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Para Meta-Golden Statistical Manifolds with Dual Semi-Conjugate Connection

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Abstract

In this paper, a new class of statistical manifolds, called meta-golden statistical manifolds, is introduced, and the geometry of these manifolds equipped with a dual semi-conjugate connection is investigated.

Keywords: *Statistical Manifolds, Meta-Golden Structures, Dual Semi-Conjugate Connections*
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1. Introduction

During the 1980s, the concept of statistical structure was introduced, marking the inception of its pivotal role in the establishment of a novel research domain known as information geometry which investigates probability and information through the lens of differential geometry. Within the realm of probability measures, a specific Riemannian metric, known as the Fisher metric, and a set of affine connections naturally emerge.

Statistical manifolds serve as geometric abstractions employed to represent information, falling within the domain of Information Geometry. This field utilizes the tools of differential geometry to investigate statistical inference, information loss, and estimation. Let M be a manifold with non-degenerate metric g and a torsion free affine connection ∇. If ∇g is symmetric, the triple (M, g, ∇) is denoted as a statistical manifold. On this statistical manifold, we introduce another connection known as the conjugate connection. The exploration and analysis of this concept have been extensively conducted within the framework of information geometry ([1],[23]). A substantial body of work has been dedicated to the study of statistical manifolds ([1], [5], [19], [23], [24]).

The Golden Ratio (also called divine ratio, golden mean, golden proportion or golden section) is known by people from ancient times. Its first definition is given by Euclid. Golden ratio is a proportion that a line segment into a major subsegment and minor subsegment in a such way that both the ratio of whole segment and major subsegment and the ratio of major subsegment and minor subsegment must be equal the number ϕ. The number $\phi = \frac{1+\sqrt{5}}{2} \simeq 1.618...$ is the positive root of the equation $x^2 - x - 1 = 0$. The Golden Ratio is used in a number of departments such that art, architecture, aesthetic e.c. So the objects having the golden ratio are searched in nature by people. Logarithmic spiral was thought one of them. But Bartlett [2] who is art professor (Towson University, USA) showed that it is a perfect example of the objects having Meta golden ratio χ which is the letter following ϕ in the Greek alphabet. Huylebrouck [15] focus on geometric properties of Meta-Golden Chi ratio.

A number of researchers introduce geometric properties of geometric structures endowed with Golden structure ([6], [8], [11], [12], [13], [14], [20]), Silver Structure [21] or Meta-Golden structure [22]. Hretcanu [14] present a generalization of whole metallic structures. Some researchers apply this ambiance to statistical manifolds by inspiring this ideas ([3], [17], [19]).

In this paper, we investigate Para Meta-Golden structures on statistical manifolds. By introducing a new connection, known as the dual semi-conjugate connection, for an arbitrary 1-form φ, we first demonstrate that this new connection ensures the satisfaction of statistical structure conditions. Subsequently, we derive some geometrical properties of Meta-Golden statistical manifolds for this connection.

2. Preliminaries

Let ∇^g be Levi-Civita connection and ∇ be a torsion free affine connection on (M, g) Riemannian manifold. $(0,3)$ type totally symmetric tensor field C satisfying

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$$C(X,Y,Z) = (\nabla_X g)(Y,Z) \tag{2.1}$$

for $\forall X,Y,Z \in \Gamma(T(M))$ is called cubic form. And the pair (∇, g) is named statistical structure. So (M,g) Riemannian manifold having a (∇, g) statistical structure is called statistical manifold and showed with (M,g,∇). The dual of ∇ is showed $\nabla^*$and satisfying

$$Xg(Y,Z) = g(\nabla_X Y, Z) + g(Y, \nabla^*_X Z) \tag{2.2}$$

for $\forall X,Y,Z \in \Gamma(T(M))$ and $(\nabla^*)^* = \nabla$. It is easily see that the cubic form C is provided

$$C(X,Y,Z) = -(\nabla^*_X g)(Y,Z) \tag{2.3}$$

for any $X,Y,Z \in \Gamma(T(M))$ and if (∇, g) is an statistical structure then (∇^*, g) is so.

Definition 2.1. *Let (M,g,∇) be a statistical manifold. $(1,2)$ type tensor field K is given by*

$$K(X,Y) = \nabla_X Y - \nabla^g_X Y \tag{2.4}$$

and called difference tensor. By writing $K_X Y$ instead of $K(X,Y)$ we obtain

$$K_X = \nabla_X - \nabla^g_X \tag{2.5}$$

and so K_X determined a $(1,1)$ tensor field.

Since ∇ and ∇^g are torsion free, K is symmetric and we have

$$g(K_X Y, Z) = g(K_X Z, Y), \qquad K_X Y = K_Y X$$

for $\forall X,Y,Z \in \Gamma(T(M))$. Because of ∇^g Levi-Civita connection is metric compatible (i.e. $\nabla^g g = 0$) we have

$$(\nabla_X g)(Y,Z) = (K_X g)(Y,Z)$$

and so we easily obtain following equation

$$(K_X g)(Y,Z) = -g(K_X Y, Z) - g(Y, K_X Z). \tag{2.6}$$

From (2.1) we have

$$C(X,Y,Z) = -2g(K_X Y, Z). \tag{2.7}$$

Finally using (2.3) we can have

$$K_X = -\nabla^*_X + \nabla^g_X \tag{2.8}$$

and using (2.8) and (2.5) we also have

$$\nabla^g_X Y = \frac{1}{2}(\nabla_X Y + \nabla^*_X Y) \qquad \text{and} \qquad K_X = \frac{1}{2}(\nabla_X - \nabla^*_X). \tag{2.9}$$

Definition 2.2. *[3] The triple (M,g,β) consisting of a semi-Riemannian manifold endowed with an endomorphism $\beta \in T^1_1$ is called para-golden Riemannian manifold if*

$$\beta^2 = \beta + I, \quad g(\beta X, Y) + g(X, \beta Y) = 0 \tag{2.10}$$

for any $X,Y,Z \in \Gamma(T(M))$, where I is identity map. The endomorphism β satisfying (2.10) is called golden structure.

We note that if β golden structure is symmetric i.e. $g(\beta X, Y) = g(X, \beta Y)$ then (M,g,β) is called *golden Riemannian manifold*. As in the almost para-Hermitian geometry, we consider the w tensor field

$$w(X,Y) = g(X, \beta Y) \tag{2.11}$$

which is skew-symmetric and called fundamental form.

Definition 2.3. *[22] Let (M,g,β) be golden Riemannian manifold. If there is an endomorphism τ on M which is satisfy*

$$\beta\tau^2 = \beta + \tau \tag{2.12}$$

then τ and (M,g,β,τ) are called a Meta-Golden structure and an almost Meta-Golden manifold, respectively.

Theorem 2.4. *[22] Let (M,g,β) be golden Riemannian manifold and τ be an endomorphism on M. τ is an Meta golden structure if and only if*

$$\tau^2 = \beta\tau - \tau + I \tag{2.13}$$

where I denotes the identity map.

Definition 2.5. *[22] Nijenhuis tensor field of τ Meta-Golden structure is given by*

$$N_\tau(X,Y) = \tau^2[X,Y] + [\tau X, \tau Y] - \tau[\tau X, Y] - \tau[X, \tau Y] \tag{2.14}$$

for $\forall X,Y \in \Gamma(T(M))$.

Lemma 2.6. *[22] Let (M,g,β,τ) be a almost Meta-Golden manifold. Nijenhuis tensor field of τ is given by*

$$N_\tau(X,Y) = (\nabla_{\tau X}\tau)Y - (\nabla_{\tau Y}\tau)X + \tau(\nabla_Y \tau)X - \tau(\nabla_X \tau)Y \tag{2.15}$$

for $\forall X,Y \in \Gamma(T(M))$.

If $N_\tau = 0$, then τ Meta golden structure is named integrable and (M,g,β,τ) is called Meta-Golden manifold.

Corollary 2.7. *[22] Let (M,g,β,τ) be an almost Meta-Golden manifold. If $\nabla\tau = 0$, in this case τ Meta-Golden structure is integrable and (M,g,β,τ) is called Meta-Golden manifold.*

Definition 2.8. *[5] Let ∇ and $\acute{\nabla}$ be two linear connections on (M,g) semi-Riemannian manifold. For ρ $1-$form,*

i. If $\acute{\nabla}_X Y = \nabla_X Y + \rho(X)Y + X\rho(Y)$ then ∇ and $\acute{\nabla}$ is called projectively equivalent
ii. If $\acute{\nabla}_X Y = \nabla_X Y - g(X,Y)\rho^\#$ then and $\acute{\nabla}$ is called dual-projectively equivalent
where $\rho^\#$ is the vector field g-dual to ρ i.e. $\rho(X) = g(X,\rho^\#)$

Proposition 2.9. *[3] Let (M,g,∇) be a statistical manifold being* $\dim M \geqslant 2$ *and ∇^* its conjugate connection. Then ∇ and ∇^* are neither projectively equivalent nor dual-projectively equivalent.*

Definition 2.10. *[5] Let (M,g) be a semi-Riemannian manifold, ∇ be an affine connection on M and φ be a 1-form on M.*

i. The generalized conjugate connection $\overset{\sim *}{\nabla}$ of ∇ with respect to g by φ is defined by

$$Xg(Y,Z) = g(\nabla_X Y, Z) + g\left(Y, \overset{\sim *}{\nabla}_X Z\right) - \varphi(X)g(Y,Z)$$

ii. The semi-conjugate connection $\overset{\approx *}{\nabla}$ of ∇ with respect to g by φ is defined by

$$Xg(Y,Z) = g(\nabla_X Y, Z) + g\left(Y, \overset{\approx *}{\nabla}_X Z\right) + \varphi(Z)g(X,Y)$$

iii. The dual semi-conjugate connection $\overset{\backsimeq *}{\nabla}$ of ∇ with respect to g by φ is defined by

$$Xg(Y,Z) = g(\nabla_X Y, Z) + g\left(Y, \overset{\backsimeq *}{\nabla}_X Z\right) - \varphi(X)g(Y,Z) - \varphi(Y)g(X,Z)$$

Proposition 2.11. *[5] Let (M,g) be a semi-Riemannian manifold, ∇ be an affine connection on M and ∇^* be standard conjugate of ∇. Suppose that ∇' affine connection be dual-projectively equivalent to ∇. Then*

i. The generalized conjugate connection $\overset{\sim *}{\nabla'}$ of ∇' is projectively equivalent to ∇^*.
ii. The semi-conjugate connection $\overset{\approx *}{\nabla'}$ of ∇' is congruent with ∇^* e.g. $\nabla^* = \overset{\approx *}{\nabla'}$.
iii. The dual semi-conjugate connection $\overset{\backsimeq *}{\nabla'}$ of ∇' is given by

$$\overset{\backsimeq *}{\nabla'}_X Y = \nabla^*_X Y + g(X,Y)\varphi^\# + \varphi(Y)X + \varphi(X)Y.$$

3. Para Meta-Golden Statistical Manifolds

Definition 3.1. *[3] Let (M,g,β) be a para-golden Riemannian manifold. If an endomorphism τ on M which is satisfied*

$$\tau^2 = \beta\tau - \tau + I, \quad g(\tau X, Y) + g(X, \tau Y) = 0 \tag{3.1}$$

then (M,g,β,τ) is called a para Meta-Golden manifold.

Definition 3.2. *If (M,g,∇) is a statistical manifold and (M,g,β,τ) is a para Meta-Golden manifold then (M,g,∇,β,τ) is called a para Meta-Golden statistical manifold. Also if a para Meta-Golden statistical manifold is of a dual connection ∇^* then it is called strong para Meta-Golden statistical manifold.*

Definition 3.3. *If endomorphism τ is covariant constant for linear connection ∇ i.e. $\nabla\tau = 0$, then the connection ∇ is called τ-connection.*

Theorem 3.4. *Let (M,g,∇,β,τ) strong para Meta-Golden statistical manifold and ∇^* be a conjugate connection of ∇. Then,*

i. *$(M,g,\nabla^*,\beta,\tau)$ is also a para Meta-Golden statistical manifold*
ii. *If ∇ is a τ-connection then ∇^* is so.*

Proof. i. It is clear from definition of para Meta-Golden manifold.
ii. Since (M,g,∇) is a statistical manifold we get

$$g(\tau X,Y)+g(X,\tau Y)=0$$

$$Zg(\tau X,Y)+Zg(X,\tau Y)=0$$

$$g(\nabla_Z\tau X,Y)+g(\tau X,\nabla^*_Z Y)+g(\nabla_Z X,\tau Y)+g(X,\nabla^*_Z\tau Y)=0$$

$$g(X,(\nabla^*_Z\tau)Y)=0$$

and hence we have $\nabla^*_Z\tau=0$ because of $X,Y\neq 0$. As a consequence we get $\nabla^*=0$ namely ∇^* is a τ-connection. □

Definition 3.5. *If a skew symmetric fundamental form w that is given by $w(X,Y)=g(X,\tau Y)$ is a covariant constant for ∇ i.e. $\nabla w=0$, then (M,g,∇,β,τ) is called Meta-Golden holomorphic statistical manifold.*

Theorem 3.6. *The para Meta-Golden statistical manifold (M,g,∇,β,τ) is Meta-Golden holomorphic statistical manifold if and only if*

$$(\nabla_X g)\circ(I\times\tau)=-g\circ(I\times\nabla_X\tau). \tag{3.2}$$

In particular, if ∇ is τ-connection then

$$(\nabla_X g)\circ(I\times\tau)=0 \tag{3.3}$$

for any $X\in\Gamma(T(M))$.

Proof. Let (M,g,∇,β,τ) be a Meta-Golden holomorphic statistical manifold. Then $\nabla w=0$ is hold. Calculating ∇ covariant derivative of w and applying (2.11) we get

$$\begin{aligned}(\nabla_X w)(Y,Z) &= \nabla_X w(Y,Z)-w(\nabla_X Y,Z)-w(Y,\nabla_X Z)\\ &= (\nabla_X g)(Y,\tau Z)-g(Y,\tau(\nabla_X Z))+g(Y,\nabla_X(\tau Z))\end{aligned}$$

and therefore we have

$$(\nabla_X w)(Y,Z)=(\nabla_X g)(Y,\tau Z)+g(Y,(\nabla_X\tau)Z). \tag{3.4}$$

As $\nabla w=0$ is hold this mean

$$(\nabla_X g)(Y,\tau Z)=-g(Y,(\nabla_X\tau)Z).$$

notably if ∇ is τ-connection then we get

$$(\nabla_X g)\circ(I\times\tau)=0.$$

□

Corollary 3.7. *Let (M,g,∇,β,τ) be a strong para Meta-Golden statistical manifold. Then the following relation is hold:*

$$(\nabla_X w+\nabla^*_X w)(Y,Z)=g(Y,(\nabla_X\tau+\nabla^*_X\tau)Z). \tag{3.5}$$

Proof. Due to (3.4) we obtain

$$(\nabla_X w)(Y,Z)=(\nabla_X g)(Y,\tau Z)+g(Y,(\nabla_X\tau)Z)$$

$$(\nabla^*_X w)(Y,Z)=(\nabla^*_X g)(Y,\tau Z)+g(Y,(\nabla^*_X\tau)Z).$$

Adding these relations and using (2.3), (3.5) is easily yielded. □

Corollary 3.8. *Let (M,g,∇,β,τ) be a strong Meta-Golden holomorphic statistical manifold. If ∇ is τ-connection then $\nabla^*w=0$.*

Proof. Suppose that (M,g,∇,β,τ) be a strong Meta-Golden holomorphic statistical manifold and ∇ is τ-connection i.e. $\nabla w=0$ and $\nabla\tau=0$. Because of ∇ is τ-connection, $\nabla^*\tau=0$ is hold. Hence from 3.5 we can easily obtain $\nabla^*w=0$. □

4. Dual Semi-conjugate Connection on Meta-golden Statistical Manifolds

In this section, by defining a new connection called dual semi-cojugate connection on para meta golden statistical manifold (M,g,∇,β,τ) we will show that this is a statistical structure on M. And some geometrical properties of a para Meta-Golden statistical structure endowed with dual semi-conjugate connection will be investigated.
Suppose that (M,g,∇,β,τ) is a para Meta-Golden statistical manifold, ∇ is an affine connection, ∇^g is an Levi-Civita connection and φ is a 1-form, $\varphi^\#$ is g-dual of φ i.e. $\varphi(X)=g\left(X,\varphi^\#\right)$. If ∇ and ∇^g is dual-projectively equivalent then we get

$$\nabla_X Y=\nabla^g_X Y-g(X,Y)\varphi^\#. \tag{4.1}$$

If ∇^φ is a dual-semi conjugate connection of ∇ then we have

$$Xg(Y,Z)=g(\nabla_X Y,Z)+g\left(Y,\nabla^\varphi_X Z\right)-\varphi(X)g(Y,Z)-\varphi(Y)g(X,Z). \tag{4.2}$$

By using (4.1) in (4.2) so as to find the dual-semi conjugate connection of para Meta-Golden statistical manifold (M,g,∇,β,τ) we obtain

$$Xg(Y,Z)=g\left(\nabla^g_X Y-g(X,Y)\varphi^\#,Z\right)+g\left(Y,\nabla^\varphi_X Z\right)-\varphi(X)g(Y,Z)-\varphi(Y)g(X,Z),$$

$$Xg(Y,Z)-g\left(\nabla^g_X Y,Z\right)=g\left(Y,\nabla^\varphi_X Z\right)-g(X,Y)g\left(\varphi^\#,Z\right)-\varphi(X)g(Y,Z)-\varphi(Y)g(X,Z),$$

$$g\left(Y,\nabla^g_X Z\right)=g\left(Y,\nabla^\varphi_X Z\right)-\varphi(Z)g(X,Y)-\varphi(X)g(Y,Z)-g(X,Z)g\left(Y,\varphi^\#\right),$$

and so

$$\nabla^\varphi_X Z=\nabla^g_X Z+\varphi(X)Z+\varphi(Z)X+g(X,Z)\varphi^\# \tag{4.3}$$

which is the dual-semi conjugate connection sought.

Lemma 4.1. *The $(0,3)$-type tensor field C^φ which is given by*

$$C^\varphi(X,Y,Z)=\left(\nabla^\varphi_X g\right)(Y,Z) \tag{4.4}$$

is a cubic form.

Proof. Using (4.3) in (4.4) we get

$$\begin{aligned} C^\varphi(X,Y,Z)&=\left(\nabla^\varphi_X g\right)(Y,Z) && (4.5)\\ &=Xg(Y,Z)-g\left(\nabla^\varphi_X Y,Z\right)-g\left(Y,\nabla^\varphi_X Z\right) && (4.6)\\ &=\left(\nabla^g_X g\right)(Y,Z)-2\left[\varphi(X)g(Y,Z)+\varphi(Y)g(X,Z)+\varphi(Z)g(X,Y)\right] && (4.7)\\ &=-2\sum_{cyclic}\left[\varphi(X)g(Y,Z)\right]. && (4.8)\end{aligned}$$

and hence C^φ is totally symmetric tensor field. □

Lemma 4.2. $(\nabla^\varphi)^*=\nabla^{-\varphi}$ *is hold.*

Proof. If we put $-\varphi$ instead of φ in (4.4), then we obtain

$$C^{-\varphi}(X,Y,Z)=2\sum_{cyclic}\left[\varphi(X)g(Y,Z)\right]. \tag{4.9}$$

From (2.3) this imply that $(\nabla^\varphi)^*=\nabla^{-\varphi}$. □

Lemma 4.3. ∇^φ *is torsion-free.*

Proof. Definition of torsion tensor we get

$$\begin{aligned} T^\varphi(X,Y)&=\nabla^\varphi_X Y-\nabla^\varphi_Y X-[X,Y] && (4.10)\\ &=\nabla^g_X Y-\nabla^g_Y X-[X,Y] && (4.11)\\ &=0. && (4.12)\end{aligned}$$

□

Lemma 4.4. *Let $\varphi^\#$ which is $g-$dual of φ be a geodesic vector field for ∇^g i.e. $\nabla^g_{\varphi^\#}\varphi^\#=0$. Then $\varphi^\#$ is a geodesic vector field for ∇^φ if and only if it is a null vector field i.e. $\left\|\varphi^\#\right\|=0$.*

Proof. From (4.3) we get

$$\begin{aligned}\nabla^{\varphi}_{\varphi^{\#}}\varphi^{\#} &= \nabla^{g}_{\varphi^{\#}}\varphi^{\#}+\varphi\left(\varphi^{\#}\right)\varphi^{\#}+\varphi\left(\varphi^{\#}\right)\varphi^{\#}+g\left(\varphi^{\#},\varphi^{\#}\right)\varphi^{\#}\\ &= 3g\left(\varphi^{\#},\varphi^{\#}\right)\varphi^{\#}\\ &= 3\left\|\varphi^{\#}\right\|_{g}^{2}\varphi^{\#}.\end{aligned}$$

From geodesic vector field definition we obtain $\left\|\varphi^{\#}\right\|=0$. □

Lemma 4.5. *∇^{φ} covariant derivative of Meta-Golden structure τ is given by*

$$\begin{aligned}\left(\nabla^{\varphi}_{X}\tau\right)Y &= \left(\nabla^{g}_{X}\tau\right)Y+(\varphi\circ\tau)(Y)X-\varphi(Y)\tau(X)\\ &\quad+g(X,\tau(Y))\varphi^{\#}-g(X,Y)\tau\left(\varphi^{\#}\right).\end{aligned}\tag{4.13}$$

Proof. Since τ is an endomorphism $\left(\nabla^{\varphi}_{X}\tau\right)Y=\nabla^{\varphi}_{X}\tau(Y)-\tau\left(\nabla^{\varphi}_{X}Y\right)$ is satisfied. Using (4.3) in this relation we easily get the assertion. □

Corollary 4.6. *If we take $X=Y$ in ∇^{φ} covariant derivative of Meta-Golden structure τ we get*

$$\left(\nabla^{\varphi}_{X}\tau\right)X=\left(\nabla^{g}_{X}\tau\right)X+(\varphi\circ\tau)(X)X-\varphi(X)\tau(X)-\|X\|_{g}^{2}\tau\left(\varphi^{\#}\right).$$

Hence $\left(\nabla^{\varphi}_{X}-\nabla^{g}_{X}\right)\tau(X)$ belong to kernel of φ for any X vector field satisfied $\varphi(X)=0$.

Proof. Since we take $X=Y$ in (4.13) we can easily obtain that

$$\left(\nabla^{\varphi}_{X}\tau\right)X=\left(\nabla^{g}_{X}\tau\right)X+(\varphi\circ\tau)(X)X-\varphi(X)\tau(X)-\|X\|_{g}^{2}\tau\left(\varphi^{\#}\right)$$

is hold. From $\nabla^{\varphi}_{X}\tau(X)=\tau\left(\nabla^{\varphi}_{X}X\right)+\left(\nabla^{\varphi}_{X}\tau\right)X$ and $\nabla^{g}_{X}\tau(X)=\tau\left(\nabla^{g}_{X}X\right)+\left(\nabla^{g}_{X}\tau\right)X$ we get

$$\begin{aligned}\nabla^{\varphi}_{X}\tau(X)-\tau\left(\nabla^{\varphi}_{X}X\right) &= \nabla^{g}_{X}\tau(X)-\tau\left(\nabla^{g}_{X}X\right)+(\varphi\circ\tau)(X)X\\ &\quad-\varphi(X)\tau(X)-\|X\|_{g}^{2}\tau\left(\varphi^{\#}\right)\\ \left(\nabla^{\varphi}_{X}-\nabla^{g}_{X}\right)(\tau(X)) &= \tau\left(\nabla^{\varphi}_{X}X\right)-\tau\left(\nabla^{g}_{X}X\right)+(\varphi\circ\tau)(X)X\\ &\quad-\varphi(X)\tau(X)-\|X\|_{g}^{2}\tau\left(\varphi^{\#}\right).\end{aligned}$$

From definition of kernel, by applying φ to both sides of above equation and using (3.1) second identity we acquire

$$\begin{aligned}\varphi\left[\left(\nabla^{\varphi}_{X}-\nabla^{g}_{X}\right)(\tau(X))\right] &= \varphi\left(\tau\left(\nabla^{\varphi}_{X}X\right)\right)-\varphi\left(\tau\left(\nabla^{g}_{X}X\right)\right)+(\varphi\circ\tau)(X)\varphi(X)\\ &\quad-\varphi(X)\varphi(\tau(X))-\|X\|_{g}^{2}\varphi\left(\tau\left(\varphi^{\#}\right)\right)\\ &= -g\left(\nabla^{\varphi}_{X}X,\tau\left(\varphi^{\#}\right)\right)+g\left(\nabla^{g}_{X}X,\tau\left(\varphi^{\#}\right)\right)\\ &\quad-\|X\|_{g}^{2}g\left(\tau\left(\varphi^{\#}\right),\varphi^{\#}\right)\\ &= -g\left(\nabla^{\varphi}_{X}X-\nabla^{g}_{X}X,\tau\left(\varphi^{\#}\right)\right)\end{aligned}$$

By using (4.3) and (3.1) second identity in this last equation

$$\varphi\left[\left(\nabla^{\varphi}_{X}-\nabla^{g}_{X}\right)(\tau(X))\right]=-g\left(2\varphi(X)X,\tau\left(\varphi^{\#}\right)\right)-\|X\|_{g}^{2}g\left(\varphi^{\#},\tau\left(\varphi^{\#}\right)\right)=0$$

which gives the assertion. □

Theorem 4.7. *∇^{φ} covariant derivative of fundamental form w is given by*

$$\begin{aligned}\left(\nabla^{\varphi}_{X}w\right)(Y,Z) &= g\left(Y,\left(\nabla^{g}_{X}\tau\right)Z\right)+\varphi(\tau Y)g(X,Z)-\varphi(\tau Z)g(X,Y)-2\varphi(X)g(Y,\tau Z)\\ &\quad-\varphi(Y)g(X,\tau Z)+\varphi(Z)g(X,\tau Y).\end{aligned}\tag{4.14}$$

Proof. Since the skew-symmetric fundamental form w verify the relation $w(X,Y)=g(X,\tau Y)$, by using (4.3) and (3.1) we get

$$\begin{aligned}\left(\nabla^{\varphi}_{X}w\right)(Y,Z) &= Xw(Y,Z)-w\left(\nabla^{\varphi}_{X}Y,Z\right)-w\left(Y,\nabla^{\varphi}_{X}Z\right)\\ &= Xg(Y,\tau Z)-g\left(\nabla^{g}_{X}Y,\tau Z\right)-g\left(Y,\tau\left(\nabla^{g}_{X}Z\right)\right)+g\left(Y,\left(\nabla^{g}_{X}\tau\right)Z\right)\\ &\quad-g\left(Y,\left(\nabla^{g}_{X}\tau\right)Z\right)-2\varphi(X)g(Y,\tau Z)-\varphi(Y)g(X,\tau Z)\\ &\quad-\varphi(Z)g(Y,\tau X)-g(X,Y)g\left(\varphi^{\#},\tau Z\right)-g(X,Z)g\left(Y,\tau\varphi^{\#}\right)\\ &= \left(\nabla^{g}_{X}g\right)(Y,\tau Z)+g\left(Y,\left(\nabla^{g}_{X}\tau\right)Z\right)-2\varphi(X)g(Y,\tau Z)\\ &\quad-\varphi(Y)g(X,\tau Z)-\varphi(Z)g(Y,\tau X)-g(X,Y)g\left(\varphi^{\#},\tau Z\right)\\ &\quad+g(X,Z)g\left(\tau Y,\varphi^{\#}\right)\\ &= g\left(Y,\left(\nabla^{g}_{X}\tau\right)Z\right)+\varphi(\tau Y)g(X,Z)-\varphi(\tau Z)g(X,Y)\\ &\quad-2\varphi(X)g(Y,\tau Z)-\varphi(Y)g(X,\tau Z)+\varphi(Z)g(X,\tau Y).\end{aligned}$$

□

Theorem 4.8. *If (M,g,φ,β,τ) is a strong para Meta-Golden statistical manifold endowed with τ-connection ∇^{φ} then*

$$\left(\nabla^g_X\tau\right)Y=g(X,Y)\,\tau\left(\varphi^{\#}\right)+g(\tau(X),Y)\,\varphi^{\#}+\varphi(Y)\,\tau(X)-(\varphi\circ\tau)(Y)X. \tag{4.15}$$

Proof. Let (M,g,∇,β,τ) be a strong para Meta-Golden statistical manifold endowed with τ-connection ∇^{φ}. By using (4.13) and definition of τ-connection, we easily get the assertion looked for. □

Corollary 4.9. *Let (M,g,φ,β,τ) be a strong para Meta-Golden statistical manifold endowed with τ-connection ∇^{φ}. If (M,g,φ,β,τ) is integrable i.e. $\nabla^g\tau=0$, then*

$$g(X,Y)\,\tau\left(\varphi^{\#}\right)+g\left(\tau\left(\varphi^{\#}\right),Y\right)X=-g(\tau(X),Y)\,\varphi^{\#}-g\left(\varphi^{\#},Y\right)\tau(X).$$

Proof. Since (M,g,φ,β,τ) is integrable i.e. $\nabla^g\tau=0$, from (4.15) we can easily get the relation asserted. □

Corollary 4.10. *If $\varphi^{\#}$ g-dual vector field of φ is a geodesic vector field of ∇^g, then*

$$\nabla^g_{\varphi^{\#}}\tau\left(\varphi^{\#}\right)=2\left\|\varphi^{\#}\right\|_g^2\tau\left(\varphi^{\#}\right).$$

Proof. If $\varphi^{\#}$ g-dual vector field of φ is a geodesic vector field of ∇^g, then $\nabla^g_{\varphi^{\#}}\varphi^{\#}=0$. By using $X=\varphi^{\#}$ and $Y=\varphi^{\#}$ in (4.15) we get

$$\begin{aligned}\left(\nabla^g_{\varphi^{\#}}\tau\right)\varphi^{\#} &= g\left(\varphi^{\#},\varphi^{\#}\right)\tau\left(\varphi^{\#}\right)+g\left(\tau\left(\varphi^{\#}\right),\varphi^{\#}\right)\varphi^{\#}\\ &\quad+\varphi\left(\varphi^{\#}\right)\tau\left(\varphi^{\#}\right)-(\varphi\circ\tau)\left(\varphi^{\#}\right)\varphi^{\#}.\end{aligned}$$

Since $\tau\left(\nabla^g_{\varphi^{\#}}\varphi^{\#}\right)=0$, this imply that $\nabla^g_{\varphi^{\#}}\tau\left(\varphi^{\#}\right)=\left(\nabla^g_{\varphi^{\#}}\tau\right)\varphi^{\#}+\tau\left(\nabla^g_{\varphi^{\#}}\varphi^{\#}\right)$. So we obtain

$$\nabla^g_{\varphi^{\#}}\tau\left(\varphi^{\#}\right)=2\left\|\varphi^{\#}\right\|_g^2\tau\left(\varphi^{\#}\right)$$

which gives the assertion. □

Corollary 4.11. *If (M,g,φ,β,τ) is Meta-Golden holomorphic manifold, then*

$$\begin{aligned}g\left(Y,\left(\nabla^g_X\tau\right)Z\right)+\varphi(\tau Y)\,g(X,Z)+\varphi(Z)\,g(X,\tau Y) &= 2\varphi(X)\,g(Y,\tau Z)\\ &\quad+\varphi(\tau Z)\,g(X,Y)+\varphi(Y)\,g(X,\tau Z)\end{aligned} \tag{4.16}$$

is satisfied.

Proof. If (M,g,φ,β,τ) is Meta-Golden holomorphic manifold, then $\nabla^{\varphi}w=0$ is satisfied. From (4.14) we easily get the assertion. □

Corollary 4.12. *A strong para meta-Golden statistical manifold (M,g,φ,β,τ) endowed with τ-connection ∇^g is holomorphic if and only if*

$$\varphi(\tau Y)\,g(X,Z)+\varphi(Z)\,g(X,\tau Y)=2\varphi(X)\,g(Y,\tau Z)+\varphi(\tau Z)\,g(X,Y)+\varphi(Y)\,g(X,\tau Z).$$

Proof. Since the ∇^g is τ-connection then $\nabla^g\tau=0$ and so the assertion is clear from (4.16). □

Theorem 4.13. *If (M,g,φ,β,τ) is a strong para meta-Golden statistical manifold then curvature tensor field R^{φ} is given by*

$$\begin{aligned}\left(R^{\varphi}-R^g\right)(X,Y)Z &= \left[g\left(Y,\nabla^g_X\varphi^{\#}\right)-g\left(X,\nabla^g_Y\varphi^{\#}\right)\right]Z\\ &\quad+\left[\varphi(Y)\,\varphi(Z)+\left\|\varphi^{\#}\right\|_g^2-g\left(Z,\nabla^g_Y\varphi^{\#}\right)\right]X\\ &\quad-\left[\varphi(X)\,\varphi(Z)+\left\|\varphi^{\#}\right\|_g^2-g\left(Z,\nabla^g_X\varphi^{\#}\right)\right]Y+g(Y,Z)\,\nabla^g_X\varphi^{\#}\\ &\quad-g(X,Z)\,\nabla^g_Y\varphi^{\#}+\left[\varphi(X)\,g(Y,Z)-\varphi(Y)\,g(X,Z)\right]\varphi^{\#}.\end{aligned}$$

Proof. The curvature tensor field of ∇^{φ} is given by $R^{\varphi}(X,Y)Z=\nabla^{\varphi}_X\nabla^{\varphi}_YZ-\nabla^{\varphi}_Y\nabla^{\varphi}_XZ-\nabla^{\varphi}_{[X,Y]}Z$. By using the relation (4.3) in the expression of curvature tensor field we can get the assertion. □

Theorem 4.14. *Let (M,g,φ,β,τ) be a para Meta-Golden statistical manifold. Nijenhuis tensor field of τ is given by*

$$\begin{aligned}N^{\varphi}_{\tau}(X,Y) &= \left(\nabla^{\varphi}_{\tau X}\tau\right)Y-\left(\nabla^{\varphi}_{\tau Y}\tau\right)X-\tau\left(\nabla^{\varphi}_X\tau\right)Y+\tau\left(\nabla^{\varphi}_Y\tau\right)X\\ &\quad-g\left(\tau X,\beta^2Y\right)\varphi^{\#}+g\left(\tau Y,\beta^2X\right)\varphi^{\#}.\end{aligned}$$

Proof. Since $\left(\nabla^{\varphi}_{\tau X}\tau\right)Y=\nabla^{\varphi}_{\tau X}\tau Y-\tau\left(\nabla^{\varphi}_{\tau X}Y\right)$ is hold by using the 3.1 and 4.3 in this equation we get

$$\begin{aligned}\left(\nabla^{g}_{\tau X}\tau\right)Y &= \left(\nabla^{\varphi}_{\tau X}\tau\right)Y-\varphi\left(\tau Y\right)\tau X+\varphi\left(Y\right)\tau^{2}\left(X\right)-g\left(\tau X,\beta^{2}Y\right)\varphi^{\#}\\ &\quad+g\left(X,Y\right)\varphi^{\#}+g\left(\tau X,Y\right)\tau\left(\varphi^{\#}\right)\end{aligned}$$

and

$$\begin{aligned}\left(\nabla^{g}_{\tau Y}\tau\right)X &= \left(\nabla^{\varphi}_{\tau Y}\tau\right)X-\varphi\left(\tau X\right)\tau Y+\varphi\left(X\right)\tau^{2}\left(Y\right)-g\left(\tau Y,\beta^{2}X\right)\varphi^{\#}\\ &\quad+g\left(X,Y\right)\varphi^{\#}+g\left(\tau Y,X\right)\tau\left(\varphi^{\#}\right).\end{aligned}$$

Similiarly since $\left(\nabla^{\varphi}_{X}\tau\right)Y=\nabla^{\varphi}_{X}\tau Y-\tau\left(\nabla^{\varphi}_{X}Y\right)$ is hold by using 4.3 in this equation we get

$$\begin{aligned}\tau\left(\nabla^{g}_{X}\tau\right)Y &= \tau\left(\nabla^{\varphi}_{X}\tau\right)Y-\varphi\left(\tau Y\right)\tau\left(X\right)+\varphi\left(Y\right)\tau^{2}\left(X\right)\\ &\quad-g\left(\tau Y,X\right)\tau\left(\varphi^{\#}\right)+g\left(X,Y\right)\tau^{2}\left(\varphi^{\#}\right)\end{aligned}$$

and

$$\begin{aligned}\tau\left(\nabla^{g}_{Y}\tau\right)X &= \tau\left(\nabla^{\varphi}_{Y}\tau\right)X-\varphi\left(\tau X\right)\tau\left(Y\right)+\varphi\left(X\right)\tau^{2}\left(Y\right)\\ &\quad-g\left(\tau X,Y\right)\tau\left(\varphi^{\#}\right)+g\left(X,Y\right)\tau^{2}\left(\varphi^{\#}\right).\end{aligned}$$

By using this equations in (2.15) the assertion is obtained. □

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