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Research Article

General Two- and Three-Dimensional Integral Inequalities Based a Change of Variables Methodology

Christophe Chesneau

Abstract
This article establishes new general two- and three-dimensional integral inequalities. The first result involves four functions: two main functions defined on the positive real line and two auxiliary functions defined on the unit interval. As a significant contribution, the upper bound obtained is quite simple; it is expressed only as the product of the unweighted integral norms of these functions. The main ingredient of the proof is an original change of variables methodology. The article also presents a three-dimensional extension of this result. This higher-dimensional version uses a similar structure but with nine functions: three main functions defined on the positive real line and six auxiliary functions defined on the unit interval. It retains the simplicity and sharpness of the upper bound. Both results open up new directions for applications in analysis. This claim is supported by various examples, including some based on power, logarithmic, trigonometric, and exponential functions, as well as some secondary but still general integral inequalities.

Keywords: Change of variables, Gamma function, Hardy-Hilbert-type integral inequalities, Three-dimensional integral inequalities, Two-dimensional integral inequalities
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Department of Mathematics, LMNO, University of Caen-Normandie, 14032 Caen, France, christophe.chesneau@gmail.com
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1. Introduction

Multi-dimensional integral inequalities, especially those in two and three dimensions, are fundamental tools in mathematical analysis. In particular, they are essential for understanding the behavior of integral operators and for estimating their bounds. See [1–4]. Among the classical results in two dimensions, the Hardy-Hilbert integral inequality occupies a prominent place. A precise statement is given below. Let $p>1$, $q=p/(p-1)$ satisfying the Hölder condition $1/p+1/q=1$, and $f,g:(0,+\infty)\mapsto(0,+\infty)$ be two functions; they are thus defined on the positive real line, i.e., $(0,+\infty)$, and are positive. Then the Hardy-Hilbert integral inequality states that

$$\int_0^{+\infty}\int_0^{+\infty}\frac{1}{x+y}f(x)g(y)dxdy\leq\frac{\pi}{\sin(\pi/p)}\left[\int_0^{+\infty}f^p(x)dx\right]^{1/p}\left[\int_0^{+\infty}g^q(y)dy\right]^{1/q}, \tag{1.1}$$

provided that the integrals involved converge, i.e., $\int_0^{+\infty}f^p(x)dx<+\infty$ and $\int_0^{+\infty}g^q(y)dy<+\infty$. This result features a sharp constant factor, i.e., $\pi/\sin(\pi/p)$, and the product of two unweighted integral norms of f and g with parameters p and q, respectively. For the basic details, see the classic work by G.H. Hardy in [1]. Over the years, this inequality has inspired extensive research, including numerous extensions and generalizations in higher dimensions. Notable contributions to the development of such extensions include [5–9]. Further generalizations in higher dimensions have been explored in works such as [10–14].

Despite this extensive literature, the derivation of sharp and tractable upper bounds for multidimensional integral inequalities, particularly in two and three dimensions, remains a significant challenge. Many existing results involve sophisticated constants or rely on restrictive assumptions that limit their scope. This motivates the search for new inequalities that offer both structural clarity and wide applicability.

The first inequality established in this article addresses part of this challenge. It gives a sharp and simple upper bound for a two-dimensional integral involving four functions: two main functions defined on $(0,+\infty)$ and two auxiliary functions defined on the unit interval, i.e., $(0,1)$. In a similar framework to that of the Hardy-Hilbert integral inequality, this integral is

$$\int_0^{+\infty}\int_0^{+\infty}\frac{x^{1/p}y^{1/q}}{(x+y)^2}\ell\left(\frac{x}{x+y}\right)m\left(\frac{y}{x+y}\right)f(x)g(y)dxdy,$$

where ℓ and m are the auxiliary functions. The upper bound obtained is quite manageable. It depends only on the unweighted integral norms of f, g, ℓ, and m. Explicitly, it is given by

$$\left[\int_0^1 \ell^p(t)dt\right]^{1/p}\left[\int_0^1 m^q(t)dt\right]^{1/q}\left[\int_0^{+\infty} f^p(x)dx\right]^{1/p}\left[\int_0^{+\infty} g^q(y)dy\right]^{1/q},$$

with a constant factor exactly equal to one. The proof strategy differs from the traditional approaches used in the study of Hardy-Hilbert-type inequalities. It is based on an appropriate factorization of the integrand, the Hölder integral inequality, and a special change of variables that transforms the expression into a simpler form. This change of variables methodology is the main originality of the proof.

In addition to the two-dimensional result, the article introduces a natural extension to three dimensions. This generalized inequality involves a three-dimensional integral that depends on nine functions: three main functions defined on $(0,+\infty)$ and six auxiliary functions defined on $(0,1)$. It has the following general form:

$$\int_0^{+\infty}\int_0^{+\infty}\int_0^{+\infty}\frac{1}{(x+y+z)^2}\left(\frac{x}{x+y}\right)^{1/p}\left(\frac{y}{y+z}\right)^{1/q}\left(\frac{z}{x+z}\right)^{1/r}$$

$$\times\, i\left(\frac{x}{x+y}\right)j\left(\frac{y}{y+z}\right)k\left(\frac{z}{x+z}\right)\ell\left(\frac{x+y}{x+y+z}\right)m\left(\frac{y+z}{x+y+z}\right)n\left(\frac{x+z}{x+y+z}\right)f(x)g(y)h(z)dxdydz,$$

where i, j, k, ℓ, m and n are the auxiliary functions, and r is an additional norm parameter. This extended version retains the simplicity of the two-dimensional case, still with a tractable upper bound depending only on the unweighted integral norms of the functions involved. Explicitly, it is given by

$$\left[\int_0^1 i^p(t)dt\right]^{1/p}\left[\int_0^1 \ell^p(t)dt\right]^{1/p}\left[\int_0^1 j^q(t)dt\right]^{1/q}\left[\int_0^1 m^q(t)dt\right]^{1/q}$$

$$\times\left[\int_0^1 k^r(t)dt\right]^{1/r}\left[\int_0^1 n^r(t)dt\right]^{1/r}\left[\int_0^{+\infty} f^p(x)dx\right]^{1/p}\left[\int_0^{+\infty} g^q(y)dy\right]^{1/q}\left[\int_0^{+\infty} h^r(z)dz\right]^{1/r}.$$

The proof strategy is based on a suitable factorization of the integrand, the generalized Hölder integral inequality (see [15, 16]), and an adapted change of variables that transforms the expression into a simpler form. Again, this change of variables methodology remains the main originality of the proof. The structure of the inequality allows considerable flexibility in the choice of auxiliary functions, providing a unified framework for bounding a large class of complex three-dimensional integrals.

To illustrate the scope and applicability of the results, several concrete examples are given. These include functions of the power, logarithmic, trigonometric, and exponential types. Other new secondary general inequalities are also derived. These highlight the versatility of the proposed inequalities and demonstrate their potential for further applications in analysis.

The rest of the article is as follows: Section 2 is devoted to our main two-dimensional integral result with examples and secondary results. A natural extension to three dimensions is studied in Section 3, again illustrated with examples. Section 4 concludes the article.

2. Two-Dimensional Integral Inequality Results

2.1 Main result

The theorem below gives our general two-dimensional integral inequality result. It is followed by the detailed proof and some discussion.

Theorem 2.1. *Let $p > 1$, $q = p/(p-1)$, and $f, g : (0,+\infty) \mapsto (0,+\infty)$ and $\ell, m : (0,1) \mapsto (0,+\infty)$ be four functions. Then, we have*

$$\int_0^{+\infty}\int_0^{+\infty} \frac{x^{1/p}y^{1/q}}{(x+y)^2}\ell\left(\frac{x}{x+y}\right)m\left(\frac{y}{x+y}\right)f(x)g(y)dxdy \leq \Upsilon\left[\int_0^{+\infty} f^p(x)dx\right]^{1/p}\left[\int_0^{+\infty} g^q(y)dy\right]^{1/q},$$

where

$$\Upsilon = \left[\int_0^1 \ell^p(t)dt\right]^{1/p}\left[\int_0^1 m^q(t)dt\right]^{1/q}, \tag{2.1}$$

provided that the integrals involved converge.

Proof. By a well-chosen decomposition of the integrand as the product of two main terms using $1/p + 1/q = 1$ and the Hölder integral inequality applied to those terms at the parameters p and $q = p/(p-1)$, we get

$$\begin{aligned}
&\int_0^{+\infty}\int_0^{+\infty} \frac{x^{1/p}y^{1/q}}{(x+y)^2}\ell\left(\frac{x}{x+y}\right)m\left(\frac{y}{x+y}\right)f(x)g(y)dxdy \\
&= \int_0^{+\infty}\int_0^{+\infty} \frac{x^{1/p}}{(x+y)^{2/p}}\ell\left(\frac{x}{x+y}\right)f(x) \times \frac{y^{1/q}}{(x+y)^{2/q}}m\left(\frac{y}{x+y}\right)g(y)dxdy \\
&\leq A^{1/p}B^{1/q},
\end{aligned} \tag{2.2}$$

where

$$A = \int_0^{+\infty}\int_0^{+\infty} \frac{x}{(x+y)^2}\ell^p\left(\frac{x}{x+y}\right)f^p(x)dxdy$$

and

$$B = \int_0^{+\infty}\int_0^{+\infty} \frac{y}{(x+y)^2}m^q\left(\frac{y}{x+y}\right)g^q(y)dxdy.$$

We can now find the exact expressions for A and B, starting with A. Using the Fubini-Tonelli integral theorem to permutate the two integral signs and changing the variables as $u = x/(x+y)$, so $du = [-x/(x+y)^2]dy$, $y = 0 \Rightarrow u = 1$ and $y \to +\infty \Rightarrow u = 0$, we obtain

$$\begin{aligned}
A &= \int_0^{+\infty} f^p(x)\left[\int_0^{+\infty} \frac{x}{(x+y)^2}\ell^p\left(\frac{x}{x+y}\right)dy\right]dx = \int_0^{+\infty} f^p(x)\left[\int_0^1 \ell^p(u)du\right]dx \\
&= \left[\int_0^1 \ell^p(t)dt\right]\left[\int_0^{+\infty} f^p(x)dx\right].
\end{aligned} \tag{2.3}$$

In a similar way, but with the change of variables $v = y/(x+y)$ with $dv = [-y/(x+y)^2]dx$, $x = 0 \Rightarrow v = 1$ and $x \to +\infty \Rightarrow v = 0$, we get

$$\begin{aligned}
B &= \int_0^{+\infty} g^q(y)\left[\int_0^{+\infty} \frac{y}{(x+y)^2}m^q\left(\frac{y}{x+y}\right)dx\right]dy = \int_0^{+\infty} g^q(y)\left[\int_0^1 m^q(v)dv\right]dy \\
&= \left[\int_0^1 m^q(t)dt\right]\left[\int_0^{+\infty} g^q(y)dy\right].
\end{aligned} \tag{2.4}$$

Combining Equations (2.2), (2.3) and (2.4), and using $1/p+1/q=1$, we get

$$\int_0^{+\infty}\int_0^{+\infty}\frac{x^{1/p}y^{1/q}}{(x+y)^2}\ell\left(\frac{x}{x+y}\right)m\left(\frac{y}{x+y}\right)f(x)g(y)dxdy$$

$$\leq\left\{\left[\int_0^1\ell^p(t)dt\right]\left[\int_0^{+\infty}f^p(x)dx\right]\right\}^{1/p}\left\{\left[\int_0^1 m^q(t)dt\right]\left[\int_0^{+\infty}g^q(y)dy\right]\right\}^{1/q}$$

$$=\left[\int_0^1\ell^p(t)dt\right]^{1/p}\left[\int_0^1 m^q(t)dt\right]^{1/q}\left[\int_0^{+\infty}f^p(x)dx\right]^{1/p}\left[\int_0^{+\infty}g^q(y)dy\right]^{1/q}$$

$$=\Upsilon\left[\int_0^{+\infty}f^p(x)dx\right]^{1/p}\left[\int_0^{+\infty}g^q(y)dy\right]^{1/q},$$

where Υ is indicated in Equation (2.1). This concludes the proof of Theorem 2.1. □

The main interest of this theorem is the generality of the two-dimensional integral. It includes two auxiliary functions, ℓ and m, which provide additional flexibility. Another major strength is the simplicity of the upper bound. It depends only on the integral norms of the functions involved. Given the wide variety of known integral formulas (see [17]), the structure of this upper bound allows for easy adaptation. In particular, it allows the derivation of tractable two-dimensional integral inequalities. These may have useful applications in operator theory and related areas.

In the rest of this section, we support these claims with several examples considering different types of auxiliary functions, and with established secondary results derived more or less directly from Theorem 2.1.

2.2 Examples

Some specific examples of applications of Theorem 2.1 are given below. They deal with different functions ℓ and m.

Example 1. Applying Theorem 2.1 with $\ell(t)=t^\alpha$, $\alpha>0$, and $m(t)=t^\beta$, $\beta>0$, we get

$$\int_0^{+\infty}\int_0^{+\infty}\frac{x^{\alpha+1/p}y^{\beta+1/q}}{(x+y)^{\alpha+\beta+2}}f(x)g(y)dxdy=\int_0^{+\infty}\int_0^{+\infty}\frac{x^{1/p}y^{1/q}}{(x+y)^2}\ell\left(\frac{x}{x+y}\right)m\left(\frac{y}{x+y}\right)f(x)g(y)dxdy$$
$$\leq\Upsilon\left[\int_0^{+\infty}f^p(x)dx\right]^{1/p}\left[\int_0^{+\infty}g^q(y)dy\right]^{1/q},$$

where

$$\Upsilon=\left[\int_0^1\ell^p(t)dt\right]^{1/p}\left[\int_0^1 m^q(t)dt\right]^{1/q}=\left(\int_0^1 t^{\alpha p}dt\right)^{1/p}\left(\int_0^1 t^{\beta q}dt\right)^{1/q}=\frac{1}{(\alpha p+1)^{1/p}(\beta q+1)^{1/q}}.$$

This upper bound is thus determined in a straightforward manner, despite the relative complexity of the two main two-dimensional integrals. In summary, thanks to Theorem 2.1, we have established that

$$\int_0^{+\infty}\int_0^{+\infty}\frac{x^{\alpha+1/p}y^{\beta+1/q}}{(x+y)^{\alpha+\beta+2}}f(x)g(y)dxdy\leq\frac{1}{(\alpha p+1)^{1/p}(\beta q+1)^{1/q}}\left[\int_0^{+\infty}f^p(x)dx\right]^{1/p}\left[\int_0^{+\infty}g^q(y)dy\right]^{1/q}.$$

This inequality can also be manipulated to derive other integral inequality results. For example, if we take $p=2$, $\alpha=\gamma/2$, $\gamma>0$, and $\beta=\gamma/2=\alpha$, it simplifies to

$$\int_0^{+\infty}\int_0^{+\infty}\frac{\sqrt{xy}}{(x+y)^2}\left(\frac{\sqrt{xy}}{x+y}\right)^\gamma f(x)g(y)dxdy\leq\frac{1}{\gamma+1}\sqrt{\int_0^{+\infty}f^2(x)dx}\sqrt{\int_0^{+\infty}g^2(y)dy}.$$

This result has the advantage of being tractable with a simple constant factor. For example, considering γ as a variable and integrating both sides for $\gamma \in (0,\tau)$ with $\tau > 0$, and using the Fubini-Tonelli integral theorem, we get

$$\int_0^{+\infty}\int_0^{+\infty} \frac{\sqrt{xy}}{(x+y)^2} \times \frac{1}{\log[(x+y)/\sqrt{xy}]}\left[1-\left(\frac{\sqrt{xy}}{x+y}\right)^{\tau}\right] f(x)g(y)dxdy$$

$$= \int_0^{+\infty}\int_0^{+\infty} \frac{\sqrt{xy}}{(x+y)^2}\left[\int_0^{\tau}\left(\frac{\sqrt{xy}}{x+y}\right)^{\gamma} d\gamma\right] f(x)g(y)dxdy$$

$$= \int_0^{\tau}\left[\int_0^{+\infty}\int_0^{+\infty} \frac{\sqrt{xy}}{(x+y)^2}\left(\frac{\sqrt{xy}}{x+y}\right)^{\gamma} f(x)g(y)dxdy\right] d\gamma$$

$$\leq \left[\int_0^{\tau} \frac{1}{\gamma+1} d\gamma\right]\sqrt{\int_0^{+\infty} f^2(x)dx}\sqrt{\int_0^{+\infty} g^2(y)dy}$$

$$= \log(\tau+1)\sqrt{\int_0^{+\infty} f^2(x)dx}\sqrt{\int_0^{+\infty} g^2(y)dy}.$$

More directly, we have

$$\int_0^{+\infty}\int_0^{+\infty} \frac{\sqrt{xy}}{(x+y)^2\log[(x+y)/\sqrt{xy}]}\left[1-\left(\frac{\sqrt{xy}}{x+y}\right)^{\tau}\right] f(x)g(y)dxdy \leq \log(\tau+1)\sqrt{\int_0^{+\infty} f^2(x)dx}\sqrt{\int_0^{+\infty} g^2(y)dy}.$$

To the best of our knowledge, this is a new two-dimensional integral inequality in the literature.

Example 2. Applying Theorem 2.1 with $\ell(t) = [-\log(t)]^{\alpha}$, $\alpha > 0$, and $m(t) = [-\log(t)]^{\beta}$, $\beta > 0$, we get

$$\int_0^{+\infty}\int_0^{+\infty} \frac{x^{1/p}y^{1/q}}{(x+y)^2}\left[-\log\left(\frac{x}{x+y}\right)\right]^{\alpha}\left[-\log\left(\frac{y}{x+y}\right)\right]^{\beta} f(x)g(y)dxdy$$

$$= \int_0^{+\infty}\int_0^{+\infty} \frac{x^{1/p}y^{1/q}}{(x+y)^2}\ell\left(\frac{x}{x+y}\right) m\left(\frac{y}{x+y}\right) f(x)g(y)dxdy$$

$$\leq \Upsilon\left[\int_0^{+\infty} f^p(x)dx\right]^{1/p}\left[\int_0^{+\infty} g^q(y)dy\right]^{1/q},$$

where, using the formula of the gamma function in [17, Entry 4.2726], i.e., $\Gamma(x) = \int_0^1 [-\log(t)]^{x-1}dt$, $x > 0$,

$$\Upsilon = \left[\int_0^1 \ell^p(t)dt\right]^{1/p}\left[\int_0^1 m^q(t)dt\right]^{1/q} = \left[\int_0^1 [-\log(t)]^{\alpha p}dt\right]^{1/p}\left[\int_0^1 [-\log(t)]^{\beta q}dt\right]^{1/q}$$
$$= \Gamma^{1/p}(\alpha p+1)\Gamma^{1/q}(\beta q+1).$$

As a more direct presentation, we have

$$\int_0^{+\infty}\int_0^{+\infty} \frac{x^{1/p}y^{1/q}}{(x+y)^2}\left[-\log\left(\frac{x}{x+y}\right)\right]^{\alpha}\left[-\log\left(\frac{y}{x+y}\right)\right]^{\beta} f(x)g(y)dxdy$$
$$\leq \Gamma^{1/p}(\alpha p+1)\Gamma^{1/q}(\beta q+1)\left[\int_0^{+\infty} f^p(x)dx\right]^{1/p}\left[\int_0^{+\infty} g^q(y)dy\right]^{1/q}.$$

We emphasize again the originality of the main two-dimensional integral and the simplicity of the constant factor.

Example 3. Trigonometric functions can also be used as auxiliary functions in Theorem 2.1. In particular, applying this theorem with $p=2$, $\ell(t)=\sin[\theta(\pi/2)t]$, $\theta\in[0,1]$, and $m(t)=\sin[\theta(\pi/2)t]=\ell(t)$, $\theta\in[0,1]$, we obtain

$$\int_0^{+\infty}\int_0^{+\infty}\frac{\sqrt{xy}}{(x+y)^2}\sin\left[\theta\frac{\pi}{2}\left(\frac{x}{x+y}\right)\right]\sin\left[\theta\frac{\pi}{2}\left(\frac{y}{x+y}\right)\right]f(x)g(y)dxdy$$

$$=\int_0^{+\infty}\int_0^{+\infty}\frac{x^{1/p}y^{1/q}}{(x+y)^2}\ell\left(\frac{x}{x+y}\right)m\left(\frac{y}{x+y}\right)f(x)g(y)dxdy$$

$$\leq \Upsilon\left[\int_0^{+\infty}f^p(x)dx\right]^{1/p}\left[\int_0^{+\infty}g^q(y)dy\right]^{1/q}$$

$$=\Upsilon\sqrt{\int_0^{+\infty}f^2(x)dx}\sqrt{\int_0^{+\infty}g^2(y)dy},$$

where

$$\Upsilon=\left[\int_0^{+\infty}f^p(x)dx\right]^{1/p}\left[\int_0^{+\infty}g^q(y)dy\right]^{1/q}=\sqrt{\int_0^1\sin^2\left(\theta\frac{\pi}{2}t\right)dt}\sqrt{\int_0^1\sin^2\left(\theta\frac{\pi}{2}t\right)dt}$$

$$=\int_0^1\sin^2\left(\theta\frac{\pi}{2}t\right)dt=\frac{1}{2}\left[1-\frac{\sin(\theta\pi)}{\theta\pi}\right].$$

Therefore, we have

$$\int_0^{+\infty}\int_0^{+\infty}\frac{\sqrt{xy}}{(x+y)^2}\sin\left[\theta\frac{\pi}{2}\left(\frac{x}{x+y}\right)\right]\sin\left[\theta\frac{\pi}{2}\left(\frac{y}{x+y}\right)\right]f(x)g(y)dxdy$$

$$\leq\frac{1}{2}\left[1-\frac{\sin(\theta\pi)}{\theta\pi}\right]\sqrt{\int_0^{+\infty}f^2(x)dx}\sqrt{\int_0^{+\infty}g^2(y)dy}.$$

We can also express the constant factor in terms of the sine cardinal function, as $(1/2)[1-\mathrm{sinc}(\theta\pi)]$, with $\mathrm{sinc}(a)=\sin(a)/a$ for $a\neq 0$, and $\mathrm{sinc}(0)=1$.

Example 4. Complementing the logarithmic functions considered in the second example, exponential functions can be investigated. Applying Theorem 2.1 with $\ell(t)=e^{\alpha t}$, $\alpha>0$, and $m(t)=e^{\beta t}$, $\beta>0$, we get

$$\int_0^{+\infty}\int_0^{+\infty}\frac{x^{1/p}y^{1/q}}{(x+y)^2}e^{(\alpha x+\beta y)/(x+y)}f(x)g(y)dxdy=\int_0^{+\infty}\int_0^{+\infty}\frac{x^{1/p}y^{1/q}}{(x+y)^2}\ell\left(\frac{x}{x+y}\right)m\left(\frac{y}{x+y}\right)f(x)g(y)dxdy$$

$$\leq\Upsilon\left[\int_0^{+\infty}f^p(x)dx\right]^{1/p}\left[\int_0^{+\infty}g^q(y)dy\right]^{1/q},$$

where

$$\Upsilon=\left[\int_0^1\ell^p(t)dt\right]^{1/p}\left[\int_0^1 m^q(t)dt\right]^{1/q}=\left(\int_0^1 e^{\alpha pt}dt\right)^{1/p}\left(\int_0^1 e^{\beta qt}dt\right)^{1/q}=\frac{1}{(\alpha p)^{1/p}(\beta q)^{1/q}}\left(e^{\alpha p}-1\right)^{1/p}\left(e^{\beta q}-1\right)^{1/q}.$$

Thus, we have

$$\int_0^{+\infty}\int_0^{+\infty}\frac{x^{1/p}y^{1/q}}{(x+y)^2}e^{(\alpha x+\beta y)/(x+y)}f(x)g(y)dxdy\leq\frac{1}{(\alpha p)^{1/p}(\beta q)^{1/q}}\left(e^{\alpha p}-1\right)^{1/p}\left(e^{\alpha q}-1\right)^{1/q}\left[\int_0^{+\infty}f^p(x)dx\right]^{1/p}\left[\int_0^{+\infty}g^q(y)dy\right]^{1/q}.$$

2.3 Secondary results

We can use Theorem 2.1 to establish other new general two-dimensional integral inequality results. Three such results are presented and proved below.

The proposition below can be viewed as a variant of Theorem 2.1.

Proposition 2.2. *Let $p>1$, $q=p/(p-1)$, and $f,g:(0,+\infty)\mapsto(0,+\infty)$ and $\ell,m:(0,1)\mapsto(0,+\infty)$ be four functions. Then, we have*

$$\int_0^{+\infty}\int_0^{+\infty}\frac{1}{(x+y)^2}\ell\left(\frac{x}{x+y}\right)m\left(\frac{y}{x+y}\right)f(x)g(y)dxdy\le\Upsilon\left[\int_0^{+\infty}\frac{1}{x}f^p(x)dx\right]^{1/p}\left[\int_0^{+\infty}\frac{1}{y}g^q(y)dy\right]^{1/q},$$

where Υ is given by Equation (2.1), provided that the integrals involved converge.

Proof. We can express the main two-dimensional integral as follows:

$$\int_0^{+\infty}\int_0^{+\infty}\frac{1}{(x+y)^2}\ell\left(\frac{x}{x+y}\right)m\left(\frac{y}{x+y}\right)f(x)g(y)dxdy=\int_0^{+\infty}\int_0^{+\infty}\frac{x^{1/p}y^{1/q}}{(x+y)^2}\ell\left(\frac{x}{x+y}\right)m\left(\frac{y}{x+y}\right)f_\dagger(x)g_\dagger(y)dxdy,$$

where

$$f_\dagger(x)=\frac{1}{x^{1/p}}f(x),\quad g_\dagger(y)=\frac{1}{y^{1/q}}g(y).$$

Applying Theorem 2.1 to these functions, we get

$$\begin{aligned}\int_0^{+\infty}\int_0^{+\infty}\frac{x^{1/p}y^{1/q}}{(x+y)^2}\ell\left(\frac{x}{x+y}\right)m\left(\frac{y}{x+y}\right)f_\dagger(x)g_\dagger(y)dxdy&\le\Upsilon\left[\int_0^{+\infty}f_\dagger^p(x)dx\right]^{1/p}\left[\int_0^{+\infty}g_\dagger^q(y)dy\right]^{1/q}\\&=\Upsilon\left[\int_0^{+\infty}\frac{1}{x}f^p(x)dx\right]^{1/p}\left[\int_0^{+\infty}\frac{1}{y}g^q(y)dy\right]^{1/q},\end{aligned}$$

where Υ is given by Equation (2.1). So we have

$$\int_0^{+\infty}\int_0^{+\infty}\frac{1}{(x+y)^2}\ell\left(\frac{x}{x+y}\right)m\left(\frac{y}{x+y}\right)f(x)g(y)dxdy\le\Upsilon\left[\int_0^{+\infty}\frac{1}{x}f^p(x)dx\right]^{1/p}\left[\int_0^{+\infty}\frac{1}{y}g^q(y)dy\right]^{1/q}.$$

This ends the proof of Proposition 2.2. □

This result thus relativizes the importance of the power functions $x^{1/p}$ and $y^{1/q}$ in Theorem 2.1; they can be transposed to the integral norms of f and g, leading to appropriate weighted integral norms with suitable definitions of the weight functions.

The proposition below gives a framework that unifies the Hardy-Hilbert integral inequality and Theorem 2.1.

Proposition 2.3. *Let $p>1$, $q=p/(p-1)$, $\sigma\in[0,1]$, and $f,g:(0,+\infty)\mapsto(0,+\infty)$ and $\ell,m:(0,1)\mapsto(0,+\infty)$ be four functions. Then, we have*

$$\begin{aligned}&\int_0^{+\infty}\int_0^{+\infty}\frac{x^{(1-\sigma)/p}y^{(1-\sigma)/q}}{(x+y)^{2-\sigma}}\ell^{1-\sigma}\left(\frac{x}{x+y}\right)m^{1-\sigma}\left(\frac{y}{x+y}\right)f(x)g(y)dxdy\\&\le\frac{\pi^\sigma}{\sin^\sigma(\pi/p)}\Upsilon^{1-\sigma}\left[\int_0^{+\infty}f^p(x)dx\right]^{1/p}\left[\int_0^{+\infty}g^q(y)dy\right]^{1/q},\end{aligned}$$

where Υ is given by Equation (2.1), provided that the integrals involved converge.

Proof. The case $\sigma=0$ corresponds to Theorem 2.1, and the case $\sigma=1$ corresponds to the Hardy-Hilbert integral inequality as recalled in Equation (1.1). So let us assume $\sigma\in(0,1)$. We can express the main two-dimensional integral as follows:

$$\begin{aligned}&\int_0^{+\infty}\int_0^{+\infty}\frac{x^{(1-\sigma)/p}y^{(1-\sigma)/q}}{(x+y)^{2-\sigma}}\ell^{1-\sigma}\left(\frac{x}{x+y}\right)m^{1-\sigma}\left(\frac{y}{x+y}\right)f(x)g(y)dxdy\\&=\int_0^{+\infty}\int_0^{+\infty}\left[\frac{1}{x+y}f(x)g(y)\right]^\sigma\times\left[\frac{x^{1/p}y^{1/q}}{(x+y)^2}\ell\left(\frac{x}{x+y}\right)m\left(\frac{y}{x+y}\right)f(x)g(y)\right]^{1-\sigma}dxdy.\end{aligned}$$

Using the Hölder integral inequality applied to the two main terms at the parameters $1/\sigma$ and $1/(1-\sigma)$, the Hardy-Hilbert integral inequality and Theorem 2.1, we get

$$\int_0^{+\infty}\int_0^{+\infty}\left[\frac{1}{x+y}f(x)g(y)\right]^{\sigma}\times\left[\frac{x^{1/p}y^{1/q}}{(x+y)^2}\ell\left(\frac{x}{x+y}\right)m\left(\frac{y}{x+y}\right)f(x)g(y)\right]^{1-\sigma}dxdy$$

$$\leq\left[\int_0^{+\infty}\int_0^{+\infty}\frac{1}{x+y}f(x)g(y)dxdy\right]^{\sigma}\left[\int_0^{+\infty}\int_0^{+\infty}\frac{x^{1/p}y^{1/q}}{(x+y)^2}\ell\left(\frac{x}{x+y}\right)m\left(\frac{y}{x+y}\right)f(x)g(y)dxdy\right]^{1-\sigma}$$

$$\leq\left\{\frac{\pi}{\sin(\pi/p)}\left[\int_0^{+\infty}f^p(x)dx\right]^{1/p}\left[\int_0^{+\infty}g^q(y)dy\right]^{1/q}\right\}^{\sigma}\left\{\Upsilon\left[\int_0^{+\infty}f^p(x)dx\right]^{1/p}\left[\int_0^{+\infty}g^q(y)dy\right]^{1/q}\right\}^{1-\sigma}$$

$$=\frac{\pi^{\sigma}}{\sin^{\sigma}(\pi/p)}\Upsilon^{1-\sigma}\left[\int_0^{+\infty}f^p(x)dx\right]^{1/p}\left[\int_0^{+\infty}g^q(y)dy\right]^{1/q}.$$

So we have

$$\int_0^{+\infty}\int_0^{+\infty}\frac{x^{(1-\sigma)/p}y^{(1-\sigma)/q}}{(x+y)^{2-\sigma}}\ell^{1-\sigma}\left(\frac{x}{x+y}\right)m^{1-\sigma}\left(\frac{y}{x+y}\right)f(x)g(y)dxdy\leq\frac{\pi^{\sigma}}{\sin^{\sigma}(\pi/p)}\Upsilon^{1-\sigma}\left[\int_0^{+\infty}f^p(x)dx\right]^{1/p}\left[\int_0^{+\infty}g^q(y)dy\right]^{1/q}.$$

This concludes the proof of Proposition 2.3. □

As indicated in the proof, the case $\sigma=0$ corresponds to Theorem 2.1, and the case $\sigma=1$ corresponds to the Hardy-Hilbert integral inequality. To the best of our knowledge, all intermediate cases lead to new two-dimensional integral inequalities.

The proposition below presents a different formulation of Theorem 2.1, dealing with only one main function.

Proposition 2.4. *Let $p>1$, $q=p/(p-1)$, $\sigma\in[0,1]$, and $f:(0,+\infty)\mapsto(0,+\infty)$ and $\ell,m:(0,1)\mapsto(0,+\infty)$ be three functions. Then the inequality in Theorem 2.1 is equivalent to the following inequality:*

$$\int_0^{+\infty}\left[\int_0^{+\infty}\frac{x^{1/p}y^{1/q}}{(x+y)^2}\ell\left(\frac{x}{x+y}\right)m\left(\frac{y}{x+y}\right)f(x)dx\right]^p dy\leq\Upsilon^p\int_0^{+\infty}f^p(x)dx,$$

where Υ is given by Equation (2.1), provided that the integrals involved converge.

Proof. We start by proving that Theorem 2.1 implies the stated inequality. We can write the main two-dimensional integral term as follows:

$$\int_0^{+\infty}\left[\int_0^{+\infty}\frac{x^{1/p}y^{1/q}}{(x+y)^2}\ell\left(\frac{x}{x+y}\right)m\left(\frac{y}{x+y}\right)f(x)dx\right]^p dy$$

$$=\int_0^{+\infty}\left[\int_0^{+\infty}\frac{x^{1/p}y^{1/q}}{(x+y)^2}\ell\left(\frac{x}{x+y}\right)m\left(\frac{y}{x+y}\right)f(x)dx\right]^{p-1}\times\left[\int_0^{+\infty}\frac{x^{1/p}y^{1/q}}{(x+y)^2}\ell\left(\frac{x}{x+y}\right)m\left(\frac{y}{x+y}\right)f(x)dx\right]dy$$

$$=\int_0^{+\infty}\int_0^{+\infty}\frac{x^{1/p}y^{1/q}}{(x+y)^2}\ell\left(\frac{x}{x+y}\right)m\left(\frac{y}{x+y}\right)f(x)g_\star(y)dxdy,\tag{2.5}$$

where

$$g_\star(y)=\left[\int_0^{+\infty}\frac{x^{1/p}y^{1/q}}{(x+y)^2}\ell\left(\frac{x}{x+y}\right)m\left(\frac{y}{x+y}\right)f(x)dx\right]^{p-1}.$$

Applying Theorem 2.1 to the functions f and $g_\star$, we obtain

$$\int_0^{+\infty}\int_0^{+\infty}\frac{x^{1/p}y^{1/q}}{(x+y)^2}\ell\left(\frac{x}{x+y}\right)m\left(\frac{y}{x+y}\right)f(x)g_\star(y)dxdy\leq\Upsilon\left[\int_0^{+\infty}f^p(x)dx\right]^{1/p}\left[\int_0^{+\infty}g_\star^q(y)dy\right]^{1/q}.\tag{2.6}$$

Let us now investigate the second integral term of this upper bound. Since $q(p-1)=p$, we get

$$\int_0^{+\infty} g_\star^q(y)dy = \int_0^{+\infty}\left[\int_0^{+\infty}\frac{x^{1/p}y^{1/q}}{(x+y)^2}\ell\left(\frac{x}{x+y}\right)m\left(\frac{y}{x+y}\right)f(x)dx\right]^{q(p-1)}dy$$
$$=\int_0^{+\infty}\left[\int_0^{+\infty}\frac{x^{1/p}y^{1/q}}{(x+y)^2}\ell\left(\frac{x}{x+y}\right)m\left(\frac{y}{x+y}\right)f(x)dx\right]^{p}dy. \tag{2.7}$$

Combining Equations (2.5), (2.6) and (2.7), we get

$$\int_0^{+\infty}\left[\int_0^{+\infty}\frac{x^{1/p}y^{1/q}}{(x+y)^2}\ell\left(\frac{x}{x+y}\right)m\left(\frac{y}{x+y}\right)f(x)dx\right]^{p}dy$$
$$\leq \Upsilon\left[\int_0^{+\infty}f^p(x)dx\right]^{1/p}\left\{\int_0^{+\infty}\left[\int_0^{+\infty}\frac{x^{1/p}y^{1/q}}{(x+y)^2}\ell\left(\frac{x}{x+y}\right)m\left(\frac{y}{x+y}\right)f(x)dx\right]^{p}dy\right\}^{1/q}.$$

Simplifying both sides and using $1/p+1/q=1$, we have

$$\left\{\int_0^{+\infty}\left[\int_0^{+\infty}\frac{x^{1/p}y^{1/q}}{(x+y)^2}\ell\left(\frac{x}{x+y}\right)m\left(\frac{y}{x+y}\right)f(x)dx\right]^{p}dy\right\}^{1/p}\leq \Upsilon\left[\int_0^{+\infty}f^p(x)dx\right]^{1/p},$$

which implies that

$$\int_0^{+\infty}\left[\int_0^{+\infty}\frac{x^{1/p}y^{1/q}}{(x+y)^2}\ell\left(\frac{x}{x+y}\right)m\left(\frac{y}{x+y}\right)f(x)dx\right]^{p}dy\leq \Upsilon^p\int_0^{+\infty}f^p(x)dx.$$

This is the desired inequality.

Let us now assume that this inequality holds and implies Theorem 2.1.

We can express the main two-dimensional integral inequality of Theorem 2.1 as follows:

$$\int_0^{+\infty}\int_0^{+\infty}\frac{x^{1/p}y^{1/q}}{(x+y)^2}\ell\left(\frac{x}{x+y}\right)m\left(\frac{y}{x+y}\right)f(x)g(y)dxdy=\int_0^{+\infty}\left[\int_0^{+\infty}\frac{x^{1/p}y^{1/q}}{(x+y)^2}\ell\left(\frac{x}{x+y}\right)m\left(\frac{y}{x+y}\right)f(x)\right]g(y)dxdy.$$

Applying the Hölder integral inequality to the two main terms with respect to y at the parameters p and q, and using the supposed inequality, we get

$$\int_0^{+\infty}\left[\int_0^{+\infty}\frac{x^{1/p}y^{1/q}}{(x+y)^2}\ell\left(\frac{x}{x+y}\right)m\left(\frac{y}{x+y}\right)f(x)\right]g(y)dxdy$$
$$\leq\left\{\int_0^{+\infty}\left[\int_0^{+\infty}\frac{x^{1/p}y^{1/q}}{(x+y)^2}\ell\left(\frac{x}{x+y}\right)m\left(\frac{y}{x+y}\right)f(x)\right]^p dy\right\}^{1/p}\left[\int_0^{+\infty}g^q(y)dy\right]^{1/q}$$
$$=\left\{\Upsilon^p\int_0^{+\infty}f^p(x)dx\right\}^{1/p}\left[\int_0^{+\infty}g^q(y)dy\right]^{1/q}$$
$$=\Upsilon\left[\int_0^{+\infty}f^p(x)dx\right]^{1/p}\left[\int_0^{+\infty}g^q(y)dy\right]^{1/q}.$$

So we have

$$\int_0^{+\infty}\int_0^{+\infty}\frac{x^{1/p}y^{1/q}}{(x+y)^2}\ell\left(\frac{x}{x+y}\right)m\left(\frac{y}{x+y}\right)f(x)g(y)dxdy\leq\Upsilon\left[\int_0^{+\infty}f^p(x)dx\right]^{1/p}\left[\int_0^{+\infty}g^q(y)dy\right]^{1/q},$$

which is the inequality in Theorem 2.1. The equivalence is shown, which concludes the proof of Proposition 2.4. □

This result is of particular interest in operator theory, as it gives a guarantee of continuity in the sense of the integral norm for operators of the following form:

$$\mathscr{T}(f)(y) = \left[\int_0^{+\infty} \frac{x^{1/p}y^{1/q}}{(x+y)^2}\ell\left(\frac{x}{x+y}\right)m\left(\frac{y}{x+y}\right)f(x)dx\right]^p,$$

where ℓ and m can be adapted to different mathematical scenarios.

The rest of the article is devoted to a new three-dimensional perspective of integral inequality, inspired by our two-dimensional results.

3. Three-Dimensional Integral Inequality Results

3.1 Main result

The theorem below can be seen as a natural three-dimensional extension of Theorem 2.1. The addition of one dimension also allows for the use of more auxiliary functions while still having a manageable upper bound. The detailed proof, a complementary version of the theorem, and some discussion follow.

Theorem 3.1. *Let $p>1$, $q>1$ and $r = pq/(pq-p-q)$, and $f,g,h:(0,+\infty)\mapsto(0,+\infty)$ and $i,j,k,\ell,m,n:(0,1)\mapsto(0,+\infty)$ be nine functions. Then, we have*

$$\int_0^{+\infty}\int_0^{+\infty}\int_0^{+\infty}\frac{1}{(x+y+z)^2}\left(\frac{x}{x+y}\right)^{1/p}\left(\frac{y}{y+z}\right)^{1/q}\left(\frac{z}{x+z}\right)^{1/r}$$
$$\times\, i\left(\frac{x}{x+y}\right)j\left(\frac{y}{y+z}\right)k\left(\frac{z}{x+z}\right)\ell\left(\frac{x+y}{x+y+z}\right)m\left(\frac{y+z}{x+y+z}\right)n\left(\frac{x+z}{x+y+z}\right)f(x)g(y)h(z)dxdydz$$
$$\leq \Xi\left[\int_0^{+\infty}f^p(x)dx\right]^{1/p}\left[\int_0^{+\infty}g^q(y)dy\right]^{1/q}\left[\int_0^{+\infty}h^r(z)dz\right]^{1/r},$$

where

$$\Xi = \left[\int_0^1 i^p(t)dt\right]^{1/p}\left[\int_0^1 \ell^p(t)dt\right]^{1/p}\left[\int_0^1 j^q(t)dt\right]^{1/q}\left[\int_0^1 m^q(t)dt\right]^{1/q}\left[\int_0^1 k^r(t)dt\right]^{1/r}\left[\int_0^1 n^r(t)dt\right]^{1/r}, \tag{3.1}$$

provided that the integrals involved converge.

Proof. By a well-chosen decomposition of the integrand as the product of three main terms using $1/p+1/q+1/r=1$, and the generalized Hölder integral inequality applied to those terms at the parameters p, q, and $r = pq/(pq-p-q)$, we get

$$\int_0^{+\infty}\int_0^{+\infty}\int_0^{+\infty}\frac{1}{(x+y+z)^2}\left(\frac{x}{x+y}\right)^{1/p}\left(\frac{y}{y+z}\right)^{1/q}\left(\frac{z}{x+z}\right)^{1/r}$$
$$\times\, i\left(\frac{x}{x+y}\right)j\left(\frac{y}{y+z}\right)k\left(\frac{z}{x+z}\right)\ell\left(\frac{x+y}{x+y+z}\right)m\left(\frac{y+z}{x+y+z}\right)n\left(\frac{x+z}{x+y+z}\right)f(x)g(y)h(z)dxdydz$$
$$=\int_0^{+\infty}\int_0^{+\infty}\int_0^{+\infty}\frac{1}{(x+y+z)^{2/p}}\left(\frac{x}{x+y}\right)^{1/p}i\left(\frac{x}{x+y}\right)\ell\left(\frac{x+y}{x+y+z}\right)f(x)$$
$$\times\frac{1}{(x+y+z)^{2/q}}\left(\frac{y}{y+z}\right)^{1/q}j\left(\frac{y}{y+z}\right)m\left(\frac{y+z}{x+y+z}\right)g(y)$$
$$\times\frac{1}{(x+y+z)^{2/r}}\left(\frac{z}{x+z}\right)^{1/r}k\left(\frac{z}{x+z}\right)n\left(\frac{x+z}{x+y+z}\right)h(z)dxdydz$$
$$\leq C^{1/p}D^{1/q}E^{1/r}, \tag{3.2}$$

where

$$C=\int_0^{+\infty}\int_0^{+\infty}\int_0^{+\infty}\frac{1}{(x+y+z)^2}\times\frac{x}{x+y}i^p\left(\frac{x}{x+y}\right)\ell^p\left(\frac{x+y}{x+y+z}\right)f(x)dxdydz,$$

$$D=\int_0^{+\infty}\int_0^{+\infty}\int_0^{+\infty}\frac{1}{(x+y+z)^2}\times\frac{y}{y+z}j^q\left(\frac{y}{y+z}\right)m^q\left(\frac{y+z}{x+y+z}\right)g(y)dxdydz$$

and

$$E=\int_0^{+\infty}\int_0^{+\infty}\int_0^{+\infty}\frac{1}{(x+y+z)^2}\times\frac{z}{x+z}k^r\left(\frac{z}{x+z}\right)n^r\left(\frac{x+z}{x+y+z}\right)h(z)dxdydz.$$

For more details on the generalized Hölder integral inequality, we refer to [15, 16].

We can now find the exact expressions for C, D, and E, starting with C. Applying the Fubini-Tonelli integral theorem to permutate the three integral signs, introducing the term $1=(x+y)/(x+y)$ and changing the variables as $u=(x+y)/(x+y+z)$, so $du=[-(x+y)/(x+y+z)^2]dz$, $z=0\Rightarrow u=1$ and $z\to+\infty\Rightarrow u=0$, followed by the change of variables $v=x/(x+y)$, so $dv=[-x/(x+y)^2]dy$, $y=0\Rightarrow v=1$ and $y\to+\infty\Rightarrow v=0$, we get

$$\begin{aligned}C&=\int_0^{+\infty}f(x)\left\{\int_0^{+\infty}\frac{x}{(x+y)^2}i^p\left(\frac{x}{x+y}\right)\left[\int_0^{+\infty}\frac{x+y}{(x+y+z)^2}\ell^p\left(\frac{x+y}{x+y+z}\right)dz\right]dy\right\}dx\\&=\int_0^{+\infty}f(x)\left\{\int_0^{+\infty}\frac{x}{(x+y)^2}i^p\left(\frac{x}{x+y}\right)\left[\int_0^{1}\ell^p(u)du\right]dy\right\}dx\\&=\left[\int_0^1\ell^p(u)du\right]\int_0^{+\infty}f(x)\left[\int_0^{+\infty}\frac{x}{(x+y)^2}i^p\left(\frac{x}{x+y}\right)dy\right]dx\\&=\left[\int_0^1\ell^p(u)du\right]\int_0^{+\infty}f(x)\left[\int_0^1 i^p(v)dv\right]dx\\&=\left[\int_0^1\ell^p(t)dt\right]\left[\int_0^1 i^p(t)dt\right]\int_0^{+\infty}f(x)dx.\end{aligned}\tag{3.3}$$

In a similar way, but with the introduction of the term $1=(y+z)/(y+z)$, the change of variables $u=(y+z)/(x+y+z)$ with $du=[-(y+z)/(x+y+z)^2]dx$, $x=0\Rightarrow u=1$ and $x\to+\infty\Rightarrow u=0$, and the change of variables $v=y/(y+z)$ with $dv=[-y/(y+z)^2]dz$, $z=0\Rightarrow v=1$ and $z\to+\infty\Rightarrow v=0$, we get

$$\begin{aligned}D&=\int_0^{+\infty}g(y)\left\{\int_0^{+\infty}\frac{y}{(y+z)^2}j^q\left(\frac{y}{y+z}\right)\left[\int_0^{+\infty}\frac{y+z}{(x+y+z)^2}m^q\left(\frac{y+z}{x+y+z}\right)dx\right]dz\right\}dy\\&=\int_0^{+\infty}g(y)\left\{\int_0^{+\infty}\frac{y}{(y+z)^2}j^q\left(\frac{y}{y+z}\right)\left[\int_0^{1}m^q(u)du\right]dz\right\}dy\\&=\left[\int_0^1 m^q(u)du\right]\int_0^{+\infty}g(y)\left[\int_0^{+\infty}\frac{y}{(y+z)^2}j^q\left(\frac{y}{y+z}\right)dz\right]dy\\&=\left[\int_0^1 m^q(u)du\right]\int_0^{+\infty}g(y)\left[\int_0^1 j^q(v)dv\right]dy\\&=\left[\int_0^1 m^q(t)dt\right]\left[\int_0^1 j^q(t)dt\right]\int_0^{+\infty}g(y)dy.\end{aligned}\tag{3.4}$$

Adopting a similar approach, but with the introduction of the term $1=(x+z)/(x+z)$, the change of variables $u=(x+z)/(x+y+z)$ with $du=[-(x+z)/(x+y+z)^2]dy$, $y=0\Rightarrow u=1$ and $y\to+\infty\Rightarrow u=0$, and the change of variables $v=z/(x+z)$

with $dv = [-z/(x+z)^2]dx$, $x = 0 \Rightarrow v = 1$ and $x \to +\infty \Rightarrow v = 0$, we get

$$
\begin{aligned}
E &= \int_0^{+\infty} h(z) \left\{ \int_0^{+\infty} \frac{z}{(x+z)^2} k^r \left(\frac{z}{x+z} \right) \left[\int_0^{+\infty} \frac{x+z}{(x+y+z)^2} n^r \left(\frac{x+z}{x+y+z} \right) dy \right] dx \right\} dz \\
&= \int_0^{+\infty} h(z) \left\{ \int_0^{+\infty} \frac{z}{(x+z)^2} k^r \left(\frac{z}{x+z} \right) \left[\int_0^1 n^r(u) du \right] dx \right\} dz \\
&= \left[\int_0^1 n^r(u) du \right] \int_0^{+\infty} h(z) \left[\int_0^{+\infty} \frac{z}{(x+z)^2} k^r \left(\frac{z}{x+z} \right) dx \right] dz \\
&= \left[\int_0^1 n^r(u) du \right] \int_0^{+\infty} h(z) \left[\int_0^1 k^r(v) dv \right] dz \\
&= \left[\int_0^1 n^r(t) dt \right] \left[\int_0^1 k^r(t) dt \right] \int_0^{+\infty} h(z) dz. \qquad (3.5)
\end{aligned}
$$

Combining Equations (3.2), (3.3), (3.4) and (3.5), and using $1/p + 1/q + 1/r = 1$, we get

$$
\begin{aligned}
&\int_0^{+\infty} \int_0^{+\infty} \int_0^{+\infty} \frac{1}{(x+y+z)^2} \left(\frac{x}{x+y} \right)^{1/p} \left(\frac{y}{y+z} \right)^{1/q} \left(\frac{z}{x+z} \right)^{1/r} \\
&\times i\left(\frac{x}{x+y} \right) j\left(\frac{y}{y+z} \right) k\left(\frac{z}{x+z} \right) \ell\left(\frac{x+y}{x+y+z} \right) m\left(\frac{y+z}{x+y+z} \right) n\left(\frac{x+z}{x+y+z} \right) f(x)g(y)h(z)dxdydz \\
&\leq \left\{ \left[\int_0^1 \ell^p(t) dt \right] \left[\int_0^1 i^p(t) dt \right] \int_0^{+\infty} f(x) dx \right\}^{1/p} \left\{ \left[\int_0^1 m^q(t) dt \right] \left[\int_0^1 j^q(t) dt \right] \int_0^{+\infty} g(y) dy \right\}^{1/q} \\
&\times \left\{ \left[\int_0^1 n^r(t) dt \right] \left[\int_0^1 k^r(t) dt \right] \int_0^{+\infty} h(z) dz \right\}^{1/r} \\
&= \left[\int_0^1 i^p(t) dt \right]^{1/p} \left[\int_0^1 \ell^p(t) dt \right]^{1/p} \left[\int_0^1 j^q(t) dt \right]^{1/q} \left[\int_0^1 m^q(t) dt \right]^{1/q} \\
&\times \left[\int_0^1 k^r(t) dt \right]^{1/r} \left[\int_0^1 n^r(t) dt \right]^{1/r} \left[\int_0^{+\infty} f^p(x) dx \right]^{1/p} \left[\int_0^{+\infty} g^q(y) dy \right]^{1/q} \left[\int_0^{+\infty} h^r(z) dz \right]^{1/r} \\
&= \Xi \left[\int_0^{+\infty} f^p(x) dx \right]^{1/p} \left[\int_0^{+\infty} g^q(y) dy \right]^{1/q} \left[\int_0^{+\infty} h^r(z) dz \right]^{1/r},
\end{aligned}
$$

where Ξ is indicated in Equation (3.1). This concludes the proof of Theorem 3.1. □

Another version of Theorem 3.1 can be presented by thoroughly changing the order of the variables x, y, and z. It is given below.

Theorem 3.2. *Let $p > 1$, $q > 1$ and $r = pq/(pq - p - q)$, and $f, g, h : (0, +\infty) \mapsto (0, +\infty)$ and $i, j, k, \ell, m, n : (0, 1) \mapsto (0, +\infty)$ be nine functions. Then, we have*

$$
\begin{aligned}
&\int_0^{+\infty} \int_0^{+\infty} \int_0^{+\infty} \frac{1}{(x+y+z)^2} \left(\frac{x}{x+z} \right)^{1/p} \left(\frac{y}{x+y} \right)^{1/q} \left(\frac{z}{y+z} \right)^{1/r} \\
&\times i\left(\frac{x}{x+z} \right) j\left(\frac{y}{x+y} \right) k\left(\frac{z}{y+z} \right) \ell\left(\frac{x+z}{x+y+z} \right) m\left(\frac{x+y}{x+y+z} \right) n\left(\frac{y+z}{x+y+z} \right) f(x)g(y)h(z)dxdydz \\
&\leq \Xi \left[\int_0^{+\infty} f^p(x) dx \right]^{1/p} \left[\int_0^{+\infty} g^q(y) dy \right]^{1/q} \left[\int_0^{+\infty} h^r(z) dz \right]^{1/r},
\end{aligned}
$$

where Ξ is given by Equation (3.1), provided that the integrals involved converge.

Proof. The proof is almost identical to that of Theorem 3.1. There are only slight modifications in the management of the variables x, y, and z. For the sake of redundancy, we omit the full development. □

The main interest of Theorems 3.1 and 3.2 is the generality of the three-dimensional integral. They include six auxiliary functions, i, j, k, ℓ, m and n, which provide additional flexibility. Another major strength is the simplicity of the upper bound. It depends only on the unweighted integral norms of the functions involved, allowing tractable three-dimensional integral inequalities to be derived. These can have useful applications in operator theory dealing with three-dimensional operators and related areas.

In the rest of this section, several examples are given involving different types of auxiliary functions.

3.2 Examples

Some specific examples of applications of Theorem 3.1 are given below. They deal with different functions i, j, k, ℓ, m and n. Similar examples can be given for Theorem 3.2. We omit them for the sake of redundancy.

Example 1. We start with the use of standard power functions. Applying Theorem 3.1 with $i(t) = t^{\alpha}$, $\alpha > 0$, $j(t) = t^{\beta}$, $\beta > 0$, $k(t) = t^{\gamma}$, $\gamma > 0$, $\ell(t) = t^{\kappa}$, $\kappa > 0$, $m(t) = t^{\theta}$, $\theta > 0$, and $n(t) = t^{\upsilon}$, $\upsilon > 0$, we get

$$\int_0^{+\infty}\int_0^{+\infty}\int_0^{+\infty} \frac{x^{1/p}y^{1/q}z^{1/r}(x+y)^{\kappa-\alpha-1/p}(y+z)^{\theta-\beta-1/q}(x+z)^{\upsilon-\gamma-1/r}}{(x+y+z)^{2+\kappa+\theta+\upsilon}} f(x)g(y)h(z)dxdydz$$

$$= \int_0^{+\infty}\int_0^{+\infty}\int_0^{+\infty} \frac{1}{(x+y+z)^2}\left(\frac{x}{x+y}\right)^{1/p}\left(\frac{y}{y+z}\right)^{1/q}\left(\frac{z}{x+z}\right)^{1/r}$$

$$\times i\left(\frac{x}{x+y}\right) j\left(\frac{y}{y+z}\right) k\left(\frac{z}{x+z}\right) \ell\left(\frac{x+y}{x+y+z}\right) m\left(\frac{y+z}{x+y+z}\right) n\left(\frac{x+z}{x+y+z}\right) f(x)g(y)h(z)dxdydz$$

$$\leq \Xi \left[\int_0^{+\infty} f^p(x)dx\right]^{1/p}\left[\int_0^{+\infty} g^q(y)dy\right]^{1/q}\left[\int_0^{+\infty} h^r(z)dz\right]^{1/r},$$

where

$$\Xi = \left[\int_0^1 i^p(t)dt\right]^{1/p}\left[\int_0^1 \ell^p(t)dt\right]^{1/p}\left[\int_0^1 j^q(t)dt\right]^{1/q}\left[\int_0^1 m^q(t)dt\right]^{1/q}\left[\int_0^1 k^r(t)dt\right]^{1/r}\left[\int_0^1 n^r(t)dt\right]^{1/r}$$

$$= \left(\int_0^1 t^{\alpha p}dt\right)^{1/p}\left(\int_0^1 t^{\kappa p}dt\right)^{1/p}\left(\int_0^1 t^{\beta q}dt\right)^{1/q}\left(\int_0^1 t^{\theta q}dt\right)^{1/q}\left(\int_0^1 t^{\gamma r}dt\right)^{1/r}\left(\int_0^1 t^{\upsilon r}dt\right)^{1/r}$$

$$= \frac{1}{(\alpha p+1)^{1/p}(\kappa p+1)^{1/p}(\beta q+1)^{1/q}(\theta q+1)^{1/q}(\gamma r+1)^{1/r}(\upsilon r+1)^{1/r}}.$$

As a result, we have

$$\int_0^{+\infty}\int_0^{+\infty}\int_0^{+\infty} \frac{x^{1/p}y^{1/q}z^{1/r}(x+y)^{\kappa-\alpha-1/p}(y+z)^{\theta-\beta-1/q}(x+z)^{\upsilon-\gamma-1/r}}{(x+y+z)^{2+\kappa+\theta+\upsilon}} f(x)g(y)h(z)dxdydz$$

$$\leq \frac{1}{(\alpha p+1)^{1/p}(\kappa p+1)^{1/p}(\beta q+1)^{1/q}(\theta q+1)^{1/q}(\gamma r+1)^{1/r}(\upsilon r+1)^{1/r}}$$

$$\times \left[\int_0^{+\infty} f^p(x)dx\right]^{1/p}\left[\int_0^{+\infty} g^q(y)dy\right]^{1/q}\left[\int_0^{+\infty} h^r(z)dz\right]^{1/r}.$$

This provides a new manageable three-dimensional integral inequality, with numerous adjustable parameters, which can be adapted to different contexts.

Example 2. Applying Theorem 3.1 with $i(t)=1$, $j(t)=1$, $k(t)=1$, $\ell(t)=[-\log(t)]^{\alpha}$, $\alpha>0$, $m(t)=[-\log(t)]^{\beta}$, $\beta>0$, and $n(t)=[-\log(t)]^{\gamma}$, $\gamma>0$, we get

$$\int_0^{+\infty}\int_0^{+\infty}\int_0^{+\infty}\frac{1}{(x+y+z)^2}\left(\frac{x}{x+y}\right)^{1/p}\left(\frac{y}{y+z}\right)^{1/q}\left(\frac{z}{x+z}\right)^{1/r}$$

$$\times\left[-\log\left(\frac{x+y}{x+y+z}\right)\right]^{\alpha}\left[-\log\left(\frac{y+z}{x+y+z}\right)\right]^{\beta}\left[-\log\left(\frac{x+z}{x+y+z}\right)\right]^{\gamma}f(x)g(y)h(z)dxdydz$$

$$=\int_0^{+\infty}\int_0^{+\infty}\int_0^{+\infty}\frac{1}{(x+y+z)^2}\left(\frac{x}{x+y}\right)^{1/p}\left(\frac{y}{y+z}\right)^{1/q}\left(\frac{z}{x+z}\right)^{1/r}$$

$$\times\, i\left(\frac{x}{x+y}\right)j\left(\frac{y}{y+z}\right)k\left(\frac{z}{x+z}\right)\ell\left(\frac{x+y}{x+y+z}\right)m\left(\frac{y+z}{x+y+z}\right)n\left(\frac{x+z}{x+y+z}\right)f(x)g(y)h(z)dxdydz$$

$$\leq \Xi\left[\int_0^{+\infty}f^p(x)dx\right]^{1/p}\left[\int_0^{+\infty}g^q(y)dy\right]^{1/q}\left[\int_0^{+\infty}h^r(z)dz\right]^{1/r},$$

where

$$\Xi=\left[\int_0^1 i^p(t)dt\right]^{1/p}\left[\int_0^1 \ell^p(t)dt\right]^{1/p}\left[\int_0^1 j^q(t)dt\right]^{1/q}\left[\int_0^1 m^q(t)dt\right]^{1/q}\left[\int_0^1 k^r(t)dt\right]^{1/r}\left[\int_0^1 n^r(t)dt\right]^{1/r}$$

$$=\left[\int_0^1[-\log(t)]^{\alpha p}dt\right]^{1/p}\left[\int_0^1[-\log(t)]^{\beta q}dt\right]^{1/q}\left[\int_0^1[-\log(t)]^{\gamma r}dt\right]^{1/r}$$

$$=\Gamma^{1/p}(\alpha p+1)\Gamma^{1/q}(\beta q+1)\Gamma^{1/r}(\gamma r+1).$$

More directly, we have

$$\int_0^{+\infty}\int_0^{+\infty}\int_0^{+\infty}\frac{1}{(x+y+z)^2}\left(\frac{x}{x+y}\right)^{1/p}\left(\frac{y}{y+z}\right)^{1/q}\left(\frac{z}{x+z}\right)^{1/r}$$

$$\times\left[-\log\left(\frac{x+y}{x+y+z}\right)\right]^{\alpha}\left[-\log\left(\frac{y+z}{x+y+z}\right)\right]^{\beta}\left[-\log\left(\frac{x+z}{x+y+z}\right)\right]^{\gamma}f(x)g(y)h(z)dxdydz$$

$$\leq\Gamma^{1/p}(\alpha p+1)\Gamma^{1/q}(\beta q+1)\Gamma^{1/r}(\gamma r+1)\left[\int_0^{+\infty}f^p(x)dx\right]^{1/p}\left[\int_0^{+\infty}g^q(y)dy\right]^{1/q}\left[\int_0^{+\infty}h^r(z)dz\right]^{1/r}.$$

We emphasize the crucial role of the gamma function in the constant factor and the relative complexity of the integrand.

Example 3. Theorem 3.1 can involve trigonometric functions. For example, applying it with $p=3$, $q=3$, $i(t)=1$, $j(t)=1$, $k(t)=1$, $\ell(t)=\sin[\theta(\pi/2)t]$, $\theta\in[0,1]$, $m(t)=\sin[\theta(\pi/2)t]=\ell(t)$, $\theta\in[0,1]$, and $n(t)=\sin[\theta(\pi/2)t]=\ell(t)=m(t)$, $\theta\in[0,1]$, we get

$$\int_0^{+\infty}\int_0^{+\infty}\int_0^{+\infty}\frac{1}{(x+y+z)^2}\left(\frac{x}{x+y}\right)^{1/3}\left(\frac{y}{y+z}\right)^{1/3}\left(\frac{z}{x+z}\right)^{1/3}$$

$$\times\sin\left[\theta\frac{\pi}{2}\left(\frac{x+y}{x+y+z}\right)\right]\sin\left[\theta\frac{\pi}{2}\left(\frac{y+z}{x+y+z}\right)\right]\sin\left[\theta\frac{\pi}{2}\left(\frac{x+z}{x+y+z}\right)\right]f(x)g(y)h(z)dxdydz$$

$$=\int_0^{+\infty}\int_0^{+\infty}\int_0^{+\infty}\frac{1}{(x+y+z)^2}\left(\frac{x}{x+y}\right)^{1/p}\left(\frac{y}{y+z}\right)^{1/q}\left(\frac{z}{x+z}\right)^{1/r}$$

$$\times\, i\left(\frac{x}{x+y}\right)j\left(\frac{y}{y+z}\right)k\left(\frac{z}{x+z}\right)\ell\left(\frac{x+y}{x+y+z}\right)m\left(\frac{y+z}{x+y+z}\right)n\left(\frac{x+z}{x+y+z}\right)f(x)g(y)h(z)dxdydz$$

$$\leq \Xi \left[\int_0^{+\infty} f^p(x)dx\right]^{1/p} \left[\int_0^{+\infty} g^q(y)dy\right]^{1/q} \left[\int_0^{+\infty} h^r(z)dz\right]^{1/r}$$

$$= \Xi \left[\int_0^{+\infty} f^3(x)dx\right]^{1/3} \left[\int_0^{+\infty} g^3(y)dy\right]^{1/3} \left[\int_0^{+\infty} h^3(z)dz\right]^{1/3},$$

where

$$\Xi = \left[\int_0^1 i^p(t)dt\right]^{1/p} \left[\int_0^1 \ell^p(t)dt\right]^{1/p} \left[\int_0^1 j^q(t)dt\right]^{1/q} \left[\int_0^1 m^q(t)dt\right]^{1/q} \left[\int_0^1 k^r(t)dt\right]^{1/r} \left[\int_0^1 n^r(t)dt\right]^{1/r}$$
$$= \left[\int_0^1 \sin^3\left(\theta\frac{\pi}{2}t\right)dt\right]^{1/3} \left[\int_0^1 \sin^3\left(\theta\frac{\pi}{2}t\right)dt\right]^{1/3} \left[\int_0^1 \sin^3\left(\theta\frac{\pi}{2}t\right)dt\right]^{1/3}$$
$$= \int_0^1 \sin^3\left(\theta\frac{\pi}{2}t\right)dt = \frac{8}{3\theta\pi}\sin^4\left(\theta\frac{\pi}{4}\right)\left[2+\cos\left(\theta\frac{\pi}{2}\right)\right].$$

Therefore, we have

$$\int_0^{+\infty}\int_0^{+\infty}\int_0^{+\infty} \frac{1}{(x+y+z)^2}\left(\frac{x}{x+y}\right)^{1/3}\left(\frac{y}{y+z}\right)^{1/3}\left(\frac{z}{x+z}\right)^{1/3}$$

$$\times \sin\left[\theta\frac{\pi}{2}\left(\frac{x+y}{x+y+z}\right)\right]\sin\left[\theta\frac{\pi}{2}\left(\frac{y+z}{x+y+z}\right)\right]\sin\left[\theta\frac{\pi}{2}\left(\frac{x+z}{x+y+z}\right)\right] f(x)g(y)h(z)dxdydz$$

$$\leq \frac{8}{3\theta\pi}\sin^4\left(\theta\frac{\pi}{4}\right)\left[2+\cos\left(\theta\frac{\pi}{2}\right)\right]\left[\int_0^{+\infty} f^3(x)dx\right]^{1/3}\left[\int_0^{+\infty} g^3(y)dy\right]^{1/3}\left[\int_0^{+\infty} h^3(z)dz\right]^{1/3}.$$

Example 4. As a last simple example, applying Theorem 3.1 with $i(t) = e^{\alpha t}$, $\alpha > 0$, $j(t) = e^{\beta t}$, $\beta > 0$, $k(t) = e^{\gamma t}$, $\gamma > 0$, $\ell(t) = e^{\kappa t}$, $\kappa > 0$, $m(t) = e^{\theta t}$, $\theta > 0$, and $n(t) = e^{\upsilon t}$, $\upsilon > 0$, we get

$$\int_0^{+\infty}\int_0^{+\infty}\int_0^{+\infty} \frac{1}{(x+y+z)^2}\left(\frac{x}{x+y}\right)^{1/p}\left(\frac{y}{y+z}\right)^{1/q}\left(\frac{z}{x+z}\right)^{1/r}$$

$$\times e^{\alpha x/(x+y)+\beta y/(y+z)+\gamma z/(x+z)+[(\kappa+\upsilon)x+(\kappa+\theta)y+(\theta+\upsilon)z]/(x+y+z)} f(x)g(y)h(z)dxdydz$$

$$= \int_0^{+\infty}\int_0^{+\infty}\int_0^{+\infty} \frac{1}{(x+y+z)^2}\left(\frac{x}{x+y}\right)^{1/p}\left(\frac{y}{y+z}\right)^{1/q}\left(\frac{z}{x+z}\right)^{1/r}$$

$$\times i\left(\frac{x}{x+y}\right) j\left(\frac{y}{y+z}\right) k\left(\frac{z}{x+z}\right) \ell\left(\frac{x+y}{x+y+z}\right) m\left(\frac{y+z}{x+y+z}\right) n\left(\frac{x+z}{x+y+z}\right) f(x)g(y)h(z)dxdydz$$

$$\leq \Xi \left[\int_0^{+\infty} f^p(x)dx\right]^{1/p} \left[\int_0^{+\infty} g^q(y)dy\right]^{1/q} \left[\int_0^{+\infty} h^r(z)dz\right]^{1/r},$$

where

$$\Xi = \left[\int_0^1 i^p(t)dt\right]^{1/p} \left[\int_0^1 \ell^p(t)dt\right]^{1/p} \left[\int_0^1 j^q(t)dt\right]^{1/q} \left[\int_0^1 m^q(t)dt\right]^{1/q} \left[\int_0^1 k^r(t)dt\right]^{1/r} \left[\int_0^1 n^r(t)dt\right]^{1/r}$$

$$= \left(\int_0^1 e^{\alpha pt}dt\right)^{1/p}\left(\int_0^1 e^{\kappa pt}dt\right)^{1/p}\left(\int_0^1 e^{\beta qt}dt\right)^{1/q}\left(\int_0^1 e^{\theta qt}dt\right)^{1/q}\left(\int_0^1 e^{\gamma rt}dt\right)^{1/r}\left(\int_0^1 e^{\upsilon rt}dt\right)^{1/r}$$

$$= \frac{1}{(\alpha p)^{1/p}(\kappa p)^{1/p}(\beta q)^{1/q}(\theta q)^{1/q}(\gamma r)^{1/r}(\upsilon r)^{1/r}}$$

$$\times (e^{\alpha p}-1)^{1/p}(e^{\kappa p}-1)^{1/p}(e^{\alpha q}-1)^{1/q}\left(e^{\theta q}-1\right)^{1/q}(e^{\gamma r}-1)^{1/r}(e^{\upsilon r}-1)^{1/r}.$$

More directly, we have

$$\int_0^{+\infty}\int_0^{+\infty}\int_0^{+\infty}\frac{1}{(x+y+z)^2}\left(\frac{x}{x+y}\right)^{1/p}\left(\frac{y}{y+z}\right)^{1/q}\left(\frac{z}{x+z}\right)^{1/r}$$
$$\times e^{\alpha x/(x+y)+\beta y/(y+z)+\gamma z/(x+z)+[(\kappa+\upsilon)x+(\kappa+\theta)y+(\theta+\upsilon)z]/(x+y+z)}f(x)g(y)h(z)dxdydz$$
$$\leq \frac{1}{(\alpha p)^{1/p}(\kappa p)^{1/p}(\beta q)^{1/q}(\theta q)^{1/q}(\gamma r)^{1/r}(\upsilon r)^{1/r}}$$
$$\times\left(e^{\alpha p}-1\right)^{1/p}\left(e^{\kappa p}-1\right)^{1/p}\left(e^{\alpha q}-1\right)^{1/q}\left(e^{\theta q}-1\right)^{1/q}\left(e^{\gamma r}-1\right)^{1/r}\left(e^{\upsilon r}-1\right)^{1/r}$$
$$\times\left[\int_0^{+\infty}f^p(x)dx\right]^{1/p}\left[\int_0^{+\infty}g^q(y)dy\right]^{1/q}\left[\int_0^{+\infty}h^r(z)dz\right]^{1/r}.$$

To the best of our knowledge, this is again a new three-dimensional integral inequality in the literature. We can also think of using the exponential function to establish various inequalities of the Laplace transform of three-dimensional functions.

4. Conclusion

In conclusion, this article offers new tools to the theory of two- and three-dimensional integral inequalities by establishing two general theorems. This is characterized by the presence of several auxiliary functions. The first theorem focuses on the two-dimensional case and gives a simple upper bound for two-dimensional integrals of a certain form. This upper bound is based on the integral norms of the function involved. The second theorem can be seen as a natural extension of the first to three dimensions. It still provides tractable, sharp, and general upper bounds. They may lead to further developments in mathematical analysis in three dimensions. The theory has been illustrated by several examples dealing with specific auxiliary functions. Some complementary results beyond the standard framework have also been established.

Future work may explore refined inequalities under additional structural conditions. Applications to operator theory, functional analysis, and partial differential equations are also anticipated. Furthermore, the flexibility of the approach suggests possible generalizations to higher dimensions. We will explore these perspectives in future articles.

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