ON SOME NEW INEQUALITIES OF HERMITE-HADAMARD TYPE INVOLVING HARMONICALLY CONVEX FUNCTIONS VIA FRACTIONAL INTEGRALS

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ABSTRACT. In this paper, some new results related to the left-hand side of the Hermite-Hadamard type inequality for harmonically convex functions using Riemann Liouville fractional integrals are obtained.

1. INTRODUCTION

Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a convex function defined on the interval I of real numbers and $a, b \in I$ with $a < b$. The following inequality holds

$$(1.1) \qquad f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}.$$

In [4], the author introduced the class of harmonically convex functions, defined as follows.

Definition 1.1. Let $I \subseteq \mathbb{R} \backslash \{0\}$ be a real interval. A function $f : I \to \mathbb{R}$ is said to be harmonically convex, if

$$f\left(\frac{xy}{tx + (1-t)y}\right) \leq tf(y) + (1-t)f(x)$$

for all $x, y \in I$ and $t \in [0, 1]$. If the inequality in (1.1) is reversed, then f is said to be harmonically concave.

We recall the following special functions and inequality
(1) The Beta function:

$$\beta(x, y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} = \int_0^1 t^{x-1} (1-t)^{y-1} \, dt, \quad x, y > 0,$$

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(2) The hypergeometric function:

$$_2F_1\left(a,b;c;z\right) = \frac{1}{\beta\left(b, c-b\right)} \int_0^1 t^{b-1}\left(1-t\right)^{c-b-1}\left(1-zt\right)^{-a} dt, \ c > b > 0, \ |z| < 1.$$

In the following we will give some necessary definitions and mathematical preliminaries of fractional calculus theory which are used further in this paper. More details, one can consult ([1],[2],[7]).

Definition 1.2. Let $f \in L\left[a, b\right]$. The Riemann-Liouville integrals $J_{a+}^{\alpha}f$ and $J_{b-}^{\alpha}f$ of oder $\alpha > 0$ with $a \geq 0$ are defined by

$$J_{a+}^{\alpha}f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t)dt, \ x > a$$

and

$$J_{b-}^{\alpha}f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (t-x)^{\alpha-1} f(t)dt, \ x < b$$

respectively, where $\Gamma(\alpha)$ is the Gamma function defined by $\Gamma(\alpha) = \int_0^{\infty} e^{-t}t^{\alpha-1}dt$ and $J_{a+}^0 f(x) = J_{b-}^0 f(x) = f(x)$.

Because of the wide application of Hermite-Hadamard type inequalities and fractional integrals, many researchers extend their studies to Hermite-Hadamard type inequalities involving fractional integrals not limited to integer integrals. Recently, more and more Hermite-Hadamard inequalities involving fractional integrals have been obtained for different classes of functions; see ([5],[6],[9],[10],[11]).

In [3], Iscan proved a variant of Hermite-Hadamard inequality which holds for the harmonically convex functions in fractional integral forms as follows:

Theorem 1.1. *Let $f : I \subseteq (0, \infty) \to \mathbb{R}$ be a function such that $f \in L[a, b]$, where $a, b \in I$ with $a < b$. If f is a harmonically convex function on $[a, b]$, then the following inequalities for fractional integrals hold:*

$$f\left(\frac{2ab}{a+b}\right) \leq \frac{\Gamma(\alpha+1)}{2}\left(\frac{ab}{b-a}\right)^{\alpha}\left\{J_{1/a-}^{\alpha}\left(f\circ g\right)(1/b) + J_{1/b+}^{\alpha}\left(f\circ g\right)(1/a)\right\}$$

$$\leq \frac{f(a) + f(b)}{2}$$

with $\alpha > 0$.

Lemma 1.1. *([8],[12]) For $0 < \alpha \leq 1$ and $0 \leq a < b$, we have*

$$|a^{\alpha} - b^{\alpha}| \leq (b-a)^{\alpha}.$$

Lemma 1.2. *([3]) Let $f : I \subseteq (0, \infty) \to \mathbb{R}$ be a differentiable function on I° such that $f' \in L[a, b]$, where $a, b \in I$ with $a < b$. Then the following equality for fractional integrals holds:*

$$I_f\left(g; \alpha, a, b\right)$$

$$= \frac{ab\left(b-a\right)}{2} \int_0^1 \frac{\left[t^{\alpha} - (1-t)^{\alpha}\right]}{\left(ta + (1-t)b\right)^2} f'\left(\frac{ab}{ta + (1-t)b}\right) dt.$$

where

$$I_f\left(g;\alpha,a,b\right)$$

$$= \frac{f(a)+f(b)}{2} - \frac{\Gamma(\alpha+1)}{2}\left(\frac{ab}{b-a}\right)^{\alpha}\left\{J_{1/a-}^{\alpha}\left(f\circ g\right)\left(1/b\right) + J_{1/b+}^{\alpha}\left(f\circ g\right)\left(1/a\right)\right\}.$$

with $\alpha > 0, g\left(x\right) = 1/x$ *and* Γ *is Euler Gamma function.*

In [3], Iscan proved the following theorems using the above Lemma 1.1 and Lemma 1.2.

Theorem 1.2. *Let* $f : I \subseteq (0,\infty) \to \mathbb{R}$ *be a differentiable function on* I° *such that* $f' \in L[a,b]$, *where* $a,b \in I^{\circ}$ *with* $a < b$. *If* $|f'|^{q}$ *is harmonically convex on* $[a,b]$ *for some fixed* $q \geq 1$, *then the following inequality for fractional integrals holds:*

$$|I_f\left(g;\alpha,a,b\right)|$$

$$\leq \frac{ab\left(b-a\right)}{2}C_1^{1-1/q}(\alpha;a,b)\left(C_2(\alpha;a,b)\left|f'(b)\right|^{q} + C_3(\alpha;a,b)\left|f'(a)\right|^{q}\right)^{1/q}$$

where

$$C_1(\alpha;a,b) = \frac{b^{-2}}{\alpha+1}\left[{}_2F_1\left(2,1;\alpha+2;1-\frac{a}{b}\right) + {}_2F_1\left(2,\alpha+1;\alpha+2;1-\frac{a}{b}\right)\right],$$

$$C_2(\alpha;a,b) = \frac{b^{-2}}{\alpha+2}\left[\frac{{}_2F_1\left(2,2;\alpha+3;1-\frac{a}{b}\right)}{\alpha+1} + {}_2F_1\left(2,\alpha+2;\alpha+3;1-\frac{a}{b}\right)\right],$$

$$C_3(\alpha;a,b) = \frac{b^{-2}}{\alpha+1}\left[{}_2F_1\left(2,1;\alpha+3;1-\frac{a}{b}\right) + \frac{{}_2F_1\left(2,\alpha+1;\alpha+3;1-\frac{a}{b}\right)}{\alpha+1}\right].$$

Theorem 1.3. *Let* $f : I \subseteq (0,\infty) \to \mathbb{R}$ *be a differentiable function on* I° *such that* $f' \in L[a,b]$, *where* $a,b \in I^{\circ}$ *with* $a < b$. *If* $|f'|^{q}$ *is harmonically convex on* $[a,b]$ *for some fixed* $q \geq 1$, *then the following inequality for fractional integrals holds:*

$$|I_f\left(g;\alpha,a,b\right)|$$

$$\leq \frac{ab\left(b-a\right)}{2}C_1^{1-1/q}(\alpha;a,b)\left(C_2(\alpha;a,b)\left|f'(b)\right|^{q} + C_3(\alpha;a,b)\left|f'(a)\right|^{q}\right)^{1/q},$$

where

$$C_1\left(\alpha;a,b\right)$$

$$= \frac{b^{-2}}{\alpha+1}\left[{}_2F_1\left(2,\alpha+1;\alpha+2;1-\frac{a}{b}\right) -{}_2F_1\left(2,1;\alpha+2;1-\frac{a}{b}\right)\right.$$

$$\left. +{}_2F_1\left(2,1;\alpha+2;\frac{1}{2}\left(1-\frac{a}{b}\right)\right)\right],$$

$$C_2\left(\alpha;a,b\right)$$

$$= \frac{b^{-2}}{\alpha+2}\left[{}_2F_1\left(2,\alpha+2;\alpha+3;1-\frac{a}{b}\right) - \frac{1}{\alpha+1}\,{}_2F_1\left(2,2;\alpha+3;1-\frac{a}{b}\right)\right.$$

$$\left. +\frac{1}{2\left(\alpha+1\right)}\,{}_2F_1\left(2,2;\alpha+3;\frac{1}{2}\left(1-\frac{a}{b}\right)\right)\right],$$

$$C_3\left(\alpha; a, b\right)$$

$$= \frac{b^{-2}}{\alpha+2}\left[\frac{1}{\alpha+1}\,{}_2F_1\left(2, \alpha+1; \alpha+3; 1-\frac{a}{b}\right) -_2F_1\left(2, 1; \alpha+3; 1-\frac{a}{b}\right)\right.$$

$$\left.+_2F_1\left(2, 1; \alpha+3; \frac{1}{2}\left(1-\frac{a}{b}\right)\right)\right]$$

and $0 < \alpha \leq 1$.

Theorem 1.4. *Let $f : I \subseteq (0, \infty) \rightarrow \mathbb{R}$ be a differentiable function on I° such that $f' \in L[a, b]$, where $a, b \in I^\circ$ with $a < b$. If $|f'|^q$ is harmonically convex on $[a, b]$ for some fixed $q > 1$, then the following inequality for fractional integrals holds:*

$$|I_f\left(g; \alpha, a, b\right)|$$

$$\leq \frac{a\left(b-a\right)}{2b}\left(\frac{1}{\alpha p+1}\right)^{1/p}\left(\frac{|f'(b)|^q + |f'(a)|^q}{2}\right)^{1/q}$$

$$\times\left[{}_2F_1^{1/p}\left(2p, 1; \alpha p+2; 1-\frac{a}{b}\right) +_2F_1^{1/p}\left(2p, \alpha p+1; \alpha p+2; 1-\frac{a}{b}\right)\right],$$

where $1/p + 1/q = 1$.

Theorem 1.5. *Let $f : I \subseteq (0, \infty) \rightarrow \mathbb{R}$ be a differentiable function on I° such that $f' \in L[a, b]$, where $a, b \in I^\circ$ with $a < b$. If $|f'|^q$ is harmonically convex on $[a, b]$ for some fixed $q > 1$, then the following inequality for fractional integrals holds:*

$$|I_f\left(g; \alpha, a, b\right)|$$

$$\leq \frac{b-a}{2\left(ab\right)^{1-1/p}}L_{2p-2}^{2-2/p}(a,b)\left(\frac{1}{\alpha q+1}\right)^{1/q}\left(\frac{|f'(b)|^q + |f'(a)|^q}{2}\right)^{1/q},$$

where $1/p + 1/q = 1$ and $L_{2p-2}(a,b) = \left(\frac{b^{2p-1}-a^{2p-1}}{(2p-1)(b-a)}\right)^{1/(2p-2)}$ is $2p-2$-Logarithmic mean.

Theorem 1.6. *Let $f : I \subseteq (0, \infty) \rightarrow \mathbb{R}$ be a differentiable function on I° such that $f' \in L[a, b]$, where $a, b \in I^\circ$ with $a < b$. If $|f'|^q$ is harmonically convex on $[a, b]$ for some fixed $q > 1$, then the following inequality for fractional integrals holds:*

$$|I_f\left(g; \alpha, a, b\right)|$$

$$\leq \frac{a\left(b-a\right)}{2b}\left(\frac{1}{\alpha p+1}\right)^{1/p}$$

$$\times\left(\frac{{}_2F_1\left(2q, 2; 3; 1-\frac{a}{b}\right)|f'(b)|^q +_2F_1\left(2q, 1; 3; 1-\frac{a}{b}\right)|f'(a)|^q}{2}\right)^{1/q},$$

where $1/p + 1/q = 1$.

In this paper, new identity for fractional integrals have been defined. By using of this identity we obtained some new results related to the left-hand side of the Hermite-Hadamard type inequality for harmonically convex functions via Riemann Liouville fractional integral.

2. Main Results

Let $f : I \subseteq (0, \infty) \to \mathbb{R}$ be a differentiable function on I°, the interior of I. Throughout this section we will take

$$
\begin{aligned}
& K_f(g; \alpha, a, b) \\
= \ & f\left(\frac{2ab}{a+b}\right) - \frac{\Gamma(\alpha+1)}{2}\left(\frac{ab}{b-a}\right)^\alpha \left\{ J^\alpha_{\frac{1}{a}-}(f \circ g)(1/b) + J^\alpha_{\frac{1}{b}+}(f \circ g)(1/a) \right\}
\end{aligned}
$$

where $a, b \in I$, with $a < b$, $\alpha > 0$, $g(x) = \frac{1}{x}$ and Γ is Euler Gamma function.

In order to prove our main results, we need the following Lemma:

Lemma 2.1. *Let $f : I \subseteq (0, \infty) \to \mathbb{R}$ be a differentiable function on I° such that $f' \in L[a, b]$, where $a, b \in I$ with $a < b$. Then the following equality for fractional integral holds :*

$$(2.1) \qquad\qquad \mathcal{K}_f(g; \alpha, a, b) = \frac{1}{2} \sum_{k=1}^{3} I_k$$

where

$$
\begin{aligned}
I_1 \ &= \ ab(b-a) \int_0^{1/2} f'\left(\frac{ab}{A_t}\right) \frac{dt}{A_t^2} \\
I_2 \ &= \ -ab(b-a) \int_{1/2}^{1} f'\left(\frac{ab}{A_t}\right) \frac{dt}{A_t^2} \\
I_3 \ &= \ -ab(b-a) \int_0^1 [(1-t)^\alpha - t^\alpha] f'\left(\frac{ab}{A_t}\right) \frac{dt}{A_t^2}
\end{aligned}
$$

and $A_t = ta + (1-t)b$.

Proof. Calculating I_1, I_2 and I_3 we have

$$
\begin{aligned}
I_1 \ &= \ ab(b-a) \int_0^{1/2} f'\left(\frac{ab}{A_t}\right) \frac{dt}{A_t^2} \\
&= \ \int_0^{1/2} df\left(\frac{ab}{A_t}\right) \\
&= \ \left. f\left(\frac{ab}{A_t}\right) \right|_0^{1/2} \\
&= \ f\left(\frac{2ab}{a+b}\right) - f(a),
\end{aligned}
$$

$$I_2 = -ab(b-a)\int_{1/2}^{1} f'\left(\frac{ab}{A_t}\right)\frac{dt}{A_t^2}$$

$$= -\int_{1/2}^{1} df\left(\frac{ab}{A_t}\right)$$

$$= -f\left(\frac{ab}{A_t}\right)\Big|_{1/2}^{1}$$

$$= f\left(\frac{2ab}{a+b}\right) - f(b)$$

and

$$I_3 = -ab(b-a)\int_{0}^{1} [(1-t)^\alpha - t^\alpha] f'\left(\frac{ab}{A_t}\right)\frac{dt}{A_t^2}$$

$$= -\int_{0}^{1} [(1-t)^\alpha - t^\alpha] df\left(\frac{ab}{A_t}\right)$$

$$= -\int_{0}^{1} (1-t)^\alpha df\left(\frac{ab}{A_t}\right) + \int_{0}^{1} t^\alpha df\left(\frac{ab}{A_t}\right)$$

$$= I_3^* + I_3^{**}.$$

By integrating by part in I_3^*, we get

$$I_3^* = -\int_{0}^{1} (1-t)^\alpha df\left(\frac{ab}{A_t}\right)$$

$$= (1-t)^\alpha f\left(\frac{ab}{A_t}\right)\Big|_{0}^{1} - \alpha\int_{0}^{1} (1-t)^{\alpha-1} f\left(\frac{ab}{A_t}\right) dt$$

$$= f(a) - \alpha\int_{0}^{1} (1-t)^{\alpha-1} f\left(\frac{ab}{A_t}\right) dt.$$

Here, by the changes of variables $u = \frac{ab}{A_t}$, we get

$$I_3^* = f(a) - \alpha\int_{a}^{b} \left(\frac{ab}{b-a}\right)^{\alpha-1} \left(\frac{1}{u} - \frac{1}{b}\right)^{\alpha-1} f(u)\frac{ab}{b-a}\frac{1}{u^2} du$$

$$= f(a) - \alpha\left(\frac{ab}{b-a}\right)^{\alpha}\int_{a}^{b} \left(\frac{1}{u} - \frac{1}{b}\right)^{\alpha-1}\frac{1}{u^2} f(u) du$$

and again by the changes of variables $u = \frac{1}{t}$, we get

$$
\begin{aligned}
I_3^* &= f(a) - \alpha \left(\frac{ab}{b-a}\right)^\alpha \int_{1/a}^{1/b} \left(t - \frac{1}{b}\right)^{\alpha-1} t^2 \left(-\frac{1}{t^2}\right) f(\tfrac{1}{t}) dt \\
&= f(a) + \alpha \left(\frac{ab}{b-a}\right)^\alpha \int_{1/a}^{1/b} \left(t - \frac{1}{b}\right)^{\alpha-1} f \circ g(t) dt \\
&= f(a) - \alpha \left(\frac{ab}{b-a}\right)^\alpha \Gamma(\alpha) \frac{1}{\Gamma(\alpha)} \int_{1/b}^{1/a} \left(t - \frac{1}{b}\right)^{\alpha-1} f \circ g(t) dt \\
&= f(a) - \Gamma(\alpha+1) \left(\frac{ab}{b-a}\right)^\alpha J_{1/a-}^\alpha f \circ g(1/b).
\end{aligned}
$$

Similarly, we get $I_3^{**} = \int_0^1 t^\alpha df\left(\frac{ab}{A_t}\right) = f(b) - \Gamma(\alpha+1) \left(\frac{ab}{b-a}\right)^\alpha J_{1/b+}^\alpha f \circ g(1/a)$. By adding I_1, I_2 and I_3, the desired result is obtained. $\qquad\square$

Using this Lemma, we can obtain the following inequalities.

Theorem 2.1. *Let $f : I \subseteq (0,\infty) \to \mathbb{R}$ a differentiable increasing function on I° such that $f' \in L[a,b]$, where $a, b \in I^\circ$ with $a < b$. If $(f')^q$ is harmonically convex on $[a,b]$ for some fixed $q \geq 1$, then the following inequality for fractional integrals holds:*

$$
\begin{aligned}
&|\mathcal{K}_f (g; \alpha, a, b)| \\
&\leq \quad \frac{f(b) - f(a)}{2} + \frac{ab(b-a)}{2} C_1^{1-1/q}(\alpha; a, b) \\
&\qquad \times \left[C_2(\alpha; a, b)(f'(b))^q + C_3(\alpha; a, b)(f'(a))^q\right]^{1/q},
\end{aligned}
$$

where

$$
\begin{aligned}
C_1(\alpha; a, b) &= \frac{b^{-2}}{\alpha+1} \left[{}_2F_1\left(2, 1; \alpha+2; 1 - \frac{a}{b}\right) + {}_2F_1\left(2, \alpha+1; \alpha+2; 1 - \frac{a}{b}\right)\right], \\
C_2(\alpha; a, b) &= \frac{b^{-2}}{\alpha+2} \left[\frac{{}_2F_1\left(2, 2; \alpha+3; 1 - \frac{a}{b}\right)}{\alpha+1} + {}_2F_1\left(2, \alpha+2; \alpha+3; 1 - \frac{a}{b}\right)\right], \\
C_3(\alpha; a, b) &= \frac{b^{-2}}{\alpha+1} \left[{}_2F_1\left(2, 1; \alpha+3; 1 - \frac{a}{b}\right) + \frac{{}_2F_1\left(2, \alpha+1; \alpha+3; 1 - \frac{a}{b}\right)}{\alpha+1}\right].
\end{aligned}
$$

Proof. Let $A_t = ta + (1-t)b$. From Lemma 2.1, using the property of the modulus , the power mean inequality and harmonically convexity of $(f')^q$, we find

$$
\begin{aligned}
|\mathcal{K}_f (g; \alpha, a, b)| &\leq \quad \frac{1}{2} \{|I_1| + |I_2| + |I_3|\} \\
&= \quad \frac{ab(b-a)}{2} \left[\int_0^{1/2} f'\left(\frac{ab}{A_t}\right) \frac{dt}{A_t^2} + \int_{1/2}^1 f'\left(\frac{ab}{A_t}\right) \frac{dt}{A_t^2}\right] + \frac{1}{2}|I_3| \\
&= \quad \frac{f(b) - f(a)}{2} + \frac{1}{2}|I_3|.
\end{aligned}
$$

$\qquad\square$

As in the proof of the Theorem 1.2, we have

$$
\begin{aligned}
\frac{1}{2}\,|I_3| \;&=\; \frac{ab(b-a)}{2}\left|\int\limits_0^1 \left[(1-t)^\alpha - t^\alpha\right] f'\left(\frac{ab}{A_t}\right)\frac{dt}{A_t^2}\right| \\
&\leq\; \frac{ab\,(b-a)}{2}C_1^{1-1/q}(\alpha;a,b)\left(C_2(\alpha;a,b)\,(f'(b))^q + C_3(\alpha;a,b)\,(f'(a))^q\right)^{1/q},
\end{aligned}
$$

where

$$
\begin{aligned}
C_1(\alpha;a,b) \;&=\; \frac{b^{-2}}{\alpha+1}\left[{}_2F_1\left(2,1;\alpha+2;1-\frac{a}{b}\right) +{}_2F_1\left(2,\alpha+1;\alpha+2;1-\frac{a}{b}\right)\right], \\
C_2(\alpha;a,b) \;&=\; \frac{b^{-2}}{\alpha+2}\left[\frac{{}_2F_1\left(2,2;\alpha+3;1-\frac{a}{b}\right)}{\alpha+1} +{}_2F_1\left(2,\alpha+2;\alpha+3;1-\frac{a}{b}\right)\right], \\
C_3(\alpha;a,b) \;&=\; \frac{b^{-2}}{\alpha+1}\left[{}_2F_1\left(2,1;\alpha+3;1-\frac{a}{b}\right) + \frac{{}_2F_1\left(2,\alpha+1;\alpha+3;1-\frac{a}{b}\right)}{\alpha+1}\right].
\end{aligned}
$$

The proof is competed.

Theorem 2.2. *Let* $f : I \subseteq (0,\infty) \to \mathbb{R}$ *be a differentiable increasing function on* I° *such that* $f' \in L[a,b]$, *where* $a,b \in I^\circ$ *with* $a < b$. *If* $(f')^q$ *is harmonically convex on* $[a,b]$ *for some fixed* $q \geq 1$, *then the following inequality for fractional integrals holds:*

$$
\begin{aligned}
|\mathcal{K}_f\,(g;\alpha,a,b)| \\
\leq\; \frac{f(b)-f(a)}{2} + \frac{ab\,(b-a)}{2}C_1^{1-1/q}(\alpha;a,b) \\
\times\left[C_2(\alpha;a,b)(f'(b))^q + C_3(\alpha;a,b)(f'(a))^q\right]^{1/q},
\end{aligned}
$$

where

$$
\begin{aligned}
&C_1(\alpha;a,b) \\
&=\; \frac{b^{-2}}{\alpha+1}\left[{}_2F_1\left(2,\alpha+1;\alpha+2;1-\frac{a}{b}\right) -{}_2F_1\left(2,1;\alpha+2;1-\frac{a}{b}\right)\right. \\
&\quad \left.+{}_2F_1\left(2,1;\alpha+2;\frac{1}{2}\left(1-\frac{a}{b}\right)\right)\right],
\end{aligned}
$$

$$
\begin{aligned}
&C_2(\alpha;a,b) \\
&=\; \frac{b^{-2}}{\alpha+2}\left[{}_2F_1\left(2,\alpha+2;\alpha+3;1-\frac{a}{b}\right) - \frac{1}{\alpha+1}\,{}_2F_1\left(2,2;\alpha+3;1-\frac{a}{b}\right)\right. \\
&\quad \left.+\frac{1}{2\,(\alpha+1)}\,{}_2F_1\left(2,2;\alpha+3;\frac{1}{2}\left(1-\frac{a}{b}\right)\right)\right],
\end{aligned}
$$

$$
\begin{aligned}
&C_3(\alpha;a,b) \\
&=\; \frac{b^{-2}}{\alpha+2}\left[\frac{1}{\alpha+1}\,{}_2F_1\left(2,\alpha+1;\alpha+3;1-\frac{a}{b}\right) -{}_2F_1\left(2,1;\alpha+3;1-\frac{a}{b}\right)\right. \\
&\quad \left.+{}_2F_1\left(2,1;\alpha+3;\frac{1}{2}\left(1-\frac{a}{b}\right)\right)\right]
\end{aligned}
$$

and $0 < \alpha \leq 1$.

Proof. Let $A_t = ta + (1 - t) b$. From Lemma 2.1, using the property of the modulus , the power mean inequality and the harmonically convexity of $(f')^q$, we find

$$
\begin{aligned}
|\mathcal{K}_f (g; \alpha, a, b)| &\leq \frac{1}{2} (|I_1| + |I_2| + |I_3|) \\
&\leq \frac{f(b) - f(a)}{2} + \frac{1}{2} |I_3|
\end{aligned}
$$

and

(2.2)

$$
\frac{1}{2} |I_3|
$$

$$
\leq \frac{ab(b-a)}{2} \int_0^1 \frac{|(1-t)^\alpha - t^\alpha|}{A_t^2} f'\left(\frac{ab}{A_t}\right) dt
$$

$$
\leq \frac{ab(b-a)}{2} \left(\int_0^1 \frac{|(1-t)^\alpha - t^\alpha|}{A_t^2} dt \right)^{1-\frac{1}{q}} \left(\int_0^1 \frac{|(1-t)^\alpha - t^\alpha|}{A_t^2} \left(f'\left(\frac{ab}{A_t}\right) \right)^q dt \right)^{\frac{1}{q}}
$$

$$
\leq \frac{ab(b-a)}{2} K_1^{1-1/q} \left(\int_0^1 \frac{|(1-t)^\alpha - t^\alpha|}{A_t^2} [t(f'(b))^q + (1-t)(f'(a))^q] dt \right)^{1/q}
$$

$$
\leq \frac{ab(b-a)}{2} K_1^{1-1/q} (K_2(f'(b))^q + K_3(f'(a))^q)^{1/q},
$$

where

$$
K_1 = \int_0^1 \frac{|(1-t)^\alpha - t^\alpha|}{A_t^2} dt,
$$

$$
K_2 = \int_0^1 \frac{|(1-t)^\alpha - t^\alpha|}{A_t^2} t \, dt,
$$

$$
K_3 = \int_0^1 \frac{|(1-t)^\alpha - t^\alpha|}{A_t^2} (1-t) \, dt.
$$

If K_1, K_2 and K_3 are calculated as in the proof of the Theorem 1.3, and used in the inequality (2.2), the desired result is obtained.

Theorem 2.3. *Let $f : I \subseteq (0, \infty) \to \mathbb{R}$ be a differentiable increasing function on I° such that $f' \in L[a, b]$, where $a, b \in I^\circ$ with $a < b$. If $(f')^q$ is harmonically convex on $[a, b]$ for some fixed $q > 1$, then the following inequality for fractional integrals holds:*

(2.3)
$$
|\mathcal{K}_f (g; \alpha, a, b)|
$$
$$
\leq \frac{f(b) - f(a)}{2} + \frac{a(b-a)}{2b} \left(\frac{1}{\alpha p + 1} \right)^{1/p} \left(\frac{(f'(b))^q + (f'(a))^q}{2} \right)^{1/q}
$$
$$
\times \left[{}_2F_1^{1/p} \left(2p, 1; \alpha p + 2; 1 - \frac{a}{b} \right) + {}_2F_1^{1/p} \left(2p, \alpha p + 1; \alpha p + 2; 1 - \frac{a}{b} \right) \right],
$$

where $1/p + 1/q = 1$.

$\square$

Proof. Let $A_t = ta + (1-t)b$. From Lemma 2.1, using the Hölder inequality and the harmonically convexity of $(f')^q$, we get

$$|\mathcal{K}_f(g; \alpha, a, b)|$$

$$\leq \frac{f(b) - f(a)}{2} + \frac{ab(b-a)}{2}\left\{\int_0^1 \frac{(1-t)^\alpha}{A_t^2} f'\left(\frac{ab}{A_t}\right) dt + \int_0^1 \frac{t^\alpha}{A_t^2} f'\left(\frac{ab}{A_t}\right) dt\right\}$$

$$\leq \frac{f(b) - f(a)}{2} + \frac{ab(b-a)}{2}\left\{\left(\int_0^1 \frac{(1-t)^{\alpha p}}{A_t^{2p}} dt\right)^{1/p}\left(\int_0^1 \left(f'\left(\frac{ab}{A_t}\right)\right)^q dt\right)^{1/q}\right.$$

$$(2.4)$$

$$\left. + \left(\int_0^1 \frac{t^{\alpha p}}{A_t^{2p}} dt\right)^{1/p}\left(\int_0^1 \left(f'\left(\frac{ab}{A_t}\right)\right)^q dt\right)^{1/q}\right\}$$

$$\leq \frac{f(b) - f(a)}{2} + \frac{ab(b-a)}{2}\left(K_4^{1/p} + K_5^{1/p}\right)\left(\int_0^1 [t(f'(b))^q + (1-t)(f'(a))^q] dt\right)^{\frac{1}{q}}$$

$$\leq \frac{f(b) - f(a)}{2} + \frac{ab(b-a)}{2}\left(K_4^{1/p} + K_5^{1/p}\right)\left(\frac{f'(b)^q + f'(a)^q}{2}\right)^{1/q}.$$

As in the proof of Theorem 1.4, calculating K_4 and K_5, we have

$$(2.5) \qquad K_4 = \int_0^1 \frac{(1-t)^{\alpha p}}{A_t^{2p}} dt$$

$$= \frac{b^{-2p}}{\alpha p + 1} \cdot {}_2F_1\left(2p, 1; \alpha p + 2; 1 - \frac{a}{b}\right),$$

$$(2.6) \qquad K_5 = \int_0^1 \frac{t^{\alpha p}}{A_t^{2p}} dt$$

$$= \frac{b^{-2p}}{\alpha p + 1} \cdot {}_2F_1\left(2p, \alpha p + 1; \alpha p + 2; 1 - \frac{a}{b}\right).$$

Thus , if we use (2.5) and (2.6) in (2.4), we obtain the inequality of (2.3).This completes the proof. $\square$

Theorem 2.4. *Let $f : I \subseteq (0, \infty) \to \mathbb{R}$ be a differentiable increasing function on I° such that $f' \in L[a, b]$, where $a, b \in I^\circ$ with $a < b$. If $(f')^q$ is harmonically convex on $[a, b]$ for some fixed $q > 1$, then the following inequality for fractional integrals holds:*

(2.7)
$$|\mathcal{K}_f(g;\alpha,a,b)|$$
$$\leq \frac{f(b)-f(a)}{2} + \frac{b-a}{2(ab)^{1-1/p}} L_{2p-2}^{2-2/p}(a,b) \left(\frac{1}{\alpha q+1}\right)^{1/q} \left(\frac{(f'(b))^q + (f'(a))^q}{2}\right)^{1/q},$$

where $1/p + 1/q = 1$ and $L_{2p-2}(a,b) = \left(\frac{b^{2p-1}-a^{2p-1}}{(2p-1)(b-a)}\right)^{1/(2p-2)}$ is $2p-2$-Logarithmic mean.

Proof. Let $A_t = ta + (1-t)b$. From Lemma 2.1 and Lemma 1.1, using the Hölder inequality and the Harmonically convexity of $(f')^q$, we get

$$|\mathcal{K}_f(g;\alpha,a,b)|$$

$$\leq \frac{f(b)-f(a)}{2} + \frac{ab(b-a)}{2} \int_0^1 \frac{|(1-t)^\alpha - t^\alpha|}{A_t^2} \left(f'\left(\frac{ab}{A_t}\right)\right) dt$$

$$\leq \frac{f(b)-f(a)}{2} + \frac{ab(b-a)}{2} \left(\int_0^1 \frac{1}{A_t^{2p}} dt\right)^{1/p}$$

$$\times \left(\int_0^1 |(1-t)^\alpha - t^\alpha|^q \left(f'\left(\frac{ab}{A_t}\right)\right)^q dt\right)^{1/q}$$

$$\leq \frac{f(b)-f(a)}{2} + \frac{ab(b-a)}{2} \left(\int_0^1 \frac{1}{A_t^{2p}} dt\right)^{1/p}$$

$$\times \left(\int_0^1 |1-2t|^{\alpha q} \left[t(f'(b))^q + (1-t)(f'(a))^q\right] dt\right)^{1/q}$$

(2.8)
$$\leq \frac{f(b)-f(a)}{2} + \frac{ab(b-a)}{2} K_6^{1/p} \left(K_7 |f'(b)|^q + K_8 |f'(a)|^q\right)^{1/q},$$

where as in the proof of Theorem 1.5

(2.9)
$$K_6 = \int_0^1 \frac{1}{A_t^{2p}} dt = b^{-2p} \int_0^1 \left(1 - t\left(1 - \frac{a}{b}\right)\right)^{-2p} dt$$

$$= b^{-2p} \cdot {}_2F_1\left(2p, 1; 2; 1 - \frac{a}{b}\right) = \frac{L_{2p-2}^{2p-2}(a,b)}{(ab)^{2p-1}},$$

$$(2.10) \qquad K_7 \;=\; \int_0^1 |1 - 2t|^{\alpha q}\, t\, dt$$

$$= \int_0^{1/2} (1 - 2t)^{\alpha q}\, t\, dt + \int_{1/2}^1 (2t - 1)^{\alpha q}\, t\, dt$$

$$= \frac{1}{2\,(\alpha q + 1)},$$

and

$$(2.11) \qquad K_8 \;=\; \int_0^1 |1 - 2t|^{\alpha q}\, (1 - t)\, dt$$

$$= \frac{1}{2\,(\alpha q + 1)}.$$

If we use (2.9),(2.10) and (2.11) in (2.8), we obtain the inequality of (2.7). This completes the proof.

Theorem 2.5. *Let $f : I \subseteq (0, \infty) \to \mathbb{R}$ be a differentiable increasing function on I° such that $f' \in L[a, b]$, where $a, b \in I^\circ$ with $a < b$. If $(f')^q$ is harmonically convex on $[a, b]$ for some fixed $q > 1$, then the following inequality for fractional integrals holds:*

$$(2.12) \qquad |\mathcal{K}_f\,(g; \alpha, a, b)|$$

$$\leq \frac{f(b) - f(a)}{2} + \frac{a\,(b - a)}{2b} \left(\frac{1}{\alpha p + 1} \right)^{1/p}$$

$$\times \left(\frac{{}_2F_1\left(2q, 2; 3; 1 - \frac{a}{b}\right) (f'(b))^q + {}_2F_1\left(2q, 1; 3; 1 - \frac{a}{b}\right) (f'(a))^q}{2} \right)^{1/q},$$

where $1/p + 1/q = 1$.

$\square$

Proof. Let $A_t = ta + (1 - t) b$. From Lemma 2.1 and Lemma 1.1, using the Hölder inequality and Harmonically convexity of $(f')^q$, we find

$$|\mathcal{K}_f (g; \alpha, a, b)|$$

$$\leq \frac{f(b) - f(a)}{2} + \frac{ab (b - a)}{2} \int_0^1 \frac{|(1 - t)^\alpha - t^\alpha|}{A_t^2} \left(f' \left(\frac{ab}{A_t} \right) \right) dt$$

$$\leq \frac{f(b) - f(a)}{2} + \frac{ab (b - a)}{2} \left(\int_0^1 |(1 - t)^\alpha - t^\alpha|^p \, dt \right)^{1/p}$$

$$\times \left(\int_0^1 \frac{1}{A_t^{2q}} \left(f' \left(\frac{ab}{A_t} \right) \right)^q dt \right)^{1/q}$$

$$\leq \frac{f(b) - f(a)}{2} + \frac{ab (b - a)}{2} \left(\int_0^1 |1 - 2t|^{\alpha p} \, dt \right)^{1/p}$$

$$\times \left(\int_0^1 \frac{1}{A_t^{2q}} \left[t \, (f'(b))^q + (1 - t) \, (f'(a))^q \right] dt \right)^{1/q}$$

$$(2.13) \qquad \leq \frac{f(b) - f(a)}{2} + \frac{ab (b - a)}{2} K_9^{1/p} \left(K_{10} \, (f'(b))^q + K_{11} \, (f'(a))^q \right)^{1/q},$$

where as in the proof of Theorem 1.6,

$$(2.14) \qquad\qquad K_9 = \int_0^1 |1 - 2t|^{\alpha p} \, dt = \frac{1}{\alpha p + 1}$$

$$(2.15) \qquad K_{10} = \int_0^1 t A_t^{-2q} dt = b^{-2q} \int_0^1 t \left(1 - t \left(1 - \frac{a}{b} \right) \right)^{-2q} dt$$

$$= \frac{1}{2b^{2q}} \, {}_2F_1 \left(2q, 2; 3; 1 - \frac{a}{b} \right)$$

and

$$(2.16) \qquad K_{11} = \int_0^1 (1 - t) A_t^{-2q} dt = \frac{1}{2b^{2q}} \, {}_2F_1 \left(2q, 1; 3; 1 - \frac{a}{b} \right)$$

Thus, if we use (2.14),(2.15) and (2.16) in (2.13), we obtain the inequality of (2.12). This completes the proof. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad\square$

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