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Research Article

A note on coset complexes of p-subgroups

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Abstract

This paper investigates the coset complexes of p-subgroups in finite groups. Given a finite group G and a prime p, we define $\mathscr{C}_p(G)$ as the poset of all cosets of p-subgroups of G. We construct a probability function $\mathrm{P}_p(G,s)$ with group-theoretic connections, strengthen the congruence formula of the p-local Euler characteristic of $\mathscr{C}_p(G)$, and analyze the connectivity of $\mathscr{C}_p(G)$.

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1. Introduction

All groups considered in this paper are finite. Let G be a group and p a prime. Given a positive integer s, we write

$$\phi_p(G,s) = |\{(g_1,....,g_s) \mid \ g_i \in G, 1 \le i \le s \text{ and } \langle g_1,....,g_s\rangle \text{ is a } p\text{-group}\}|.$$

Then a probability function that randomly selects s-elements from G to generate p-subgroups can be defined by

$$\mathrm{P}_p(G,s) = \frac{\phi_p(G,s)}{|G|^s}.$$

Obviously, G is a p-group if and only if $\mathrm{P}_p(G,s) = 1$. For a prime p and a group G, we denote by $\mathcal{S}_p(G)$ the poset of all nontrivial p-subgroups of G. Let

$$\mathcal{I}_p(G) = \Big\{P_1 \cap P_2 \cdots \cap P_s \mid P_i \in \mathrm{Syl}_p(G) \text{ for all } 1 \le i \le s, \text{ and } s \ge 1\Big\}$$

be the set of all intersections of some Sylow p-subgroups of G.

Theorem 1.1. *Let G be a group and p a prime. Suppose that G is not a p-group. The probability function $\mathrm{P}_p(G,s)$ is given by:*

$$\mathrm{P}_p(G,s) = -\sum_{H\in \mathcal{S}_p(G)\cup\{1\}} \frac{\mu(H,G)}{|G:H|^s} = -\sum_{H\in \mathcal{I}_p(G)} \frac{\mu(H,G)}{|G:H|^s},$$

where μ is the Möbius function of the poset $\mathcal{S}_p(G)\cup\{1,G\}$.

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It is easily observed that $\mathrm{P}_p(G,s)$ belongs to the ring of finite Dirichlet series

$$\mathbb{C}[1/2^s, 1/3^s, 1/5^s, \cdots],$$

which is a unique factorization domain and it is interesting to study the factorization of $\mathrm{P}_p(G,s)$ as in [1, 2]. We denote by $|G|_{p'}$ the largest positive integer that is coprime to p and divides the order of the group G. Theorem 1.1 implies that $1/|G|_{p'}^s$ divides $\mathrm{P}_p(G,s)$ for each finite group G and each prime p. Throughout this paper, we define

$$\mathrm{Z}_p(G,s) = |G|_{p'}^s\, \mathrm{P}_p(G,s).$$

In fact, we can observe that $\mathrm{Z}_p(G,s) \in \mathbb{Z}[1/p^s]$, that is, $\mathrm{Z}_p(G,s)$ is a polynomial function of $1/p^s$ with integer coefficients.

In [3], we denote by

$$\mathscr{C}_p(G) = \{Hx \mid H \text{ is a } p\text{-subgroup of } G, x \in G\}$$

the set of all right cosets Hx with p-subgroups H (including the identity subgroup) of G. Let $\Delta\mathscr{C}_p(G)$ be the order complex of $\mathscr{C}_p(G)$. We study the p-local Euler characteristic of $\Delta\mathscr{C}_p(G)$, which is defined by

$$\chi_p(G) := \frac{\chi(\mathscr{C}_p(G))}{|G|_{p'}},$$

where $\chi(\mathscr{C}_p(G))$ denotes the Euler characteristic of $\Delta\mathscr{C}_p(G)$.

It easily follows from [3, Theorem A] that

$$\chi_p(G) = \mathrm{Z}_p(G,-1).$$

It is worth noting here that if G is p-closed then $\chi_p(G) = 1$, and the converse is not true in general, for example, $G = S_3 \times S_3$ and $p = 2$ (see detail in [3, Theorem C]). Here we give a description on p-closed groups and p-TI-groups G with the function $\mathrm{Z}_p(G,s)$. Recall that for a prime p, a group G is said to be a p-TI-group if for every $g \in G$, either $P \cap P^g = 1$ or $P = P^g$, where P is a Sylow p-subgroup of G. Such class of groups has been described in [5, 8].

Theorem 1.2. *Let G be a group and p a prime. Then*

(1) $\mathrm{Z}_p(G,s) = 1$ if and only if G is p-closed;
(2) $\mathrm{Z}_p(G,s) = n_p - \frac{n_p - 1}{|G|_p^s}$ if and only if G is a p-TI-group.

where n_p is the number of all Sylow p-subgroups of G.

In [3, Theorem D], we prove that $\chi_p(G) \equiv 1 \pmod{p^d}$, where p^d is the smallest index of the intersection of two distinct Sylow p-subgroups P, Q of G in P. In fact, we can show a slight further result.

Theorem 1.3. *Let p be a prime and let G be a non-p-closed group. Then*

$$\chi_p(G) \equiv |\,\mathrm{Syl}_p(G)| \pmod{p^{d+1}},$$

where $p^d = \min\{|P : P \cap Q| \mid P, Q \in \mathrm{Syl}_p(G) \text{ with } P \neq Q\}$.

In [3, Theorem B], it is shown that a group G is p-closed if and only if $\mathscr{C}_p(G)$ has exactly $|G|_{p'}$ connected components. Denote the set of connected components of the poset $\mathscr{C}_p(G)$ by $\pi_0\mathscr{C}_p(G)$, for which a detailed definition can be found in Section 4. In fact, we have

Theorem 1.4. *Let G be a group and let P be a Sylow p-subgroup of G for some prime p. Then $|\pi_0\mathscr{C}_p(G)| = |G : P^G|$, where $P^G = \langle P^x | x \in G\rangle$, the normal closure of P in G.*

2. Probability function $\mathrm{P}_p(G, s)$

Let $\mathscr{C}$ be a finite poset and denote by

$$\mathrm{I}(\mathscr{C}) = \{(x, y) \in \mathscr{C} \times \mathscr{C} \mid x \leq y\}.$$

the subset of $\mathscr{C} \times \mathscr{C}$ consisting of all pairs x, y in $\mathscr{C}$ with $x \leq y$. Recall that the Möbius function μ of $\mathscr{C}$ is a function from $\mathrm{I}(\mathscr{C})$ to $\mathbb{Z}$ such that for each pair $(x, y) \in \mathrm{I}(\mathscr{C})$,

$$\sum_{x \leq z \leq y} \mu(x, z) = \delta(x, y) = \sum_{x \leq z \leq y} \mu(z, y),$$

where $\delta(x, y) = 1$ if $x = y$; and $\delta(x, y) = 0$ if $x < y$.

The following lemma was described in [4, Theorem 2.3]. For the sake of completeness, we present a proof.

Lemma 2.1. *Let $\mathfrak{X}$ be a poset consisting of some subgroups of a group G such that $G \notin \mathfrak{X}$ and all meets of some members of $\mathfrak{X}$ exist in $\mathfrak{X}$. Let μ be the Möbius function of $\overline{\mathfrak{X}} = \mathfrak{X} \cup \{G\}$. Let $H \in \mathfrak{X}$ with $\mu(H, G) \neq 0$. Then H is the meet of a certain number of maximal members of $\mathfrak{X}$.*

Proof. Assume that H is not the meet of a certain number of maximal members of $\mathfrak{X}$. We work by induction on $|G : H|$. Let M be the meet of all maximal members of $\mathfrak{X}$ which contain H. Then we have $H < M$. Write $\mathfrak{Y} = \{K \in \overline{\mathfrak{X}} \mid H < K \text{ and } \mu(K, G) \neq 0\}$. For each $K \in \mathfrak{Y}$ with $Y \neq G$, applying the induction, we get that K is the meet of some maximal members of $\mathfrak{X}$. Note that such maximal members also contains H. Hence $M \leq K$ by the choice of M. Now, by the definition of μ,

$$\begin{aligned} \mu(H, G) &= -\sum_{H < K \in \overline{\mathfrak{X}}} \mu(K, G) = -\sum_{H < K \in \mathfrak{Y}} \mu(K, G) \\ &= -\sum_{M \leq K \in \mathfrak{Y}} \mu(K, G) = -\sum_{M \leq K \in \overline{\mathfrak{X}}} \mu(K, G) = 0. \end{aligned}$$

□

Proof of Theorem 1.1. We may assume that G is not a p-group and write $\mathfrak{X} = \mathcal{S}_p(G) \cup \{1\}$ and $\overline{\mathfrak{X}} = \mathfrak{X} \cup \{G\}$. Recall that $\phi_p(G, s)$ is the number of s-tuple elements in G generating p-groups. For $K \in \overline{\mathfrak{X}}$, we set

$$\psi_p(K, s) = |\{(k_1, \cdots, k_s) \mid k_i \in K \text{ and } K = \langle k_1, \cdots, k_s \rangle \text{ is } p\text{-group}\}|.$$

Note that $\psi_p(G, s) = 0$ as G is not a p-group. For each $K \in \overline{\mathfrak{X}}$, by definition,

$$\phi_p(K, s) = \sum_{H \leq K \text{ in } \overline{\mathfrak{X}}} \psi_p(H, s).$$

Note that the above equation also holds for $K = G$ as $\psi_p(G, s) = 0$. Applying Möbius inversion formula [9, Proposition 1.2.5], we obtain that

$$\psi_p(K, s) = \sum_{H \leq K \text{ in } \overline{\mathfrak{X}}} \phi_p(H, s)\mu(H, K),$$

where μ is the Möbius function on $\overline{\mathfrak{X}}$. In particular, for $K = G$, it follows that

$$0 = \psi_p(G, s) = \sum_{H \leq G \text{ in } \overline{\mathfrak{X}}} \phi_p(H, s)\mu(H, G) = \phi_p(G, s) + \sum_{H \in \mathfrak{X}} \phi_p(H, s)\mu(H, G),$$

as $\psi_p(G, s) = 0$. For each $H \in \mathfrak{X}$, H is a p-group, which implies that $\phi_p(H, s) = |H|^s$ by definition. Hence

$$\phi_p(G, s) = -\sum_{H \in \mathfrak{X}} \phi_p(H, s)\mu(H, G) = -\sum_{H \in \mathfrak{X}} \mu(H, G)|H|^s.$$

Then it easily follows from that

$$\mathrm{P}_p(G,s)=\frac{\phi_p(G,s)}{|G|^s}=-\sum_{H\in\mathfrak{X}}\frac{\mu(H,G)}{|G:H|^s}=-\sum_{H\in\mathcal{I}_p(G)}\frac{\mu(H,G)}{|G:H|^s}.$$

The last equation follows from Lemma 2.1.

□

Proof of Theorem 1.2. We will first prove Part (1). If G is p-closed, $\mathcal{I}_p(G)$ contains only the Sylow p-subgroup of G. It easily from Theorem 1.1 and the definition of $\mathrm{Z}_p(G,s)$ that $\mathrm{Z}_p(G,s)=1$, as desired. Conversely, we may assume that $\mathrm{Z}_p(G,s)=1$. If G is not p-closed, then, by Theorem 1.1,

$$1=\mathrm{Z}_p(G,s)=-\sum_{H\in\mathcal{I}_p(G)}\frac{\mu(H,G)}{|G:H|_p^s}=-\sum_{H\in\mathcal{I}_p(G)\setminus\mathrm{Syl}_p(G)}\frac{\mu(H,G)}{|G:H|_p^s}+|\operatorname{Syl}_p(G)|,$$

where μ is the Möbius function of the poset $\mathcal{S}_p(G)\cup\{1,G\}$ and $\mu(H,G)=-1$ for $H\in\mathrm{Syl}_p(G)$. Comparing the coefficients of $\mathrm{Z}_p(G,s)$ as a polynomial of $1/p^s$, we conclude $|\operatorname{Syl}_p(G)|=1$, which is a contradiction. Hence G is p-closed, as desired.

Next we show the sufficiency of Part (2). If G is p-closed, $n_p=1$. As in the sufficiency proof of Part (1), $\mathrm{Z}_p(G,s)=1=n_p$, as required.

Assume G is not p-closed. Since G is a p-TI-group, $\mathcal{I}_p(G)=\mathrm{Syl}_p(G)\cup\{1\}$. Consequently, according to Theorem 1.1,

$$\mathrm{Z}_p(G,s)=-\sum_{H\in\mathcal{I}_p(G)}\frac{\mu(H,G)}{|G:H|_p^s}=n_p-\frac{\mu(1,G)}{|G|_p^s},$$

where μ is the Möbius function of the poset $\mathcal{S}_p(G)\cup\{1,G\}$ and $\mu(H,G)=-1$ for $H\in\mathrm{Syl}_p(G)$. By definition of μ and Lemma 2.1,

$$\mu(1,G)=-\sum_{1<K\in\mathcal{S}_p(G)\cup\{G\}}\mu(K,G)=-\sum_{1<K\in\mathcal{I}_p(G)}\mu(K,G)-1=n_p-1.$$

Hence $\mathrm{Z}_p(G,s)=n_p-(n_p-1)/|G|_p^s$, as desired.

Finally, we show the necessity of Part (2). We assume that $\mathrm{Z}_p(G,s)=n_p-(n_p-1)/|G|_p^s$, where $n_p=|\operatorname{Syl}_p(G)|$. We will assume that G is not p-closed. Then $\mathfrak{X}=\mathcal{I}_p(G)\setminus\mathrm{Syl}_p(G)\neq\varnothing$. Write $p^t=\min\{|G|_p/|H|\mid H\in\mathfrak{X}\}$ and $\mathcal{B}=\{H\in\mathfrak{X}\mid |G|_p/|H|=p^t\}$. Clearly $p^t>1$.

For each $H\in\mathcal{B}$ and $H<K\in\mathcal{I}_p(G)$, the minimality of $|G|_p/|H|$ implies that $K\in\mathrm{Syl}_p(G)$ and so $\mu(K,G)=-1$. By definition of μ and Lemma 2.1,

$$\mu(H,G)=-\sum_{H<K\in\mathcal{S}_p(G)\cup\{G\}}\mu(K,G)=-\sum_{H<K\in\mathcal{I}_p(G)}\mu(K,G)-1=n_H-1,$$

where n_H is the number of Sylow p-subgroups of G containing H. Since H is the intersection of at least two Sylow p-subgroups, $n_H\geq 2$. Hence $\mu(H,G)\geq 1$ for each $H\in\mathcal{B}$.

Viewing $\mathrm{Z}_p(G,s)$ as a polynomial in $\mathbb{Z}[1/p^s]$, the coefficients of the term $(1/p^s)^t$ in $\mathrm{Z}_p(G,s)$ is

$$\sum_{H\in\mathcal{B}}\mu(H,G)>0.$$

Since $\mathrm{Z}_p(G,s)=n_p-(n_p-1)/|G|_p^s$, comparing the non-zero coefficients, we have that $p^t=|G|_p$. The minimality of p^t implies that $\mathfrak{X}=\{1\}$. This means that $\mathcal{I}_p(G)=\mathrm{Syl}_p(G)\cup\{1\}$ and so G is a p-TI-group by definition. □

3. p-local Euler characteristic of $\mathscr{C}_p(G)$

Lemma 3.1. [7, Theorem] *Let K be a subgroup of G of order p^m, where p is a prime. If $m \le n$ and p^n dividing $|G|$, the number of subgroups of order p^n in G containing K is congruent to 1 modulo p.*

Proof of Theorem 1.3. Write $\mathfrak{X} = \mathcal{S}_p(G) \cup \{1, G\}$. Let

$$p^d = \min\{|P : P \cap Q| \mid P, Q \in \mathrm{Syl}_p(G),\ P \neq Q\}.$$

Since G is not p-closed, $p^d > 1$. Write $\mathcal{A} = \{P \cap Q \mid P, Q \in \mathrm{Syl}_p(G) \text{ and } |P : P \cap Q| = p^d\}$. For each $H \in \mathcal{A}$, as $\mu(K, G) = 0$ for all $K \in \mathfrak{X} \setminus (\mathcal{I}_p(G) \cup \{G\})$ by Lemma 2.1, we have that

$$0 = \sum_{H \le K \in \mathfrak{X}} \mu(K, G) = \mu(G, G) + \sum_{H \le K \in \mathcal{I}_p(G)} \mu(K, G) = 1 + \sum_{H \le K \in \mathcal{I}_p(G)} \mu(K, G).$$

Since $H \in \mathcal{A}$, H is the largest intersection of at least two distinct Sylow subgroups. Hence, for each $H < K \in \mathcal{I}_p(G)$, $K \in \mathrm{Syl}_p(G)$ and $\mu(K, G) = -1$ Now we will obtain

$$\mu(H, G) = -1 - \sum_{H < K \in \mathcal{I}_p(G)} \mu(K, G) = -1 - \sum_{H < K \in \mathcal{I}_p(G)} (-1) = n_H - 1,$$

where n_H is the number of Sylow p-subgroups of G containing H. Applying Lemma 3.1, we have that p divides $n_H - 1 = \mu(H, G)$ for each $H \in \mathcal{A}$.

Note that for each $K \in \mathcal{I}_p(G) \setminus (\mathcal{A} \cup \mathrm{Syl}_p(G))$, the minimality of p^d implies that p^{d+1} divides $|G|_p/|K|$. Then we have

$$\begin{aligned}
\chi_p(G) &= - \sum_{H \in \mathcal{I}_p(G)} \mu(H, G) \frac{|G|_p}{|H|} \\
&\equiv - \sum_{H \in \mathrm{Syl}_p(G)} \mu(H, G) \frac{|G|_p}{|H|} - \sum_{H \in \mathcal{A}} \mu(H, G) \frac{|G|_p}{|H|} \pmod{p^{d+1}} \\
&\equiv - \sum_{H \in \mathrm{Syl}_p(G)} (-1) - \sum_{H \in \mathcal{A}} \mu(H, G) p^d \pmod{p^{d+1}} \\
&\equiv |\mathrm{Syl}_p(G)| - \sum_{H \in \mathcal{A}} \mu(H, G) p^d \pmod{p^{d+1}} \\
&\equiv |\mathrm{Syl}_p(G)| \pmod{p^{d+1}}.
\end{aligned}$$

The last equality hold since p divides $\mu(H, G)$ for each $H \in \mathcal{A}$. □

4. Connectivity of $\mathscr{C}_p(G)$

Recall that, in a finite poset $(X, \le)$, we say there is a path from x to y (written by $x \sim y$) for $x, y \in X$ if there exist $x_0, x_1, \ldots, x_n \in X$ such that $x = x_0, x_n = y$ and either $x_i \le x_{i+1}$ or $x_i \ge x_{i+1}$ for each $i = 0, 1, \ldots, n$. Denote by

$$[x] = \{y \in X \mid y \sim x\}$$

the connected component containing x of X and by $\pi_0(X) = \{[x] \mid x \in X\}$ the set of all connected components of X. In particular, X is called connected if X has only one connected component; otherwise X is called disconnected, as studied in [6, section 5].

Now, let us consider the set of all connected components of $\mathscr{C}_p(G)$.

Lemma 4.1. *Let G be a group and P be a Sylow p-subgroup of G for some prime p. Then $\pi_0 \mathscr{C}_p(G) = \{[Px] \mid x \in G\}$.*

Proof. Since G is the union of all cosets Px with $x \in G$, for each $Qy \in \mathscr{C}_p(G)$, there exists some $x \in G$ such that $Qy \cap Px \neq \varnothing$. Let $z \in Qy \cap Px$. We obtain that $Qy = Qz$ and $Px = Pz$, moreover, $Qy \cap Py = Qz \cap Pz = (Q \cap P)z \in \mathscr{C}_p(G)$. This implies that there is a path $Qy \supseteq Qy \cap Px \subseteq Px$ in $\mathscr{C}_p(G)$. Thus $[Qy] = [Px]$, and consequently $\pi_0\mathscr{C}_p(G) = \{[Px] \mid x \in G\}$. □

Proof of Theorem 1.4. By Lemma 4.1, $\pi_0\mathscr{C}_p(G) = \{[Px] \mid x \in G\}$. We consider the action of G on $\pi_0\mathscr{C}_p(G)$ defined by $[Px] \cdot g \triangleq [Pxg]$. It is not difficult to check that such action is well-defined and transitive. Now let S be the stabilizer of $[P]$ in G. The transitivity of this action implies that $|G : S| = |\pi_0\mathscr{C}_p(G)|$. We only have to show that $S = P^G$, the normal closure of P in G.

For any $g \in P^G$, we can express g as a product $g = x_1x_2\cdots x_r$, where each x_i is a p-element of G for $1 \leq i \leq r$. Write $P_i = \langle x_i \rangle, y_i = x_{i+1}\cdots x_r$ for $1 \leq i \leq r-1$ and set $y_0 = x_1x_2\cdots x_r = g$ and $y_r = 1$. It is easy to see that $\{y_{i-1}\} \subseteq P_iy_i \supseteq \{y_i\} \subseteq P_{i+1}y_{i+1}$ for each $i \geq 1$. Hence there exists a sequence of inclusions in $\mathscr{C}_p(G)$ as follows:

$$Pg \supseteq \{g = y_0\} \subseteq P_1y_1 \supseteq y_1 \subseteq P_2y_2 \supseteq \cdots \supseteq y_{r-1} \subseteq P_r \supseteq \{y_r = 1\} \subseteq P,$$

which implies that $[P] = [Pg] = [P]g$ and so $g \in S$.

Conversely, for any $g \in S$, we have $[Pg] = [P]$. It implies the existence of a sequence of vertices T_iy_i in $\mathscr{C}_p(G)$ such that:

$$Pg = T_1y_1 \supseteq T_2y_2 \subseteq T_3y_3 \supseteq \cdots \subseteq T_{2n-1}y_{2n-1} = P.$$

From this, we can deduce that:

$$g^{-1} = y_{2n-1}^{-1}(y_{2n-1}y_{2n-2}^{-1})\cdots(y_3y_2^{-1})(y_2g^{-1}) \in \langle T_1, T_3, \ldots, T_{2n-1}\rangle \leq P^G.$$

Thus we have shown that $S = P^G$, as desired.

□

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