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A fast and reliable numerical approach for solving lake pollution problem

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Abstract

Just like air, water is an essential component of the environment and deterioration of water sources threatens all living organisms. Hence, investigating the water pollution problem is highly important. Lakes constitutes a big portion of water sources. The aim of this study is to analyze the dynamics of pollution in a system of three interconnected lakes using the Gegenbauer wavelet method. The problem is modeled by three ordinary linear differential equations representing the rate of pollution in each lake with respect to time. Time derivates are approximated by the truncated Gegenbauer wavelet series and the system of differential equations are converted into a system of algebraic equations. We show that the proposed technique is reliable and fast by comparing the numerical results with other numerical results available in the literature. We also show that the method is highly accurate and hence can also be used to solve other ecological phenomena as well.

Keywords: *Lake contamination, water pollution, gegenbauer wavelets.*

Göl kirliliği probleminin çözümü için hızlı ve güvenilir bir sayısal yaklaşım

Öz

Su, hava gibi çevrenin temel bileşenlerinden biridir ve su kaynaklarının bozulması tüm canlı organizmaları tehdit etmektedir. Bu nedenle su kirliliği sorununu araştırmak büyük önem taşımaktadır. Göller, su kaynaklarının büyük bir bölümünü oluşturmaktadır. Bu çalışmanın amacı, üç adet birbiriyle bağlantılı gölden oluşan bir sistemdeki kirlilik dinamiklerini Gegenbauer dalgacık yöntemi ile analiz etmektir. Problem, her bir göldeki kirlilik oranının zamana göre değişimini temsil eden üç doğrusal diferansiyel denklem sistemi ile modellenmiştir. Zaman türevlerine, kesikli Gegenbauer dalgacık serisiyle yaklaşılmış ve diferansiyel denklem sistemi, cebirsel denklem sistemine dönüştürülmüştür.

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Elde edilen sayısal sonuçlar, literatürde mevcut diğer sayısal sonuçlarla karşılaştırılarak önerilen tekniğin güvenilir ve hızlı olduğu gösterilmiştir. Ayrıca, yöntemin yüksek doğruluk sağladığı ve bu nedenle diğer ekolojik olayların çözümünde de kullanılabileceği ortaya konmuştur.

Anahtar kelimeler: *Göl kirlenmesi, su kirliliği, gegenbauer dalgacıkları.*

1. Introduction

Water is a source of life and is one of the most important parts of the environment. We can classify water sources as surface and ground water sources. Dams, rivers and lakes can be listed among surface water sources, and they meet the industrial, agricultural, and domestic water demand. The availability of these sources is diminishing due to discharges of untreated wastewater. Thus, it is important to observe water pollutants and their effects regularly. Since lakes constitutes a big portion of domestic water usage, pollution of them may have devastating effects on daily life. Hence, this problem has attracted attention of many researchers. Several numerical and analytical methods are introduced for the solution of the problem. These methods can be listed as follows: compartment modeling [1], q-homotopy analysis transform method [2], Bernoulli Ritz-collocation method [3], homotopy perturbation method, Laplace transform and Padé approximants [4], Bessel polynomials [5], Laplace transform method [6], Boubaker collocation method [7], modified differential transform method [8], semianalytical method [9], Laplace-Adomian decomposition method [10], Taylor series method [11] and the variational iteration method [12].

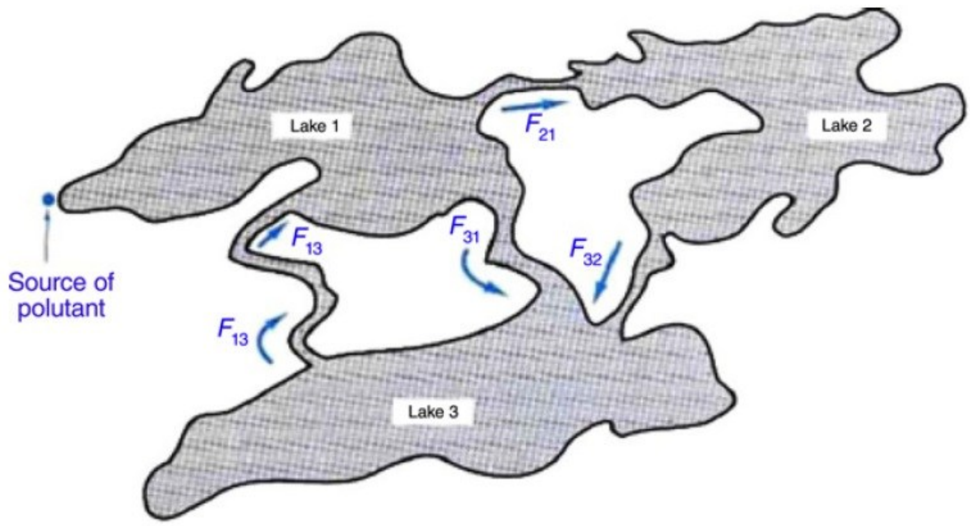

Figure 1. System of three lakes with interconnecting channels [1].

In this study, we analyze a mathematical model that describes the dynamics of pollution in a system of three inter-connected lakes given in Fig.1. Biazar *et al.* [1] introduced a series of simplifying assumptions to provide a foundation for the development of the governing equations. First, each lake is considered as a large and well-mixed separate division connected by channels that are treated as pipes, facilitating fluid flow. The direction of the fluid flow within these channels is indicated by arrows. It is assumed that the pollutant is uniformly distributed throughout each lake due to effective mixing processes. The water volume (V_k) in each lake is considered to be constant over time. The nature of the pollution is regarded as permanent, with no transformation into other forms occurring. Initially, none of the lakes contain any pollution. The rate of change in the concentration of pollutant in a lake is equal to the rate at which pollutants flow into the lake minus the rate at which pollutants flow out of the lake.

The motivation of this study is to explore the levels of pollution in each lake at any specified time. In order to do that, we solve the model by using the Gegenbauer wavelet method. Wavelet methods are known to be effective methods and result in accurate solutions [13–20]. In Section 2, we introduce the governing equations of the model and explain the problem parameters. Section 3 is reserved for the definition and application of the Gegenbauer wavelets. In the 4th section, we give the numerical results and discussion. Finally in section 5, we give the conclusion of the study.

1. Governing equations

The pollution dynamics within the three-lake system following the assumptions are given as in [1]

$$\frac{dX_1}{dt} = \frac{F_{13}}{V_3}X_3(t) - \frac{F_{31}}{V_1}X_1(t) - \frac{F_{21}}{V_1}X_1(t) + p(t)$$

$$\frac{dX_2}{dt} = \frac{F_{21}}{V_1}X_1(t) - \frac{F_{32}}{V_2}X_2(t) \qquad 0 \le t \le b \qquad (2.1)$$

$$\frac{dX_3}{dt} = \frac{F_{31}}{V_1}X_1(t) + \frac{F_{32}}{V_2}X_2(t) - \frac{F_{13}}{V_3}X_3(t)$$

with the initial conditions

$$X_1(0) = r_1, \quad X_2(0) = r_2, \quad X_3(0) = r_3.$$

Here, $p(t)$ represents the rate of pollutant discharged into Lake 1. This function may remain constant or vary over time, reflecting changes in the pollutant input. $X_k(t)$ and V_k denote the amount of pollution in the lake k $(k = 1,2,3)$ at any time $t \ge 0$ and the volume of water, respectively. The term $\frac{X_k(t)}{V_k}$ represents the concentration of pollution in lake k at any time. Furthermore F_{mk} represents the flow rate from lake k to lake m. $\frac{F_{mk}}{V_k}X_k(t)$ is the pollutant flow that V_k measures the rate at which the pollutant concentration in the lake k flows into the lake m at time t.

Since the volume of each lake is assumed to remain constant, the inflow rate to each lake must be equal to the outflow rate. Hence, the flow rate for each lake can be written as follows

$Lake$ 1: $F_{13} = F_{21} + F_{31}$

$Lake\ 2: F_{21} = F_{32}$

$Lake\ 3: F_{13} = F_{31} + F_{32}$

The flow rate $F_{13}, F_{21}, F_{31}, F_{32}$ and the water volume V_1, V_2, V_3 are appropriate constants.

2. Gegenbauer wavelets and their applications

Gegenbauer wavelet technique is a powerful numerical tool based on the Gegenbauer polynomials and has a broad applicability across fields that require accurate solutions of differential equations with complex boundary and initial conditions. Wavelet methods, including the Gegenbauer wavelet approach, are quite popular to solve complex problems requiring high precision.

2.1. Gegenbauer polynomials:

Gegenbauer polynomials, $G_m^{\gamma}(t)$, also known as the ultraspherical harmonic polynomials of order $m \in \mathbb{Z}^+$, are defined over the interval $[-1,1]$ by using the following recurrence formula [14]

$$G_{m+1}^{\gamma}(t) = \frac{1}{m+1}\left(2(m+\gamma)tG_m^{\gamma}(t) - (m+2\gamma-1)G_{m-1}^{\gamma}(t)\right), \quad \text{for } m = 1,2,\ldots$$

where $G_0^{\gamma}(t) = 1$ and $G_1^{\gamma}(t) = 2\gamma t$. Here, m is the order of the polynomial, $\gamma > -1/2$ is the ultraspherical parameter. Different values of γ offer different polynomials, i.e $\gamma = 1/2$ refers to the Legendre polynomials, $\gamma = 0$ and $\gamma = 1$ refer to the first and second types of Chebyshev polynomials, respectively. These polynomials satisfy the orthogonality relation with respect to the weight function $(1-t^2)^{\gamma-(\frac{1}{2})}$ and form an orthogonal basis for the Hilbert space $L^2[-1,1]$

$$\int_{-1}^{1} \frac{1}{(1-t^2)^{\frac{1}{2}-\gamma}} G_m^{\gamma}(t) G_n^{\gamma}(t) dt = L_m^{\gamma}(t)\delta_{mn},$$

where $L_m^{\gamma}(t) = \frac{\pi 2^{1-2\gamma}\Gamma(m+2\gamma)}{m!(m+\gamma)(\Gamma(\gamma))^2}$ is the normalizing factor and δ is the Kronecker delta function, [14].

2.2. Gegenbauer wavelets

Gegenbauer wavelets are expressed as $\psi_{r,m}(t) = \Psi_{(l,r,m,\gamma,t)}$, where $l = 1,2,\ldots$ is the level of resolution, $r = 1,2,\ldots,2^{l-1}$ is the translation parameter, and t is the normalized time. Gegenbauer wavelets are defined in [0,1] as follows [14]

$$\psi_{r,m}(t) = \begin{cases} \frac{1}{\sqrt{L_m^{\gamma}}} 2^{l/2} G_m^{\gamma}(2^l t - 2r + 1), & if \ \frac{2r-2}{2^l} \le t \le \frac{2r}{2^l} \\ 0, & otherwise. \end{cases}$$

The first few Gegenbauer wavelets calculated by taking $l = 1, M = 4$ and $\gamma = 30$ can be listed as

$$\psi_{1,0}^{30}(t) = \begin{cases} 2.49122, & 0 \le t \le 1 \\ 0, & otherwise, \end{cases}$$

$$\psi_{1,1}^{30}(t) = \begin{cases} 39.2318t - 19.6159, & 0 \le t \le 1 \\ 0, & otherwise, \end{cases}$$

$$\psi_{1,2}^{30}(t) = \begin{cases} 447.481t^2 - 447.481t + 110.066, & 0 \le t \le 1 \\ 0, & otherwise, \end{cases}$$

$$\psi_{1,3}^{30}(t) = \begin{cases} 4264.91t^3 - 6397.36t^2 + 3148.7t - 508.124, & 0 \le t \le 1 \\ 0, & otherwise. \end{cases}$$

2.3. Function approximation:

Any unknown function $f(t) =\in L^2(\mathbb{R})$ defined in [0,1] can be expanded by the Gegenbauer wavelet series as follows:

$$f(t) = \sum_{r=0}^{\infty}\sum_{m=0}^{\infty} \omega_{rm}\psi_{r,m}(t), \tag{3.1}$$

where ω_{rm} is the Gegenbauer wavelet coefficient of the form $\omega_{rm} = < f(t), \psi_{r,m}(t) >$ and $<.,.>$ is the inner product. Truncating the infinite series in Eq. (3.1) yields

$$f(t) = \sum_{r=1}^{2^{l-1}}\sum_{m=0}^{M-1} \omega_{rm}\psi_{r,m}(t) = \Omega^T\Psi(\mathrm{t}), \tag{3.2}$$

where Ω and $\Psi(\mathrm{t})$ are the $2^{l-1}M \times 1$ matrices given by

$$\Omega^T = [\omega_{1,0}, \omega_{1,1}, \ldots, \omega_{1,M-1}, \omega_{2,0}, \omega_{2,1}, \ldots, \omega_{2,M-1}, \ldots, \omega_{2^{l-1},0}, \omega_{2^{l-1},1}, \ldots, \omega_{2^{l-1},M-1}]$$

$$\Psi(\mathrm{t}) = [\psi_{1,0}(t), \psi_{1,1}(t), \ldots, \psi_{1,M-1}(t), \psi_{2,0}(t), \psi_{2,1}(t), \ldots, \psi_{2,M-1}(t)$$
$$, \ldots, \psi_{2^{l-1},0}(t), \psi_{2^{l-1},1}(t), \ldots, \psi_{2^{l-1},M-1}(t)].$$

2.4. Implementation of the method:

Gegenbauer wavelet method can be adapted to any system by expanding the solution in terms of Gegenbauer polynomials, applying the system's operator to the expansion, using orthogonality to derive a system of equations, and solving for the coefficients. Due to this flexibility and precision, it is a powerful method for approximating solutions of complex differential equations across multiple fields. In order to solve the system, we first approximate the unknown functions $X_1(t), X_2(t)$ and $X_3(t)$ by using Eq. (3.2) as follows

$$X_1(t) = \sum_{r=1}^{2^{l-1}} \sum_{m=0}^{M-1} \omega_{1,m}\psi_{r,m}(t) = \Omega_1^T \Psi(\mathrm{t}),$$

$$X_2(t) = \sum_{r=1}^{2^{l-1}} \sum_{m=0}^{M-1} \omega_{2,m}\psi_{r,m}(t) = \Omega_2^T \Psi(\mathrm{t}), \tag{3.3}$$

$$X_3(t) = \sum_{r=1}^{2^{l-1}} \sum_{m=0}^{M-1} \omega_{3,m}\psi_{r,m}(t) = \Omega_3^T \Psi(\mathrm{t}),$$

and hence,

$$X_1{}'(t) = \sum_{r=1}^{2^{l-1}} \sum_{m=0}^{M-1} \omega_{1,m}\psi'_{r,m}(t) = \Omega_1^T \Psi'(\mathrm{t}),$$

$$X_2{}'(t) = \sum_{r=1}^{2^{l-1}} \sum_{m=0}^{M-1} \omega_{2,m}\psi'_{r,m}(t) = \Omega_2^T \Psi'(\mathrm{t}), \tag{3.4}$$

$$X_3{}'(t) = \sum_{r=1}^{2^{l-1}} \sum_{m=0}^{M-1} \omega_{3,m}\psi'_{r,m}(t) = \Omega_3^T \Psi'(\mathrm{t}),$$

where $\Psi'(\mathrm{t})$ is the $2^{l-1}M \times 1$ matrices given by

$$\Psi'(\mathrm{t}) = [\psi'_{1,0}(t), \psi'_{1,1}(t), \ldots, \psi'_{1,M-1}(t), \psi'_{2,0}(t), \psi'_{2,1}(t), \ldots, \psi'_{2,M-1}(t)$$
$$, \ldots, \psi'_{2^{l-1},0}(t), \psi'_{2^{l-1},1}(t), \ldots, \psi'_{2^{l-1},M-1}(t)].$$

Afterwards, we substitute the approximations in Eq. (3.3) and (3.4) into Eq. (2.1) and we obtain

$$\Omega_1^T \Psi'(\mathrm{t}) - \frac{F_{13}}{V_3}\Omega_3^T \Psi(\mathrm{t}) + \frac{F_{31}}{V_1}\Omega_1^T \Psi(\mathrm{t}) + \frac{F_{21}}{V_1}\Omega_1^T \Psi(\mathrm{t}) - p(t) = 0$$

$$\Omega_2^T \Psi'(t) - \frac{F_{21}}{V_1}\Omega_1^T \Psi(\mathrm{t}) + \frac{F_{32}}{V_2}\Omega_2^T \Psi(\mathrm{t}) = 0 \qquad 0 \le t \le b \tag{3.5}$$

$$\Omega_3^T \Psi'(t) - \frac{F_{31}}{V_1}\Omega_1^T \Psi(\mathrm{t}) - \frac{F_{32}}{V_2}\Omega_2^T \Psi(\mathrm{t}) + \frac{F_{13}}{V_3}\Omega_3^T \Psi(\mathrm{t}) = 0$$

with the initial conditions

$$\Omega_1^T\Psi(0) = r_{1,} \quad \Omega_2^T\Psi(0) = r_{2,} \quad \Omega_3^T\Psi(0) = r_3. \tag{3.6}$$

To solve the system described by Eq. (3.5) with the initial conditions given in Eq. (3.6), we need to solve $3(2^{l-1}M \times 1)$ equations. This gives us the same number of unknown coefficients. The first three equations are based on the initial conditions Eq. (3.6). The remaining equations are obtained by substituting the standard collocation points, $t_j = \frac{(2j-1)h}{2^l M}$, where $j = 1,2,\ldots, M-1$, into Eq. (2.1).

$$\Omega_1^T\Psi'^{(t_j)} - \frac{F_{13}}{V_3}\Omega_3^T\Psi(t_j) + \frac{F_{31}}{V_1}\Omega_1^T\Psi(t_j) + \frac{F_{21}}{V_1}\Omega_1^T\Psi(t_j) - p(t_j) = 0$$

$$\Omega_2^T\Psi'(t_j) - \frac{F_{21}}{V_1}\Omega_1^T\Psi(t_j) + \frac{F_{32}}{V_2}\Omega_2^T\Psi(t_j) = 0 \qquad 0 \le t \le b$$

(3.7)

$$\Omega_3^T\Psi'(t_j) - \frac{F_{31}}{V_1}\Omega_1^T\Psi(t_j) - \frac{F_{32}}{V_2}\Omega_2^T\Psi(t_j) + \frac{F_{13}}{V_3}\Omega_3^T\Psi(t_j) = 0.$$

Once we solve the system given in Eq. (3.7), we can determine the unknown coefficients. Then, by substituting these coefficients back into Eq. (3.5), we can obtain the desired solutions.

3. Numerical results and discussions

In this section, the numerical method explained in Section 3 is implemented by using specific parameter values associated with problem (2.1), and results are presented in terms of tables and graphs. All computations are conducted by using Mathematica 12.0 on a computer with a Windows 10 64-bit operating system. The problem parameters are taken as: $p(t) = 1 + \sin(t)$, $F_{13} = 38mi^3/year$, $F_{21} = 18mi^3/year$, $F_{31} = 20mi^3/year$, $F_{32} = 18mi^3/year$, $V_1 = 2900mi^3$, $V_2 = 850mi^3$, $V_3 = 1180mi^3$. Since there is no pollution at the beginning, initial conditions for each lake are zero, [1].

In the literature, this problem is solved by using $3^{rd}, 6^{th}$ and 10^{th} degree polynomials. Hence, we also solve the problem using the same degree polynomials for comparison purposes. l is taken as 1 and γ is taken as 30 throughout the study. The 3^{rd} degree approximation polynomials

$$X_{1,3}(t) = 2.77556 \times 10^{-17} + 0.995361t + 0.522067t^2 - 0.0626506t^3,$$

$$X_{2,3}(t) = -2.1684 \times 10^{-19} - 0.0000119539t + 0.0031625t^2 + 0.000904805t^3,$$

$$X_{3,3}(t) = -2.1684 \times 10^{-19} - 0.0000130571t + 0.00351245t^2 + 0.00101764t^3,$$

the 6^{th} degree approximation polynomials

$$X_{1,6}(t) = 1.21431 \times 10^{-17} + 0.1t + 0.493439t^2 - 0.0020665t^3 - 0.0417986t^4 + 0.00325774t^5 + 0.00124991t^6,$$

$$X_{2,6}(t) = -1.87194 \times 10^{-19} - 1.05987 \times 10^{-8}t + 0.00310359t^2 + 0.0009981925t^3 -. 6.05712 \times 10^{-6}t^4 - 0.0000558183t^5 + 3.82419 \times 10^{-6}t^6,$$

$$X_{3,6}(t) = -6.43745 \times 10^{-20} - 1.18264 \times 10^{-8}t + 0.00344843t^2 + 0.00111833t^3 - 4.56587 \times 10^{-6}t^4 - 0.0000620638t^5 + 4.18187 \times 10^{-6}t^6,$$

and the 10^{th} degree approximation polynomials are obtained as follows

$$X_{1,10}(t) = -3.70255 \times 10^{-17} + 1.0t + 0.493448t^2 - 0.00211828t^3 - 0.0416507t^4 + 0.000109127t^5 + 0.00138829t^6 - 2.49434 \times 10^{-6}t^7 - 0.0000249136t^8 + 1.22844 \times 10^{-7}t^9 + 2.45148 \times 10^{-7}t^{10},$$

$$X_{2,10}(t) = -2.26634 \times 10^{-19} - 3.70657 \times 10^{-14}t + 0.00310345t^2 + 0.000999021t^3 - 8.57585 \times 10^{-6}t^4 - 0.0000516683t^5 + 2.9605 \times 10^{-7}t^6 + 1.22859 \times 10^{-6}t^7 - 3.30126 \times 10^{-9}t^8 - 1.86713 \times 10^{-8}t^9 + 7.99694 \times 10^{-10}t^{10},$$

$$X_{3,10}(t) = -2.14802 \times 10^{-20} - 4.13529 \times 10^{-14}t + 0.00344828t^2 + 0.00111926t^3 - 7.37414 \times 10^{-6}t^4 - 0.0000574384t^5 + 2.52255 \times 10^{-7}t^6 + 1.36582 \times 10^{-6}t^7 - 2.29114 \times 10^{-9}t^8 - 2.07615 \times 10^{-8}t^9 + 8.75351 \times 10^{-10}t^{10},$$

We compare the CPU time of the current method and the BRCM [3] both employing a 10^{th} degree polynomial, along with the RK4 with step size $\Delta t = 0.1$ and Mathematica default solver and present the results in Table 1. We can see from this table that, even with a high-degree polynomial, the CPU time of the proposed method is quite small compared to RK4 and Mathematica, but significantly smaller than the BRCM. This demonstrates the high efficiency of the current method.

Table 1. Comparison of the CPU time

	BRCM [3]	GWM	RK4	Mathematica
CPU	4.29	0.080411	0.011045	0.104975

Table 2 represents the results of the GWM, the Bernoulli Ritz-collocation method (BRCM) [3], the Bessel collocation method (BCM) [3] all using 10^{th} degree polynomials as well as the fourth order Runge Kutta method (RK4) with stepsize $\Delta t = 0.1$ and

Mathematica default solver. One can see from the table that the solution of the current method agrees well with the other numerical methods.

Table 2. Comparison of the BCM [5], the BRCM [3], the GWM all using 10^{th} order polynomials, together with the RK4 and Mathematica default solver for each lake

		t = 0.2	t = 0.4	t = 0.6	t = 0.8	t = 1
	RK4	0.21965447	0.47775668	0.77185868	1.09805724	1.45114967
	BCM [5]	0.21965447	0.47775668	0.77185867	1.09805723	1.45114965
$X_1(t)$	BRCM [3]	0.21965450	0.47775670	0.77185870	1.09805700	1.45115000
	GWM	0.21965447	0.47775668	0.77185867	1.09805723	1.45114965
	Mathematica	0.21965442	0.47775665	0.77185865	1.09805724	1.45114961
	RK4	1.321000×10^{-4}	5.597440×10^{-4}	1.327949×10^{-3}	2.477595×10^{-3}	4.043729×10^{-3}
	BCM [5]	1.320999×10^{-4}	5.597436×10^{-4}	$1.327949 \times$	2.477594×10^{-3}	4.043728×10^{-3}
$X_2(t)$	BRCM [3]	1.320999×10^{-4}	5.597436×10^{-4}	1.327949×10^{-3}	2.477594×10^{-3}	4.043728×10^{-3}
	GWM	1.320999×10^{-4}	5.597436×10^{-4}	1.327949×10^{-3}	2.477595×10^{-3}	4.043729×10^{-3}
	Mathematica	1.321079×10^{-4}	5.597516×10^{-4}	1.327957×10^{-3}	2.477603×10^{-3}	4.043736×10^{-3}
	RK4	1.468551×10^{-4}	6.225831×10^{-4}	1.477767×10^{-3}	2.758464×10^{-3}	4.504315×10^{-3}
	BCM [5]	1.468549×10^{-4}	6.225828×10^{-4}	1.477766×10^{-3}	2.758463×10^{-3}	4.504314×10^{-3}
$X_3(t)$	BRCM [3]	1.468549×10^{-4}	6.225828×10^{-4}	1.477766×10^{-3}	2.758463×10^{-3}	4.504314×10^{-3}
	GWM	1.468549×10^{-4}	6.225828×10^{-4}	1.477766×10^{-3}	2.758463×10^{-3}	4.504314×10^{-3}
	Mathematica	1.468639×10^{-4}	6.225917×10^{-4}	1.477775×10^{-3}	2.758472×10^{-3}	4.504323×10^{-3}

In order to validate the proposed method, we calculate the residual error of the approximate solution for each lake. $RE_{X_1(t)}$, $RE_{X_2(t)}$ and $RE_{X_3(t)}$ represent the residual error calculated by substituting the approximate solutions into Eq. (2.1)

$$RE_{X_1(t_i)} = \left|\frac{F_{13}}{V_3}X_3(t_i) - \frac{F_{31}}{V_1}X_1(t_i) - \frac{F_{21}}{V_1}X_1(t_i) + p(t_i) - \frac{dX_1(t_i)}{dt}\right|,$$

$$RE_{X_2((t_i))} = |\frac{F_{21}}{V_1}X_1((t_i)) - \frac{F_{32}}{V_2}X_2((t_i)) - \frac{dX_2(t_i)}{dt}\ |,$$

$$RE_{X_3((t_i))} = |\frac{F_{31}}{V_1}X_1((t_i)) + \frac{F_{32}}{V_2}X_2((t_i)) - \frac{F_{13}}{V_3}X_3((t_i)) - \frac{d\,X_{3(t_i)}}{dt}|.$$

Table 3 presents the comparison of the residual error of the current technique, the Boubaker polynomial method (BPM) [7] and BCM [5] using 3^{rd} degree polynomials. One can see that the results are quite close to each other. But at some points the residual error of the current method is smaller.

Table 3. Comparison of the residual errors for the BPM [7], the BCM [5] and the GWM using 3^{rd} order polynomials for each lake

		t = 0	t = 0.2	t = 0.4	t = 0.6	t = 0.8	t = 1
$RE_{X_{1,3}}(t)$	BPM [7]	0	1.96030×10^{-3}	1.10160×10^{-3}	1.61790×10^{-3}	7.36010×10^{-3}	3.18840×10^{-2}
	BCM [5]	0	1.96030×10^{-3}	1.10160×10^{-3}	1.61790×10^{-3}	7.36010×10^{-3}	3.18840×10^{-2}
	GWM	0	5.67899×10^{-4}	6.60513×10^{-4}	8.89004×10^{-4}	2.07430×10^{-3}	1.60792×10^{-2}
$RE_{X_{2,3}}(t)$	BPM [7]	0	4.56380×10^{-6}	2.60790×10^{-6}	3.91180×10^{-6}	1.82550×10^{-5}	8.14960×10^{-5}
	BCM [5]	0	4.56380×10^{-6}	2.60790×10^{-6}	3.91180×10^{-6}	1.82550×10^{-5}	8.14960×10^{-5}
	GWM	0	1.47272×10^{-6}	1.74049×10^{-6}	2.39078×10^{-6}	5.71874×10^{-6}	4.56543×10^{-5}
$RE_{X_{3,3}}(t)$	BPM [7]	0	4.97420×10^{-6}	2.84240×10^{-6}	4.26360×10^{-6}	1.98970×10^{-5}	8.88260×10^{-5}
	BCM [5]	0	4.97420×10^{-6}	2.84240×10^{-6}	4.26360×10^{-6}	1.98970×10^{-5}	8.88260×10^{-5}
	GWM	0	1.60864×10^{-6}	1.90112×10^{-6}	2.61143×10^{-6}	6.24654×10^{-6}	4.98679×10^{-5}

Once the degree of the approximation polynomial increases, the residual error of the current method decreases. Table 4 shows that the residual error of the current method is smaller than the other methods after $t = 0.4$ when the degree of the polynomial is 6. A similar behavior is observed in Table 5. At $t = 0$, the residual error of the methods are close to each other, but after this time the residual error of the current method is smaller than the other methods. Hence, we may say that the proposed method is more accurate than the BPM and the BCM.

Table 4. Comparison of the residual errors for the BPM [7], the BCM [5] and the GWM using 6^{th} order polynomials for each lake

		t = 0	t = 0.2	t = 0.4	t = 0.6	t = 0.8	t = 1
$RE_{X_{1,6}}(t)$	BPM [7]	0	4.22730×10^{-8}	4.11970×10^{-8}	6.56580×10^{-8}	2.02930×10^{-7}	1.04560×10^{-5}
	BCM [5]	0	4.22730×10^{-8}	4.11970×10^{-8}	6.56580×10^{-8}	2.02930×10^{-7}	1.04560×10^{-5}
	GWM	0	6.64391×10^{-8}	2.11896×10^{-8}	4.44089×10^{-16}	6.55893×10^{-8}	2.44337×10^{-6}
$RE_{X_{2,6}}(t)$	BPM [7]	0	6.07690×10^{-10}	5.54390×10^{-10}	8.31580×10^{-10}	2.43080×10^{-9}	1.18990×10^{-7}
	BCM [5]	0	6.07690×10^{-10}	5.54390×10^{-10}	8.31580×10^{-10}	2.43080×10^{-9}	1.18990×10^{-7}
	GWM	0	9.59807×10^{-10}	2.86258×10^{-10}	7.80964×10^{-19}	7.87210×10^{-10}	2.78344×10^{-8}
$RE_{X_{3,6}}(t)$	BPM [7]	0	6.78000×10^{-10}	6.18530×10^{-10}	9.27790×10^{-10}	2.71200×10^{-9}	1.32750×10^{-7}
	BCM [5]	0	6.78000×10^{-10}	6.18530×10^{-10}	9.27790×10^{-10}	2.71200×10^{-9}	1.32750×10^{-7}
	GWM	0	1.07099×10^{-9}	3.19418×10^{-10}	1.09606×10^{-18}	8.78399×10^{-10}	3.10587×10^{-8}

Table 5. Comparison of the residual errors for the BPM [7], the BCM [5] and the GWM using 10^{th} order polynomials for each lake

		t = 0	t = 0.2	t = 0.4	t = 0.6	t = 0.8	t = 1
	BPM [7]	0	1.26760×10^{-13}	1.21070×10^{-13}	9.85880×10^{-14}	5.07370×10^{-14}	4.89280×10^{-11}
	BCM [5]	0	1.24480×10^{-13}	2.53460×10^{-13}	3.91240×10^{-13}	5.41460×10^{-13}	4.96510×10^{-11}
$RE_{X_{1,10}}(t)$	GWM	0	1.15200×10^{-13}	4.38538×10^{-15}	5.55112×10^{-17}	1.13243×10^{-14}	2.08822×10^{-12}
	BPM [7]	0	6.61150×10^{-16}	8.09190×10^{-16}	9.37060×10^{-16}	1.01970×10^{-15}	5.45960×10^{-13}
	BCM [5]	0	1.76210×10^{-15}	3.45760×10^{-15}	5.08630×10^{-15}	6.64640×10^{-15}	5.38930×10^{-13}
$RE_{X_{2,10}}(t)$	GWM	0	1.49896×10^{-15}	5.23647×10^{-17}	1.90972×10^{-18}	1.37462×10^{-16}	2.35596×10^{-14}
	BPM [7]	0	7.45460×10^{-16}	9.14160×10^{-16}	1.06390×10^{-15}	1.16670×10^{-15}	6.09290×10^{-13}
	BCM [5]	0	1.52350×10^{-15}	2.95500×10^{-15}	4.28400×10^{-15}	5.51610×10^{-15}	6.17950×10^{-13}
$RE_{X_{3,10}}(t)$	GWM	0	1.67600×10^{-15}	6.12146×10^{-17}	1.07865×10^{-18}	1.54378×10^{-16}	2.63161×10^{-14}

Fig. (2) represents the residual error for the lake pollution model of a three-lake interconnected system, solved by using the Gegenbauer Wavelet Method (GWM) with polynomial degrees of 3, 6, and 10. Each curve in the graph corresponds to the residual error for a different polynomial, allowing a clear comparison of the solution accuracy across varying levels of approximation. The figure demonstrates that as the degree of the polynomial increases, the residual error decreases significantly, highlighting the ability to provide precise and refined solutions for the interconnected lake pollution model.

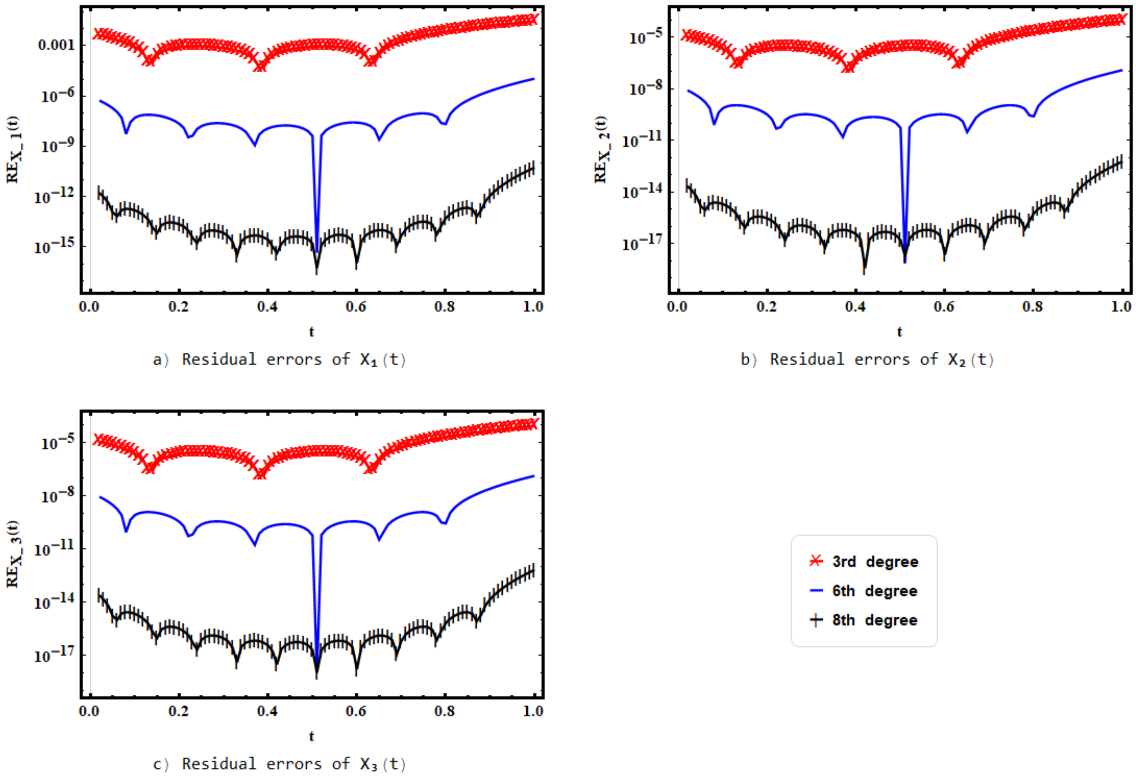

Figure 2. Residual error of $RE_{X_1(t)}, RE_{X_2(t)}$ and $RE_{X_3(t)}$ for $M = 4, M = 7$ and $M = 11$ on $0 \leq t \leq 1$ for Lake 1, Lake 2, Lake 3, respectively.

Fig. (3) illustrates the level of pollution in three lakes (Lake 1, Lake 2, and Lake 3) over a longer period of time. We compare the solution curves for the lake pollution model obtained by using 10^{th} degree polynomial and the RK4 method with step size $\Delta t = 0.1$. These graphs show the pollution dynamics over an extended time. Both methods exhibit excellent agreement throughout the time period, demonstrating that the GWM and the RK4 maintain high accuracy even over the extended time frame. This highlights the robustness and reliability of the current approach in modeling the interconnected lake pollution system.

One can see from Fig. (3) that the increase of the contamination in the first lake is significant with a sinusoidal behaviour, while the second and third lakes are contaminated as well with an exponential behaviour but the level of pollution at these lakes are much smaller than the first lake, and nearly equal to each other.

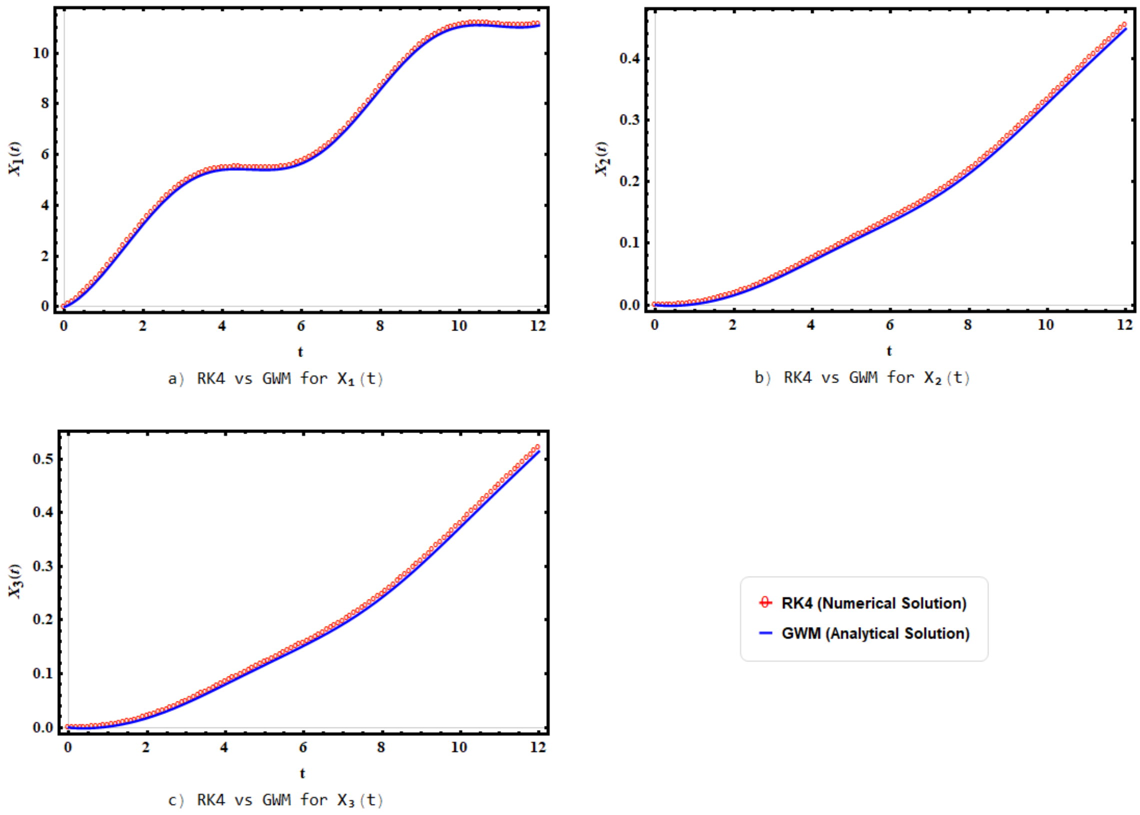

Figure 3. $X_1(t), X_2(t)$ and $X_3(t)$ of the GWM solutions with 10^{th} degree polynomials with $l = 1$ and RK4 $\Delta t = 0.1$ for $0 \leq t \leq 12$ for each of the lakes.

4. Conclusion

In this study, we investigated the pollution dynamics within a system of three interconnected lakes, employing the Gegenbauer wavelet method to effectively solve this system. Our approach is notable for its dual application across both short-term ($t = 0$ to $t = 1$) and extended time intervals ($t = 0$ to $t = 12$), offering a novel perspective on long- term pollutant behavior modeling. Comparative analysis with Mathematica default solver, the 4^{th} order Runge-Kutta method, and other techniques demonstrates the superior accuracy of the Gegenbauer wavelet method, with minimized residual errors confirming its robustness. The residual error graphs further validate the method's effectiveness, high- lighting its potential for precise long-term pollution forecasting in environmental models. Additionally, the Gegenbauer wavelet method exhibits considerable computational efficiency, even when utilizing high-degree polynomials. Despite the inherent complexity of high-degree expansions, our approach maintains practical CPU time, underscoring its suitability for large-scale or real-time applications. This computational efficiency, combined with the stability and adaptability, makes it a highly practical choice in the solution of complex systems with high precision while minimizing computational demands. This work underscores the versatility and effectiveness of the Gegenbauer wavelet method in addressing complex environmental systems, paving the way for future research on adaptive wavelet techniques in similar applications.

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