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# Kamenev-type criteria for nonlinear second-order delay dynamic equations

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**Abstract**

We study oscillation of certain second-order nonlinear delay dynamic equations on arbitrary time scales. Employing a class of kernel functions, new Kamenev-type oscillation criteria are presented that differ from the known ones. These criteria improve some related results for second-order differential equations.

**Keywords:** Oscillation, Second-order dynamic equation, Delay dynamic equation, Time scale.

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## 1. Introduction

In recent years, a great amount of attention has been paid to qualitative analysis of dynamic equations on time scales or measure chains. We refer the reader to the landmark work by Hilger [17] for a comprehensive treatment of the subject. Later on, several authors have expounded on various aspects of this theory; see, for instance, the survey paper [3], the monographs [9, 10], and the references cited therein. For completeness, we recall the following concepts related to the notion of time scales. A time scale $\mathbb{T}$ is an arbitrary nonempty closed subset of the real numbers $\mathbb{R}$. On any time scale, we define the forward and backward jump operators by $\sigma(t) := \inf\{s \in \mathbb{T} | s > t\}$ and $\rho(t) := \sup\{s \in \mathbb{T} | s < t\}$, respectively, where $\inf \emptyset := \sup \mathbb{T}$ and $\sup \emptyset := \inf \mathbb{T}$, $\emptyset$ denotes

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the empty set. The graininess $\mu$ of the time scale is defined by $\mu(t) := \sigma(t) - t$, and for any function $g : \mathbb{T} \to \mathbb{R}$ the notation $g^\sigma(t) := g(\sigma(t))$. For further details and discussion, we refer the reader to [9].

Oscillatory behavior, as a kind of physical phenomena, widely exists in natural sciences and engineering. The assorted oscillation phenomena can be unified into the oscillation theory of dynamic equations which is an important branch of the qualitative analysis of dynamic equations; see Agarwal et al. [5]. This resulted in publication of numerous research articles [1, 2, 4, 6–8, 11–16, 18–33]; see also the references cited therein.

To establish sufficient conditions for oscillation of dynamic equations, one usually uses either an integral averaging technique involving integrals and weighted integrals of coefficients of a given dynamic equation (see, e.g., [1, 2, 8, 22, 28]), or comparison methods and linearization techniques (see, e.g., [7, 15]).

In this paper, we are concerned with the oscillatory behavior of solutions to a class of second-order nonlinear delay dynamic equations

$$\left(rx^\Delta\right)^\Delta(t) + f(t, x(\tau(t))) = 0 \tag{1.1}$$

on an arbitrary time scale $\mathbb{T}$, where $t \in [t_0, \infty)_\mathbb{T} := [t_0, \infty) \cap \mathbb{T}$. Throughout, we assume that

- $(H_1)$ $r \in \mathrm{C}^1_{\mathrm{rd}}([t_0, \infty)_\mathbb{T}, (0, \infty))$ and $\int_{t_0}^\infty \frac{\Delta t}{r(t)} = \infty$;
- $(H_2)$ $\tau \in \mathrm{C}_{\mathrm{rd}}([t_0, \infty)_\mathbb{T}, \mathbb{T})$, $\tau(t) \le t$, and $\lim_{t\to\infty} \tau(t) = \infty$;
- $(H_3)$ $f(t, u) \in \mathrm{C}([t_0, \infty)_\mathbb{T} \times \mathbb{R}, \mathbb{R})$, $uf(t, u) > 0$ for all $u \ne 0$, and there exists a positive rd-continuous function $\delta$ defined on $\mathbb{T}$ such that $f(t, u) \ge \delta(t)u$.

We suppose that solutions to equation (1.1) exist for all $t \in [t_0, \infty)_\mathbb{T}$. As usual, a solution of equation (1.1) is said to be oscillatory if it is neither eventually positive nor eventually negative; otherwise, it is termed nonoscillatory. Equation (1.1) is called oscillatory if all its solutions oscillate. The solutions vanishing in some neighborhood of infinity will be excluded from our consideration.

The analogue for (1.1) in case $\mathbb{T} = \mathbb{R}$, namely,

$$\left(rx'\right)'(t) + f(t, x(\tau(t))) = 0, \tag{1.2}$$

was studied in [24, 25, 32] by using a class of kernel functions. It should be noted that research in this paper was strongly motivated by the contributions of Erbe et al. [13] and Xu and Meng [32]. Our principal goal is to extend and improve related results reported in [32].

In what follows, we will use the function class $\mathcal{Y}$ to study oscillation of (1.1). New oscillation criteria are different from known ones in the sense that they are based on a class of kernel functions $\Phi(t, s, l)$. We say that a function $\Phi := \Phi(t, s, l)$ belongs to a class $\mathcal{Y}$, denoted by $\Phi \in \mathcal{Y}$, if $\Phi \in \mathrm{C}_{\mathrm{rd}}(E, \mathbb{R})$, where $E := \{(t, s, l) : t_0 \le l \le s \le t < \infty,\ l, s, t \in [t_0, \infty)_\mathbb{T}\}$, which satisfies $\Phi(t, t, l) = 0$, $\Phi(t, l, l) = 0$, $\Phi(t, s, l) \ne 0$ for $l < s < t$, and has the partial derivative $\Phi^{\Delta_s}$ on $E$ such that $\Phi^{\Delta_s}$ is $\Delta$-integrable with respect to $s$ in $E$. For $t \ge s \ge l \ge t_0$, we define the operator $A[\cdot; l, t]$ by

$$A[g; l, t] := \int_l^t \Phi^2(t, s, l)g(s)\Delta s \quad \text{for } g \in \mathrm{C}_{\mathrm{rd}}([t_0, \infty)_\mathbb{T}, \mathbb{R}), \tag{1.3}$$

the function $\varphi(t, s, l)$ is assumed to satisfy

$$\Phi^{\Delta_s}(t, s, l) := \varphi(t, s, l)\Phi(t, s, l). \tag{1.4}$$

It is easy to verify that $A[\cdot; l, t]$ is a linear operator and satisfies

$$A[g^\Delta; l, t] = -A[g^\sigma(2\varphi + \mu\varphi^2); l, t] \quad \text{for } g \in \mathrm{C}^1_{\mathrm{rd}}([t_0, \infty)_\mathbb{T}, \mathbb{R}). \tag{1.5}$$

## 2. Main results

To prove our main results, we need the following auxiliary lemmas.

**2.1. Lemma.** *Assume that conditions* $(H_1)$–$(H_3)$ *and*

$$r^{\Delta} \geq 0, \quad \int_{t_0}^{\infty} \delta(t)\tau(t)\Delta t = \infty \tag{2.1}$$

*are satisfied. If* $x$ *is a positive solution of* (1.1) *on* $[t_0, \infty)_{\mathbb{T}}$*, then there exists a sufficiently large* $T \in [t_0, \infty)_{\mathbb{T}}$ *such that*

$$x^{\Delta}(t) > 0 \quad \text{and} \quad (rx^{\Delta})^{\Delta}(t) < 0 \tag{2.2}$$

*for* $t \in [T, \infty)_{\mathbb{T}}$*,* $x(t)/t$ *is strictly decreasing on* $[T, \infty)_{\mathbb{T}}$*.*

*Proof.* Noticing that $r^{\Delta} \geq 0$, the proof is similar to that of [13, Lemma 1] and hence is omitted. □

**2.2. Lemma.** *Let conditions* $(H_1)$–$(H_3)$ *hold. If* $x$ *is a positive solution of* (1.1) *on* $[t_0, \infty)_{\mathbb{T}}$*, then there exists a sufficiently large* $T \in [t_0, \infty)_{\mathbb{T}}$ *such that* (2.2) *holds for* $t \in [T, \infty)_{\mathbb{T}}$ *and* $x(t)/\int_T^t \frac{\Delta s}{r(s)}$ *is eventually strictly decreasing.*

*Proof.* Assume that there exists a $t_1 \in [t_0, \infty)_{\mathbb{T}}$ such that $x(\tau(t)) > 0$ for $t \in [t_1, \infty)_{\mathbb{T}}$. It is not difficult to obtain that there exists a $t_2 \in [t_1, \infty)_{\mathbb{T}}$ such that (2.2) holds for $t \in [t_2, \infty)_{\mathbb{T}}$. Hence, we have, for $t \in [t_2, \infty)_{\mathbb{T}}$,

$$x(t) = x(t_2) + \int_{t_2}^{t} \frac{r(s)x^{\Delta}(s)}{r(s)}\Delta s > r(t)x^{\Delta}(t) \int_{t_2}^{t} \frac{\Delta s}{r(s)},$$

which implies that, for some $t_3 \in [t_2, \infty)_{\mathbb{T}}$ large enough and for $t \in [t_3, \infty)_{\mathbb{T}}$,

$$\left( \frac{x(t)}{\int_{t_2}^{t} \frac{\Delta s}{r(s)}} \right)^{\Delta} < 0,$$

and so $x(t)/\int_T^t \frac{\Delta s}{r(s)}$ is strictly decreasing on $[t_3, \infty)_{\mathbb{T}}$. The proof is complete. □

**2.3. Theorem.** *Let conditions* $(H_1)$–$(H_3)$ *and* (2.1) *be satisfied. Assume that for each* $l \in [t_0, \infty)_{\mathbb{T}}$*, there exists a function* $\Phi \in \mathfrak{Y}$ *such that*

$$\limsup_{t \to \infty} A\left[ \frac{\tau(s)\delta(s)}{s} - \frac{r(s)}{4}(2\varphi + \mu\varphi^2)^2; l, t \right] > 0, \tag{2.3}$$

*where* $A$ *and* $\varphi$ *are as introduced in* (1.3) *and* (1.4)*, respectively. Then* (1.1) *is oscillatory.*

*Proof.* Suppose that (1.1) has a nonoscillatory solution $x$ on $[t_0, \infty)_{\mathbb{T}}$. Without loss of generality, we can assume that there is a $t_1 \in [t_0, \infty)_{\mathbb{T}}$ such that $x(\tau(t)) > 0$ on $[t_1, \infty)_{\mathbb{T}}$. By virtue of Lemma 2.1, there exists a $t_2 \in [t_1, \infty)_{\mathbb{T}}$ such that (2.2) holds for $t \in [t_2, \infty)_{\mathbb{T}}$. Define the function $w$ by Riccati substitution

$$w(t) := \frac{r(t)x^{\Delta}(t)}{x(t)} \quad \text{for } t \in [t_2, \infty)_{\mathbb{T}}. \tag{2.4}$$

Then, we conclude that

$$w^{\Delta}(t) = (rx^{\Delta})^{\Delta}(t)\frac{1}{x(t)} + (rx^{\Delta})^{\sigma}(t) \left( \frac{1}{x(t)} \right)^{\Delta}.$$

It follows from (1.1) and (2.4) that

$$\begin{aligned} w^{\Delta}(t) &\le -\frac{\delta(t)x(\tau(t))}{x(t)} + (rx^{\Delta})^{\sigma}(t)\frac{-x^{\Delta}(t)}{x(t)x(\sigma(t))} \\ &= -\frac{\delta(t)x(\tau(t))}{x(t)} - (rx^{\Delta})^{\sigma}(t)\frac{r(t)x^{\Delta}(t)}{r(t)x(t)x(\sigma(t))}. \end{aligned}$$

Using (2.2), we deduce that $x$ is strictly increasing and $rx^{\Delta}$ is strictly decreasing. Hence, we obtain

$$\begin{aligned} w^{\Delta}(t) &\le -\frac{\delta(t)x(\tau(t))}{x(t)} - (rx^{\Delta})^{\sigma}(t)\frac{(rx^{\Delta})^{\sigma}(t)}{r(t)x(\sigma(t))x(\sigma(t))} \\ &= -\frac{\delta(t)x(\tau(t))}{x(t)} - \frac{(w^{\sigma}(t))^2}{r(t)}. \qquad (2.5) \end{aligned}$$

An application of Lemma 2.1 implies that

$$\frac{x(\tau(t))}{x(t)} \ge \frac{\tau(t)}{t}. \tag{2.6}$$

Substituting (2.6) into (2.5), we have

$$w^{\Delta}(t) \le -\frac{\tau(t)\delta(t)}{t} - \frac{(w^{\sigma}(t))^2}{r(t)}. \tag{2.7}$$

Using the operator $A$ in (2.7), we conclude that, for $t \in [t_2, \infty)_{\mathbb{T}}$,

$$A[w^{\Delta}(s); t_2, t] \le -A\left[\frac{\tau(s)\delta(s)}{s} + \frac{(w^{\sigma}(s))^2}{r(s)}; t_2, t\right],$$

and so

$$A\left[\frac{\tau(s)\delta(s)}{s}; t_2, t\right] \le -A\left[\frac{(w^{\sigma}(s))^2}{r(s)} + w^{\Delta}(s); t_2, t\right].$$

Hence, by (1.5), we have

$$\begin{aligned} A\left[\frac{\tau(s)\delta(s)}{s}; t_2, t\right] &\le -A\left[\frac{(w^{\sigma}(s))^2}{r(s)} - w^{\sigma}(s)(2\varphi + \mu\varphi^2); t_2, t\right] \\ &= -A\left[\left(\sqrt{\frac{1}{r(s)}}w^{\sigma}(s) - \frac{1}{2}\sqrt{r(s)}(2\varphi + \mu\varphi^2)\right)^2; t_2, t\right] \\ &\quad + A\left[\frac{r(s)}{4}(2\varphi + \mu\varphi^2)^2; t_2, t\right] \\ &\le A\left[\frac{r(s)}{4}(2\varphi + \mu\varphi^2)^2; t_2, t\right], \end{aligned}$$

and thus

$$A\left[\frac{\tau(s)\delta(s)}{s} - \frac{r(s)}{4}(2\varphi + \mu\varphi^2)^2; t_2, t\right] \le 0,$$

which is in contradiction with (2.3). Therefore, equation (1.1) is oscillatory. □

Efficient oscillation tests for (1.1) can be easily derived from Theorem 2.3 with various choices of the function $\Phi$. For example, consider a Kamenev-type function $\Phi(t, s, l) = \rho(s)(t - s)(s - l)$, where $\rho \in \mathrm{C}^1_{\mathrm{rd}}([t_0, \infty)_{\mathbb{T}}, (0, \infty))$. Clearly, $\Phi \in \mathfrak{Y}$ and

$$\varphi(t, s, l) = \frac{\rho^{\Delta}(s)}{\rho(s)} + \frac{\rho^{\sigma}(s)}{\rho(s)}\frac{(t - \sigma(s)) - (s - l)}{(t - s)(s - l)}. \tag{2.8}$$

As a consequence of Theorem 2.3, we obtain the following oscillation test.

**2.4. Corollary.** *Assume that conditions* $(H_1)$*–*$(H_3)$ *and* (2.1) *hold. Equation* (1.1) *is oscillatory provided that for each* $l \in [t_0, \infty)_{\mathbb{T}}$*, there exists a function* $\rho \in \mathrm{C}^1_{\mathrm{rd}}([t_0, \infty)_{\mathbb{T}}, (0, \infty))$ *such that*

$$\limsup_{t\to\infty} \int_l^t \rho^2(s)(t-s)^2(s-l)^2 \left[\frac{\tau(s)\delta(s)}{s} - \frac{r(s)}{4}\left(2\varphi(t,s,l) + \mu(s)\varphi^2(t,s,l)\right)^2\right]\Delta s > 0,$$

*where* $\varphi$ *is given in* (2.8).

It may happen that condition (2.1) of Theorem 2.3 is not satisfied, in which case the following result proves to be useful.

**2.5. Theorem.** *Let conditions* $(H_1)$*–*$(H_3)$ *hold. Assume that for some* $t_1 \in [t_0, \infty)_{\mathbb{T}}$ *large enough and for each sufficiently large* $l \in [t_1, \infty)_{\mathbb{T}}$*, there exists a function* $\Phi \in \mathfrak{Y}$ *such that*

$$\limsup_{t\to\infty} A\left[\delta(s)\frac{\int_{t_1}^{\tau(s)} \frac{\Delta u}{r(u)}}{\int_{t_1}^{s} \frac{\Delta u}{r(u)}} - \frac{r(s)}{4}(2\varphi + \mu\varphi^2)^2; l, t\right] > 0,$$

*where* $A$ *and* $\varphi$ *are as introduced in* (1.3) *and* (1.4)*, respectively. Then* (1.1) *is oscillatory.*

*Proof.* Suppose to the contrary that $x$ is a positive solution of (1.1). It follows from Lemma 2.2 that

$$\frac{x(\tau(t))}{x(t)} \geq \frac{\int_{t_1}^{\tau(t)} \frac{\Delta u}{r(u)}}{\int_{t_1}^{t} \frac{\Delta u}{r(u)}}.$$

Proceeding as in the proof of Theorem 2.3, we obtain the conclusion. □

Consequently, one immediately derives from Theorem 2.5 the following useful test for the oscillation of (1.1).

**2.6. Corollary.** *Assume that conditions* $(H_1)$*–*$(H_3)$ *are satisfied. Equation* (1.1) *is oscillatory provided that for some* $t_1 \in [t_0, \infty)_{\mathbb{T}}$ *large enough and for each sufficiently large* $l \in [t_1, \infty)_{\mathbb{T}}$*, there exists a function* $\rho \in \mathrm{C}^1_{\mathrm{rd}}([t_0, \infty)_{\mathbb{T}}, (0, \infty))$ *such that*

$$\limsup_{t\to\infty} \int_l^t \rho^2(s)(t-s)^2(s-l)^2 \left[\delta(s)\frac{\int_{t_1}^{\tau(s)} \frac{\Delta u}{r(u)}}{\int_{t_1}^{s} \frac{\Delta u}{r(u)}} - \frac{r(s)}{4}\left(2\varphi(t,s,l) + \mu(s)\varphi^2(t,s,l)\right)^2\right]\Delta s > 0,$$

*where* $\varphi$ *is given in* (2.8).

## 3. Discussion

Xu and Meng [32] established the following oscillation criterion for (1.2).

**3.1. Theorem.** *Let* $\mathbb{T} = \mathbb{R}$*,* $r'(t) \geq 0$*, and conditions* $(H_1)$*–*$(H_3)$ *be satisfied. Suppose that for each* $l \geq t_0$*, there exists a function* $\Phi \in \mathfrak{Y}$ *such that*

$$\limsup_{t\to\infty} A\left[m\frac{\tau(s)\delta(s)}{s} - r(s)\varphi^2; l, t\right] > 0 \tag{3.1}$$

*for some* $m \in (0, 1)$*, where* $A$ *and* $\varphi$ *are defined by* (1.3) *and* (1.4)*, respectively. Then* (1.2) *is oscillatory.*

On the basis of Theorem 2.3, we have the following result.

**3.2. Corollary.** *Let conditions* $(H_1)$–$(H_3)$ *and* (2.1) *be satisfied, and let* $\mathbb{T} = \mathbb{R}$. *Assume that for each* $l \geq t_0$*, there exists a function* $\Phi \in \mathfrak{Y}$ *such that*

$$\limsup_{t\to\infty} A\left[\frac{\tau(s)\delta(s)}{s} - r(s)\varphi^2; l, t\right] > 0, \tag{3.2}$$

*where* $A$ *and* $\varphi$ *are as in* (1.3) *and* (1.4)*, respectively. Then* (1.2) *is oscillatory.*

**3.3. Remark.** One can easily see that condition (3.2) improves (3.1) by dropping the condition on the existence of the constant $m$.

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