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Statistical Convergence on b-Metric Spaces

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Abstract. This paper investigates the idea of statistical convergence in the context of b-metric spaces. As a generalization of classical metric spaces, b-metric spaces offer a useful structure to examine the basic concepts of statistical convergence. The results indicate that b-metric spaces provide an impressive theoretical basis for both convergence theory and fixed point analysis.

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1. Introduction

The theory of summability, originally developed to extend the notion of convergence to non-convergent sequences, has gained significant attention due to its applications in various areas such as approximation theory, Fourier analysis, and functional analysis. In recent years, the generalization of classical metric spaces—such as b-metric, g-metric, cone metric, and s-metric spaces—has opened new directions for extending results in both fixed point theory and convergence analysis. Motivated by these developments, this paper explores the notion of statistical convergence within the framework of b-metric spaces.

The idea under this study arises from a growing trend in metric fixed point theory, where researchers aim to define distance functions that are more flexible than the classical metric while retaining similar structural properties. This has led to the introduction of several generalized metric spaces, including partial metric, A-metric, b-metric, and others. By modifying the way distances are measured through such generalized structures, a variety of new fixed point results have been obtained. Inspired by this, we aim to examine whether these alternative notions of distance can be effectively employed in the context of statistical convergence.

Although the measurement approach used in this paper is original, the concept of defining statistical convergence using various measures is not unique. Several studies have already attempted to connect these two domains—summability theory and generalized metrics—as seen in [1, 7, 9, 11, 12], among others. In this direction, we demonstrate that tools and techniques from fixed point theory can serve as effective instruments in the broader context of summability.

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2. Preliminaries

This part focuses on fundamental concepts, definitions and theorems that brings about the main idea of our study. Since the main theme of our study is based on b-metric spaces, we will start with the definition of b-metric function and space given by Bakhtin in 1989 [2].

Definition 2.1 ([2,4])**.** Let X be a non-empty set and $s \geq 1$ is a real number. The function $d : X \times X \to [0, \infty)$ is called b-metric if the following properties hold for all $x, y, z \in X$

(bM_1) $d(x, y) = 0$ if and only if $x = y$, (nonnegativity and self distance)
(bM_2) $d(x, y) = d(y, x)$, (symmetry)
(bM_3) $d(x, z) \leq s[d(x, y) + d(y, z)]$ (weakened triangle inequality).

A pair (X, d_b) is referred to as a b-metric space.

As can be clearly seen from the definition, b-metric space is reduced to standard metric space in case of $s = 1$. In the following example, we will demonstrate that the converse is false; specifically, we will illustrate that a metric space does not necessarily qualify as a b-metric space.

Example 2.2. Let $X = \mathbb{R}$ and the function $d : \mathbb{R} \times \mathbb{R} \to [0, \infty)$ be given by $d(x, y) = (x - y)^2$.

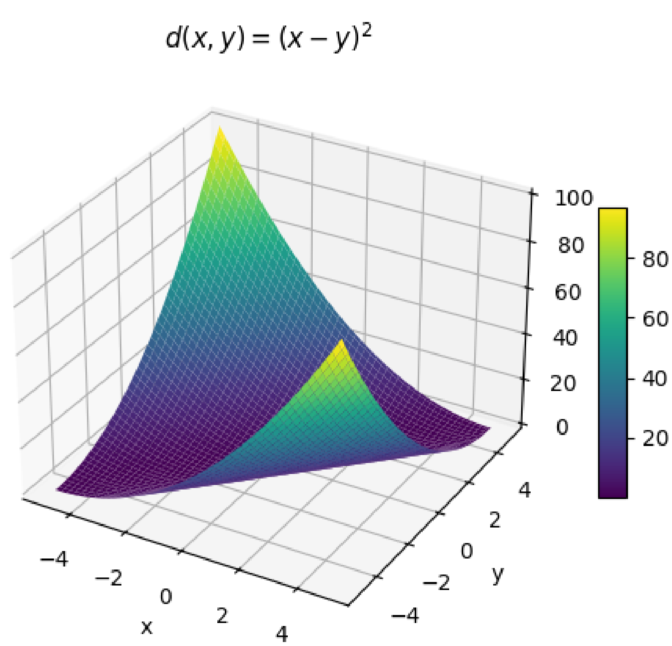

Figure 1. Graph of $d(x, y) = (x - y)^2$

d is not a metric, since the metric properity triangle inequality does not hold. For $x = -4$, $y = 2$ and $z = 3$;

$$\begin{aligned} d(x, z) = (-4 - 3)^2 = 49 > d(x, y) &+ d(y, z) \\ &= (-4 - 2)^2 + (2 - 3)^2 \\ &= 37. \end{aligned}$$

However d is a b-metric with $s = 3$. Properties (bM_1) and (bM_2) are easily seen. Let us examine (bM_3):

$$\begin{aligned}(x-z)^2 = (x-y+y-z)^2 &= (x-y)^2 + 2(x-y)(y-z) + (y-z)^2 \\ &\leq 3((x-y)^2 + (y-z)^2).\end{aligned}$$

Since $2(x-y)(y-z) \leq 2(y-z)^2$ if $(x-y) \leq (y-z)$ and using same idea $2(x-y)(y-z) \leq 2(x-y)^2$ if $(y-z) \leq (x-z)$. For all cases; we have

$$(x-z)^2 \leq 3((x-y)^2 + (y-z)^2).$$

For more information and examples about b-metric spaces see [8].

This article also includes statistical convergence concept as a major subject. We would like to remind the reader of the fundamental concepts of this topic.

The following is the definition of the natural (asymptotic) density for a subset M of positive integers [10] :

$$\delta(M) = \lim_{n\to\infty} \frac{|\{m \leq n : m \in M\}|}{n},$$

where $|M|$ denotes the *cardinality* of a subset $M \subset \mathbb{N}$.

Using the idea of natural/asymptotic density, the statistical convergence of a sequence is defined as:

Definition 2.3. (x_k) is said to be statistically convergent to a number L if, for every $\varepsilon > 0$,

$$\delta(\{k : |x_k - L| \geq \varepsilon\}) = 0, \tag{2.1}$$

or, to put equation (2.1) another way

$$\lim_{n\to\infty} \frac{|\{k \leq n : |x_k - L| \geq \varepsilon\}|}{n} = 0,$$

holds [5]. Next, we use the following notation to represent this statistical limit:

$$st - \lim_{k\to\infty} x_k = L.$$

It is commonly known that any convergent sequence statistically converges to the same limit point. However, the converse does not always hold..

Finally, we will emphasize the statistical version of the commonly recognized notion of the ”Cauchy sequence”.

Definition 2.4. [6] A sequence $x = (x_k)$ is said to be statistically Cauchy sequence if for every $\varepsilon > 0$, there exists a $N \in \mathbb{N}$ such that

$$\delta(\{k : |x_k - x_N| \geq \varepsilon\}) = 0.$$

3. Main Results

In this section, we will establish the idea of statistical convergence in b-metric spaces. Consequently, we will integrate the b-metric concept, which is in frequently use in fixed point theory, with summability theory.

Definition 3.1. Let (x_k) be a sequence in a b-metric space (X, d_b). (x_k) is said to be statistically convergent to $x \in X$ for every $\varepsilon > 0$,

$$\lim_{n\to\infty} \frac{|\{k \leq n : d_b(x_k, x) \geq \varepsilon\}|}{n} = 0,$$

or, alternatively

$$\lim_{n\to\infty} \frac{|\{k \leq n : d_b(x_k, x) \leq \varepsilon\}|}{n} = 1.$$

This limit is denoted by $st_b - \lim\limits_{k\to\infty} x_k = x$.

This present definition gives the literature a very strong framework. Choosing different s constants in b-metric space, we will refer to several definitions presented in the literature. For example, if $s = 1$, the definition of statistical convergence in metric spaces will be derived [3]. In such a case, several investigations will arrive at a specific case of this research.

Theorem 3.2. *With regards to b-metric, any convergent sequence in b-metric spaces is statistically convergent.*

Proof. Let (X, d_b) be a b-metric space and (x_n) be a sequence in X such that it converges to $x \in X$. Using the definition of convergence of a sequence in b-metric space (see [8]), for every $\varepsilon > 0$, there exists a $n_0 \in \mathbb{N}$ such that $\forall n \geq n_0$

$$d_b(x_n, x) < \varepsilon.$$

We consider the set

$$\{n \in \mathbb{N} : d_b(x_n, x) \geq \varepsilon\}. \tag{3.1}$$

We can easily see cardinality of the set (3.1) as $n_0 - 1$. Then,

$$\lim_{n\to\infty} \frac{n_0 - 1}{n} = 0.$$

Last equation implies $st_b - \lim_{n\to\infty} x_n = x$. □

The opposite of the Theorem 3.2 is false, e.g. a statistically convergent sequence is not always convergent in the b-metric spaces. This will be demonstrated in the upcoming example.

Example 3.3. Let $X = \mathbb{R}$. We will work with b-metric in Example 2.2. (x_n) be a sequence in $\mathbb{R}$ as follows:

$$x_n = \begin{cases} \sqrt{n}, \text{ if } n \text{ is a square,} \\ 0, \text{ if } n \text{ is not a square,} \end{cases}$$

that is,

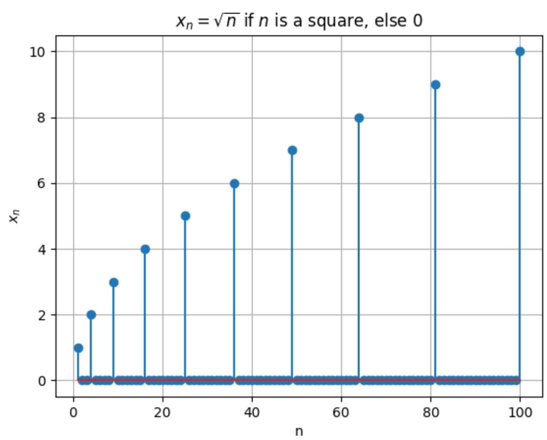

Figure 2. Illustration of the sequence x_n defined by square and non-square indices.

It is clear that (x_n) is not convergent since for all square numbers

$$d_b(x_n, 0) = (\sqrt{n} - 0)^2 = n > \varepsilon.$$

However, (x_n) is statistically convergent to 0 since

$$\lim_{n\to\infty} \frac{|\{k \leq n : d_b(x_k, 0) \geq \varepsilon\}|}{n} =$$

$$\lim_{n\to\infty} \frac{\left|\{k \leq n : k \geq \varepsilon \text{ and } k \text{ is a square natural number}\}\right|}{n} = 0.$$

The next theorem indicates the uniqueness of the statistical limit with respect to b-metric.

Theorem 3.4. *Let (X, d_b) be a b-metric space and (x_n) be a sequence in X. If $st_b - \lim_{n\to\infty} x_n = x$ and $st_b - \lim_{n\to\infty} x_n = y$, then $x = y$.*

Proof. Suppose that $st_b - \lim_{n\to\infty} x_n = x$ and $st_b - \lim_{n\to\infty} x_n = y$. For any $\varepsilon > 0$, we have following sets:

$$A = \{k \leq n : d_b(x_k, x) \geq \frac{\varepsilon}{2s}\} \text{ and } B = \{k \leq n : d_b(x_k, y) \geq \frac{\varepsilon}{2s}\},$$

where s is the constant of b-metric space. From the Definition 3.1, we have

$$\lim_{n\to\infty} \frac{\left|\left\{k \leq n : d_b(x_k, x) \geq \frac{\varepsilon}{2s}\right\}\right|}{n} = 0,$$

and

$$\lim_{n\to\infty} \frac{\left|\left\{k \leq n : d_b(x_k, y) \geq \frac{\varepsilon}{2s}\right\}\right|}{n} = 0.$$

Now, assume that for an $\varepsilon > 0$, $d_b(x, y) = \varepsilon$. Then,

$$\begin{aligned} \varepsilon &\leq d_b(x, y) \\ &\leq s(d_b(x, x_k) + d_b(y, x_k)). \end{aligned}$$

If $k \notin A \cup B$, we have

$$\begin{aligned} \varepsilon &\leq s(d_b(x, x_k) + d_b(y, x_k)) \\ &\leq s(\frac{\varepsilon}{2s} + \frac{\varepsilon}{2s}). \end{aligned}$$

This contradiction completes the proof. □

We would like to point out that we can set "almost all" notion defined on natural numbers within the perspective of b-metric spaces. In other words, for a sequence defined on an arbitrary (X, d_b), if

$$\lim_{n\to\infty} \frac{|\{k \leq n : d_b(x_k, x) < \varepsilon\}|}{n} = 1.$$

Then, we can say that the sequence $x = (x_k)$ is convergent to x for "almost all k" and we abbreviate this by " a.a.k. "

Now, we have the definition of statistical Cauchy sequence on b-metric spaces.

Definition 3.5. Let (X, d_b) be a b-metric space. The sequence $x = (x_k) \in X$ is said to be statistically Cauchy sequence on b-metric space if for every $\varepsilon > 0$, there exists a $N \in \mathbb{N}$ such that

$$\lim_{n\to\infty} \frac{|\{k \leq n : d_b(x_k, x_N) \geq \varepsilon\}|}{n} = 0.$$

The following theorem will demonstrate us the relationship between statistical Cauchy concept and statistical convergence concept in b-metric spaces.

Theorem 3.6. *Let $x = (x_k)$ be a statistically convergent sequence in (X, d_b). Then the sequence $x = (x_k)$ is statistically Cauchy on b-metric space X.*

Proof. Suppose that $st_b - \lim_{k\to\infty} x_k = L$ and choose an arbitrary $\varepsilon > 0$. Then $d_b(x_k, L) < \dfrac{\varepsilon}{2s}$ where s depends on b-metric space for almost all k. Now, By using property bM_3 of b-metric and take an N such that,

$$\begin{aligned} d_b(x_k, x_N) &\leq s(d_b(x_k, L) + d_b(L, x)) \\ &< \varepsilon. \end{aligned}$$

□

Theorem 3.7. *Let $x = (x_k)$ be a statistically convergent sequence in (X, d_b) b-metric space. Then, there is a convergent sequence (y_k) in b-metric space such that $x_k = y_k$ for almost all k.*

Proof. Suppose that $x = (x_k)$ is statistically convergent to L in b-metric sace,that is, $st_b - \lim_{k\to\infty} x_k = L$. From 3.1, we have

$$\lim_{n\to\infty} \frac{|\{k \le n : d_b(x_k, L) \le \varepsilon\}|}{n} = 1.$$

So, for every $k \in \mathbb{N}$, there exists a $n_k \in \mathbb{N}$ increasing sequence such that

$$\frac{1}{n}\, |\{k \le n : d_b(x_k, L) \le \varepsilon\}| > 1 - \frac{1}{2^k} \tag{3.2}$$

for every $k > n_k$. Now, we can choose as follows:

$$y_k := \begin{cases} x_k, & \text{if } 1 \le k \le n_1, \\ x_k, & \text{if } n_k \le k \le n_{k+1} \text{ and } d_b(x_k, L) < \dfrac{1}{2^k}, \\ L, & \text{otherwise.} \end{cases}$$

Let $\varepsilon > 0$ and choose $k \in \mathbb{N}$ such that $\frac{1}{2^k} < \varepsilon$. Then, for each $k > n_k$

$$d_b(y_k, L) = d_b(x_k, L) < \frac{1}{2^k}.$$

This implies $\lim_{k\to\infty} y_k = L$. Let $n \in \mathbb{N}$ be fixed and $n_k \le n \le n_{k+1}$, then we get

$$\{k \le n : y_k \ne x_k\} \subset \{1, 2, ..., n\} - \{k \le n : d_b(x_k, L) < \frac{1}{2^k}\}.$$

Then,

$$\lim_{n\to\infty} \frac{|\{k \le n : y_k \ne x_k\}|}{n} \le 1 - \lim_{n\to\infty} \frac{\left|\{k \le n : d_b(x_k, L) < \frac{1}{2^k}\}\right|}{n}$$

and using (3.2)

$$\begin{aligned} \lim_{n\to\infty} \frac{|\{k \le n : y_k \ne x_k\}|}{n} &\le 1 - \left(1 - \frac{1}{2^k}\right) \\ &= \frac{1}{2^k} < \varepsilon. \end{aligned}$$

Hence, we obtain $\lim_{n\to\infty} \frac{|\{k \le n : y_k \ne x_k\}|}{n} = 0$. Therefore, $x_k = y_k$ for almost all $k \in \mathbb{N}$. □

The corollary we are going to present is a direct consequence of the previous theorem.

Corollary 3.8. *In b-metric spaces, any statistically convergent sequence has a convergent subsequence.*

Conclusion

This paper examines the notion of statistical convergence in b-metric spaces, which extend classical metric structures and offer a more comprehensive framework for convergence and fixed point theorems. Several illustrative examples are provided to highlight how statistical convergence behaves differently from classical convergence in a b-metric space. The results indicate that b-metric spaces provide an effective framework to support the developments of summability methods and convergence theory. This investigation opens several avenues for future research. One possible direction is to explore statistical convergence in other generalized metric frameworks, such as dislocated metric spaces, modular metric spaces, or even fuzzy metric spaces. Another promising direction is to investigate the relationships between statistical convergence and basic fixed point theorems, with possible applications in iterative techniques, nonlinear mappings, and optimization. In summary, the integration of b-metric spaces and statistical convergence provides an efficient basis for the development of various theoretical results and practical applications, thus improving the progression of summability theory and extended metric analysis.

Conflicts of Interest

The author declares that there are no conflicts of interest regarding the publication of this article.

Authors Contribution Statement

The author has read and agreed the published version of the manuscript.

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