f^S) its tangent bundle equipped with the warped Sasaki metric and $f(x,r)=f(x)=\alpha(x)$ be a constant along the curve $x(t)$. Then the curve $C=(x(t),y(t))$ is a geodesic on TM such $\nabla_{\dot{x}}y\neq 0$ if and only if we have*

$$\begin{cases} \nabla_{\dot{x}}\dot{x} &= f(x)R(\nabla_{\dot{x}}y,y)\dot{x} \\ \nabla_{\dot{x}}\nabla_{\dot{x}}y &= 0. \end{cases} \tag{4.5}$$

Corollary 4.6. *Let (M,g) be a flat Riemannian manifold, (TM,g_f^S) its tangent bundle equipped with the warped Sasaki metric and $f(x,r)=f(x)=\alpha(x)$ be a constant along the curve $x(t)$. Then the curve $C=(x(t),y(t))$ is a geodesic on TM such $\nabla_{\dot{x}}y\neq 0$ if and only if $x(t)$ is a geodesic on M and*

$$\nabla_{\dot{x}}\nabla_{\dot{x}}y=0.$$

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