Konuralp Journal of Mathematics
Volume 3 No. 2 pp. 89–99 (2015) ©KJM

CONVERGENCE OF MULTI-STEP ITERATIVE SEQUENCE FOR NONLINEAR UNIFORMLY L-LIPSCHITZIAN MAPPINGS

MOGBADEMU, ADESANMI ALAO

Abstract. In this paper, by using the proof method of Xue, Rafiq and Zhou[19] some strong convergence results of multi-step iterative sequence are proved for nearly uniformly $L-$ Lipschitzian mappings in real Banach spaces. Our results generalise and improve some recent known results.

1. Introduction

We denote by J the normalized duality mapping from X into 2^{X^*} by

$$J(x) = \{f \in \mathrm{X}^* : \langle x, f\rangle = \|x\|^2 = \|f\|^2\},$$

where X^* denotes the dual space of real Banach space X and $\langle .,.\rangle$ denotes the generalized duality pairing between elements of X and X^*. We first recall and define some concepts as follows (see [4]):
Let K be a nonempty subset of real Banach space X.
The mapping T is said to be uniformly L- Lipschitzian if there exists a constant $L > 0$ such that

$$\|T^n x - T^n y\| \leq L\|x - y\|,$$

for any $x, y \in K$ and $\forall n \geq 1$.
The mapping T is said to be asymptotically pseudocontractive if there exists a sequence $(k_n) \subset [1, \infty)$ with $lim_{n\to\infty} k_n = 1$ and for any $x, y \in K$ there exists $j(x - y) \in J(x - y)$ such that

$$< T^n x - T^n y, j(x - y) >\leq k_n\|x - y\|^2, \forall n \geq 1.$$

The concept of asymptotically pseudocontractive mappings was introduced by Schu [17].
A mapping $T : K \to X$ is called Lipschitzian if there exists a constant $L > 0$ such that

$$\|Tx - Ty\| \leq L\|x - y\|,$$

2000 *Mathematics Subject Classification.* 47H10, 46A03.
Key words and phrases. Multistep iteration; asymptotically pseudo-contractive; uniformly $L-$ Lipschitzian; Banach spaces.

for all $x, y \in K$ and is called generalized Lipschitzian if there exists a constant $L > 0$ such that

$$\|Tx - Ty\| \leq L(\|x - y\| + 1),$$

for all $x, y \in K$.
It is obvious that the class of generalized Lipschitzian map includes the class of Lipschitz map. Moreover, every mapping with a bounded range is a generalized Lipschitzian mapping.
Sahu [18] introduced the following new class of nonlinear map which is more general than the class of generalized Lipschitzian mappings and the class of uniformly L-Lipschitzian mappings. In fact, he introduced the following class of nearly Lipschitzian: Let K be a subset of a normed space X and let $\{a_n\}_{n\geq 0}$ be a sequence in $[0, \infty)$ such that $lim_{n\to\infty} a_n = 0$.
A mapping $T : K \to K$ is called nearly Lipschitzian with respect to $\{a_n\}$ if for each $n \in N$, there exists a constant $k_n \geq 0$ such that

$$\|T^n x - T^n y\| \leq k_n(\|x - y\| + a_n), \quad \forall \quad x, y \in K. \tag{1.1}$$

Define

$$\mu(T^n) = sup\{\frac{||T^n x - T^n y||}{||x - y|| + a_n} : x, y \in K, x \neq y\}.$$

Observe that for any sequence $\{k_n\}_n \geq 1$ satisfying (1.1) $\mu(T^n) \leq k_n$ $\forall n \in N$ and that

$$\|T^n x - T^n y\| \leq \mu(T^n)(\|x - y\| + a_n), \quad \forall \quad x, y \in K$$

$\mu(T^n)$ is called the nearly Lipschitz constant of the mapping T. A nearly Lipschitzian mapping T is said to be
(i) nearly contraction if $\mu(T^n) < 1$ for all $n \in N$;
(ii) nearly nonexpansive if $\mu(T^n) = 1$ for all $n \in N$;
(iii) nearly asymptotically nonexpansive if $\mu(T^n) \geq 1$ for all $n \in N$ and $\lim_{n\to\infty} \mu(T^n) = 1$;
(iv) nearly uniformly $L-$ Lipschitzian if $\mu(T^n) \leq L$ for all $n \in N$;
(v) nearly uniformly $k-$ contraction if $\mu(T^n) \leq k < 1$ for all $n \in N$.
A nearly Lipschitzian mapping T with sequence $\{a_n\}$ is said to be nearly uniformly $L-$ Lipschitzian if $k_n = L$ for all $n \in N$.
Observe that the class of nearly uniformly $L-$ Lipschitzian mapping is more general than the class of uniformly $L-$ Lipschitzian mappings.
Example 1.1 (see Sahu[18]). Let $E = R,\ K = [0, 1]$.
Define $T : K \to K$ by

$$Tx = \{ \begin{array}{rl} 1/2, & x \in [0, 1/2), \\ 0, & x \in (1/2, 1]. \end{array}$$

It is obvious that T is not continuous, and thus not Lipschitz. However, T is nearly nonexpansive. Infact, for a real sequence $\{a_n\}_n \geq 1$ with $a_1 = \frac{1}{2}$ and $a_n \to 0$ as $n \to \infty$, we have

$$||Tx - Ty|| \leq ||x - y|| + a_1,\ \forall x, y \in K$$

and

$$||T^n x - T^n y|| \leq ||x - y|| + a_n, \ \forall x, y \in K, \ n \geq 2.$$

This is because $T^n x = \frac{1}{2}, \forall x \in [0,1], \ n \geq 2.$

Remark 1.1: The class of nearly uniformly $L-$ Lipschitzian is not necessarily continuous.

In recent years, many authors have given much attention to iterative methods for approximating fixed points of Lipschitz asymptotically type nonlinear mappings (see [1-4, 6, 9, 17, 18]).

Schu [17] proved the following theorem:

Theorem 1.1 ([17]). Let H be a Hilbert space, K be a nonempty bounded closed convex subset of H and $T : K \to K$ be completely continuous, uniformly L-Lipschitzian and asymptotically pseudocontractive mapping with a sequence $k_n \subset [1, \infty)$ satisfying the following conditions:

(i) $k_n \to 1$ as $n \to \infty$ (ii) $\sum_{n=1}^{\infty} q_n^2 - 1 < \infty$, where $q_n = 2k - 1$.

Suppose further that $\{\alpha_n\}_{n=1}^{\infty}$ and $\{\beta_n\}_{n=1}^{\infty}$ be two sequences in $[0,1]$ such that $\epsilon < \alpha_n < \beta_n \leq b, \ \ \forall n \geq 1$, where $\epsilon > 0$ and $b \in (0, L^{-2}[(1 + L^2)^{\frac{1}{2}} - 1])$ are some positive numbers. For any $x_1 \in K$, let $\{x_n\}_{n=1}^{\infty}$ be iterative sequence defined by

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T^n x_n, \quad n \geq 1.$$

Then $\{x_n\}_{n=1}^{\infty}$ converges strongly to a fixed point of T in K.

In [1], Chang extended Theorem 1.3 to a real uniformly smooth Banach space and proved the following theorem:

Theorem 1.2 ([1]). Let E be a real uniformly smooth Banach space, K be a nonempty bounded closed convex subset of E, $T : K \to K$ be an asymptotically pseudocontractive mapping with a sequence $k_n \subset [1, \infty)$ with $k_n \to 1$ and $F(T) \neq \phi$, where $F(T)$ is the set of fixed points of T in K. Let $\{\alpha_n\}_{n=0}^{\infty}$ be a sequence in $[0,1]$ satisfying the following conditions: (i) $\lim_{n\to\infty} \alpha_n = 0$ (ii)$\sum_{n=0}^{\infty} \alpha_n = \infty$. For any $x_0 \in K$, let $\{x_n\}_{n=0}^{\infty}$ be the iterative sequence defined by

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T^n x_n, \quad n \geq 0.$$

If there exists a strictly increasing function $\Phi : [0, \infty) \to [0, \infty)$ with $\Phi(0) = 0$ such that

$$< T^n x_n - \rho, j(x_n - \rho) > \leq k_n \|x_n - \rho\|^2 - \Phi(\|x_n - \rho\|), \quad n \geq 0$$

where $\rho \in F(T)$ is some fixed point of T in K, then $x_n \to \rho$ as $n \to \infty$.

Ofoedu [13] used the modified Mann iteration process introduced by Schu [17] ,

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T^n x_n, \quad n \geq 0, \tag{1.2}$$

to obtain a strong convergence theorem for uniformly Lipschitzian asymptotically pseudocontractive mapping in real Banach space setting. He proved the following theorem:

Theorem 1.3 ([13]). Let E be a real Banach space, K be a nonempty closed convex subset of E, $T : K \to K$, be a uniformly L-Lipschitzian asymptotically mappings with a sequence $k_n \subset [1, \infty)$, $k_n \to 1$ such that $\rho \in F(T)$, where $F(T)$ is the set of fixed points of T in K. Let $\{\alpha_n\}_{n=0}^{\infty}$ be a sequence in $[0,1]$ satisfying the following conditions: (i) $\sum_{n=0}^{\infty} \alpha_n = \infty$ (ii) $\sum_{n=0}^{\infty} \alpha_n^2 < \infty$ (iii) $\sum_{n=0}^{\infty} \beta_n < \infty$

(iv) $\sum_{n=0}^{\infty} \alpha_n(k_n - 1) < \infty$.
For any $x_0 \in K$, let $\{x_n\}_{n=0}^{\infty}$ be the iterative sequence defined by (1.1).
If there exists a strictly increasing function $\Phi : [0, \infty) \to [0, \infty)$ with $\Phi(0) = 0$ such that

$$< T^n x_n - \rho, j(x_n - \rho) > \leq k_n \|x_n - \rho\|^2 - \Phi(\|x_n - \rho\|)$$

for all $x \in K$, then $\{x_n\}_{n=0}^{\infty}$ converges strongly to ρ.
Obviously, this result extends Theorem 1.2 of Chang [1] from a real uniformly smooth Banach space to an arbitrary real Banach space and removes the boundedness condition imposed on K.
Chang et al.[3] used an Ishikawa iteration sequence to prove a strong convergence theorem for a pair of $L-$ Lipschitzian mappings instead of a single map used in Ofoedu [13].
Rafiq, Acu and Sofonea [15], improved the results of Chang et al. [3] in a significant more general context. They then gave an open problem whether their results can be extended for the case of three mappings which are more general than the two maps. Indeed, they proved the following theorem.
Theorem 1.3 ([15]). Let K be a nonempty closed convex subset of a real Banach space E, $T_i : K \to K$, $(i = 1, 2)$ be two uniformly L-Lipschitzian mappings with sequence $k_n \subset [1, \infty)$, $\sum_{n=1}^{\infty}(k_n - 1) < \infty$ such that $F(T_1) \cap F(T_2) \neq \phi$, where $F(T_i)$ is the set of fixed points of T_i in K and ρ be a point in $F(T_1) \cap F(T_2)$. Let $\{\alpha_n\}_{n=1}^{\infty}$ and $\{\beta_n\}_{n=1}^{\infty}$ be two sequences in $[0, 1]$ such that $\sum_{n=1}^{\infty} \alpha_n = \infty$, $lim_{n\to\infty}\alpha_n = \beta_n = 0$. For any $x_1 \in K$, let $\{x_n\}_{n=1}^{\infty}$ be a sequence iteratively defined by

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T_1^n y_n$$

$$y_n = (1 - \beta_n)x_n + \beta_n T_2^n x_n.$$

Suppose there exists a strictly increasing function $\Phi : [0, \infty) \to [0, \infty)$ with $\Phi(0) = 0$ such that

$$< T_i^n x_n - \rho, j(x_n - \rho) > \leq k_n \|x_n - \rho\|^2 - \Phi(\|x_n - \rho\|), \forall x \in K (i = 1, 2),$$

then $\{x_n\}_{n=1}^{\infty}$ converges strongly to $\rho \in F(T_1) \cap F(T_2)$.
In [10], the author established a new result on convergence of the modified Noor iteration for three nearly Lipschitzian mappings. His result extends, improves and unifies a host of recent results. Although, Mogbademu and Xue [9] had earlier obtained a strong convergence theorem for asymptotically generalized $\Phi-$ hemicontractive map in real Banach spaces using the iterative sequence generated by this map.
More recently, Xue, Rafiq and Zhou [19] employed an analytical technique to prove the convergent of an Ishikawa and Mann iterations for nonlinear mappings in uniformly smooth real Banach spaces. It is the purpose of this paper, using the style of proof by Xue, Rafiq and Zhou [19] to prove strong convergence theorems of multistep iteration scheme (1.3) for nearly uniformly Lipschitzian mappings in a real Banach space. Our results significantly generalise and improve some recent results of [1-3, 6, 13, 17, 18] in some aspects. For this, we need the following concepts and Lemmas.

The following iteration (see Rhoades and Soltuz [16]):

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n T^n y_n^1,\ n \geq 0,$$
$$y_n^i = (1-\beta_n^i)x_n + \beta_n^i T^n y_n^{k+1},\ i = 1, 2, ..., p-2,$$
$$y_n^{p-1} = (1-\beta_n^{p-1})x_n + \beta_n^{p-1} T^n x_n,\ n \geq 0, p \geq 2. \tag{1.3}$$

is called the multistep iteration sequence, where $p \geq 2$ is fixed order, $\{\alpha_n\}, \{\beta_n^i\}$ are sequences in $[0,1]$ for $i = 1, 2, ..., p-1$.
Taking $p = 3$ in (1.3) we obtain the Noor iteration scheme as follows:

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n T^n y_n^1,\ n \geq 0,$$
$$y_n^1 = (1-\beta_n^1)x_n + \beta_n^1 T^n y_n^2,$$
$$y_n^2 = (1-\beta_n^2)x_n + \beta_n^2 T^n x_n,\ n \geq 0. \tag{1.4}$$

where $\{\alpha_n\}, \{\beta_n^i\}$ are sequences in $[0,1]$ for $i = 1, 2$.
Taking $p = 2$ in (1.3) we obtain the Ishikawa iteration scheme as follows:

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n T^n y_n^1,\ n \geq 0,$$
$$y_n^1 = (1-\beta_n^1)x_n + \beta_n^1 T^n x_n, \tag{1.5}$$

where $\{\alpha_n\}, \{\beta_n^1\}$ are sequences in $[0,1]$.
In particular, if $\beta_n^1 = 0$ for $n \geq 0$ in (1.5) the sequence $\{x_n\}$ defined by

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n T^n x_n,\ n \geq 0, \tag{1.6}$$

is called the Mann iteration sequence (see [7]).
We remark that iteration (1.3) generalises the Mann, Ishikawa and Noor iteration sequences. The multistep iteration sequence (1.3) can be viewed as the predictor-corrector methods for solving nonlinear equations in Banach spaces. For the convergence analysis of the predictor-corrector and multistep iteration sequences for solving the variational inequalities and optimization problems (see Noor[11], Noor et al. [12]).
Lemma 1.1 [1, 9]. Let E be real Banach Space and $J : E \to 2^{E^*}$ be the normalized duality mapping. Then,for any $x, y \in E$

$$\|x+y\|^2 \leq \|x\|^2 + 2 < y, j(x+y) >, \forall j(x+y) \in J(x+y).$$

Lemma 1.2 [8]. Let $\Phi : [0,\infty) \to [0,\infty)$ be an increasing function with $\Phi(x) = 0 \Leftrightarrow x = 0$ and let $\{b_n\}_{n=0}^{\infty}$ be a positive real sequence satisfying

$$\sum_{n=0}^{\infty} b_n = +\infty \quad \text{and} \quad \lim_{n\to\infty} b_n = 0.$$

Suppose that $\{a_n\}_{n=0}^{\infty}$ is a nonnegative real sequence. If there exists an integer $N_0 > 0$ satisfying

$$a_{n+1}^2 < a_n^2 + o(b_n) - b_n \Phi(a_{n+1}), \quad \forall n \geq N_0$$

where $\lim_{n\to\infty} \frac{o(b_n)}{b_n} = 0$, then $\lim_{n\to\infty} a_n = 0$.

2. Main results

Theorem 2.1. Let X be a real Banach space, K be a nonempty closed convex subset of X, $T : K \to K$ be a nearly uniformly L-Lipschitzian mapping with sequence $\{a_n\}$. Let $k_n \subset [1, \infty)$ and ϵ_n be sequences with $\lim_{n\to\infty} k_n = 1$, $\lim_{n\to\infty} \epsilon_n = 0$ and $F(T) = \{\rho \in K : T\rho = \rho\}$. Let $\{\alpha_n\}_{n\geq 0}$ and $\{\beta_n^i\}_{n\geq 0}, (i = 1, 2, ..., p-1)$ be real sequences in $[0, 1]$ satisfying the following conditions: (i)$\sum_{n\geq 0} \alpha_n = \infty$ (ii) $\lim_{n\to\infty} \alpha_n, \beta_n^i = 0, (i = 1, 2, ..., p-1)$. For arbitrary $x_0 \in K$, let $\{x_n\}_{n\geq 0}$ be iteratively defined by (1.3). If there exists a strictly increasing function $\Phi : [0, \infty) \to [0, \infty)$ with $\Phi(0) = 0$ such that

$$< T^n x - T^n \rho, j(x - \rho) > \leq k_n \|x - \rho\|^2 - \Phi(\|x - \rho\|) + \epsilon_n$$

for all $x \in K$. Then, $\{x_n\}_{n\geq 0}$ converges strongly to $\rho \in F(T)$.

Proof. Since there exists a strictly increasing continuous function $\Phi : [0, \infty) \to [0, \infty)$ with $\Phi(0) = 0$ such that

$$\langle T^n x - T^n \rho, j(x - \rho)\rangle \leq k_n \|x - \rho\|^2 - \Phi(\|x - \rho\|) + \epsilon_n, \tag{2.1}$$

for $x \in K$, $\rho \in F(T)$, that is

$$\epsilon_n + \langle k_n(x - \rho) - (T^n x - \rho), j(x - \rho)\rangle \geq \Phi(\|x - \rho\|). \tag{2.2}$$

Choose some $x_0 \in K$ and $x_0 \neq Tx_0$ such that $\epsilon_n + (k_n + L)\|x_0 - \rho\|^2 + L\|x_0 - \rho\|^2 \in R(\Phi)$ and denote that $a_0 = \epsilon_n + (k_n + L)\|x_0 - \rho\|^2 + L\|x_0 - \rho\|^2$, $R(\Phi)$ is the range of Φ. Indeed, if $\Phi(a) \to +\infty$ as $a \to \infty$, then $a_0 \in R(\Phi)$; if $\sup\{\Phi(a) : a \in [0, \infty]\} = a_1 < +\infty$ with $a_1 < a_0$, then for $\rho \in K$, there exists a sequence $\{u_n\}$ in K such that $u_n \to \rho$ as $n \to \infty$ with $u_n \neq \rho$. Clearly, $Tu_n \to T\rho$ as $n \to \infty$ thus $\{u_n - Tu_n\}$ is a bounded sequence. Therefore, there exists a natural number n_0 such that $\epsilon_n + (k_n + L)\|u_n - \rho\|^2 + L\|u_n - \rho\|^2 < \frac{a_1}{2}$ for $n \geq n_0$, then we redefine $x_0 = u_{n_0}$ and $\epsilon_n + (k_n + L)\|x_0 - \rho\|^2 + L\|x_0 - \rho\|^2 \in R(\Phi)$. This is to ensure that $\Phi^{-1}(a_0)$ is well defined.
Step 1. We first show that $\{x_n\}_{n=0}^{\infty}$ is a bounded sequence.
Set $R = \Phi^{-1}(a_0)$, then from above (2.2), we obtain that $\|x_n - \rho\| \leq R$. Denote

$$B_1 = \{x \in K : \|x - \rho\| \leq R\}, \quad B_2 = \{x \in K : \|x - \rho\| \leq 2R\}. \tag{2.3}$$

Now, we want to prove that $x_n \in B_1$. If $n = 0$, then $x_0 \in B_1$. Now assume that it holds for some n, that is, $x_n \in B_1$. Suppose that, it is not the case, then $\|x_{n+1} - \rho\| > R > \frac{R}{2}$.
Since $\{a_n\} \in [0, \infty]$ with $a_n \to 0$, set $M = sup\{a_n : n \in N\}$. Denote

$$\tau_0 \;=\; min\{1, \tfrac{\Phi(\frac{R}{2})}{32R^2}, \tfrac{\Phi(\frac{R}{2})}{16R[2(L(2R+M)+R)+M]}, \tfrac{\Phi(\frac{R}{2})}{16R[L(2R+M)+R]}, \tfrac{\Phi(\frac{R}{2})}{8}\}. \tag{2.4}$$

Since $\lim_{n\to\infty} \alpha_n, \beta_n^i = 0$ for $i = 1, 2, ..., p-1$ and $\lim_{n\to\infty} k_n = 1$. Without loss of generality, let $0 \leq \alpha_n, \beta_n^i, k_n - 1, \epsilon_n \leq \tau_0$ for any $n \geq 0$. Then, we have the following estimates from (2.1) for $i = 1, 2, ..., p-1$.

$$\begin{aligned} \|y_n^{p-1} - \rho\| &\le (1-\beta_n^{p-1})\|x_n - \rho\| + \beta_n^{p-1}\|T^n x_n - \rho\| \\ &\le R + \tau_0 L(R+M) \\ &\le 2R. \end{aligned}$$

then $y^{p-1} \in B_2$. Similarly,

$$\begin{aligned} \|y_n^{p-2} - \rho\| &\le (1-\beta_n^{p-2})\|x_n - \rho\| + \beta_n^{p-2}\|T^n y_n^{p-1} - \rho\| \\ &\le R + \tau_0 L(2R+M) \\ &\le 2R. \end{aligned}$$

then $y^{p-2} \in B_2 ...$, we have

$$\begin{aligned} \|y_n^1 - \rho\| &\le (1-\beta_n^1)\|x_n - \rho\| + \beta_n^1\|T^n y_n^2 - \rho\| \\ &\le R + \tau_0 L(2R+M) \\ &\le 2R. \end{aligned}$$

then for $y^1 \in B_2$. We get

$$\begin{aligned} \|x_{n+1} - \rho\| &\le (1-\alpha_n)\|x_n - \rho\| + \alpha_n\|T^n y_n^1 - \rho\| \\ &\le R + \tau_0 L(2R+M) \\ &\le 2R. \end{aligned}$$

Therefore, we have

$$\begin{aligned} \|x_{n+1} - x_n\| &\le \alpha_n\|T^n y_n^1 - x_n\| \\ &\le \alpha_n(\|T^n y_n^1 - \rho\| + \|x_n - \rho\|) \\ &\le \tau_0(L(2R+M)+R). \end{aligned} \tag{2.5}$$

$$\begin{aligned} \|y_n^1 - x_{n+1}\| &\le \beta_n\|T^n y_n^2 - x_n\| + \alpha_n\|T^n y_n^1 - x_n\| \\ &\le \beta_n(\|T^n y_n^2 - \rho\| + \|x_n - \rho\|) \\ &\quad +\alpha_n(\|T^n y_n^1 - \rho\| + \|x_n - \rho\|) \\ &\le 2\tau_0(L(2R+M)+R). \end{aligned} \tag{2.6}$$

Using Lemma 1.1 and the above estimates, we have

$$\begin{aligned} \|x_{n+1} - \rho\|^2 &\le \|x_n - \rho\|^2 + 2\alpha_n < T^n y_n^1 - x_n, j(x_{n+1} - \rho) > \\ &= \|x_n - \rho\|^2 + 2\alpha_n < T^n x_{n+1} - x_{n+1}, j(x_{n+1} - \rho) > \\ &\quad + < x_{n+1} - x_n, j(x_{n+1} - \rho) > \\ &\quad + < T^n y_n^1 - T^n x_{n+1}, j(x_{n+1} - \rho) > \\ &\le \|x_n - \rho\|^2 + 2\alpha_n(k_n\|x_{n+1} - \rho\|^2 - \Phi(\|x_{n+1} - \rho\|) + \epsilon_n) \\ &\quad -2\alpha_n\|x_{n+1} - \rho\|^2 + 2\alpha_n L(\|y_n^1 - x_{n+1}\| + a_n)\|x_{n+1} - \rho\| \\ &\quad +2\alpha_n\|x_{n+1} - x_n\|\|x_{n+1} - \rho\| \\ &= \|x_n - \rho\|^2 + 2\alpha_n(k_n - 1)\|x_{n+1} - \rho\|^2 \\ &\quad -2\alpha_n\Phi(\|x_{n+1} - \rho\|) + 2\alpha_n\epsilon_n \\ &\quad +2\alpha_n L(\|y_n^1 - x_{n+1}\| + a_n)\|x_{n+1} - \rho\| \\ &\quad +2\alpha_n\|x_{n+1} - x_n\|\|x_{n+1} - \rho\| \\ &\le \|x_n - \rho\|^2 - 2\alpha_n\Phi(\tfrac{R}{2}) + 2\alpha_n\tfrac{\Phi(\frac{R}{2})}{32R^2}4R^2 + 2\alpha_n\tfrac{\Phi(\frac{R}{2})}{8} \end{aligned}$$

$$
\begin{array}{rl}
& +2\alpha_n L\frac{\Phi(\frac{R}{2})}{16R[2(L(2R+M)+R)+M]}2R[2(L(2R+M)+R)+M] \\
& +2\alpha_n \frac{\Phi(\frac{R}{2})}{16R[L(2R+M)+R]}2R[L(2R+M)+R] \\
\leq & \|x_n-\rho\|^2-\alpha_n\Phi(\frac{R}{2}) \\
\leq & R^2.
\end{array} \tag{2.7}
$$

which is a contradiction. Hence $\{x_n\}_{n=0}^{\infty}$ is a bounded sequence. So, $\{y_n^1\}$, $\{y^2\}$,..., $\{y_n^{p-1}\}$ are all bounded sequences.
Step 2. We want to prove $\|x_n-\rho\|\to 0$ as $n\to\infty$.
Since $\lim_{n\to\infty}\alpha_n, \beta_n^k=0$, $\lim_{n\to\infty}k_n=1$ and $\{x_n\}_{n=0}^{\infty}$ is bounded. From (2.5) and (2.6), we observed that

$$
\lim_{n\to\infty}\|x_{n+1}-x_n\|=0, \quad \lim_{n\to\infty}L\|y_n^1-x_{n+1}\|=0. \tag{2.8}
$$

So from (2.7), we have

(2.9)

$$
\begin{array}{rcl}
\|x_{n+1}-\rho\|^2 & \leq & \|x_n-\rho\|^2+2\alpha_n < T^n y_n^1-x_n, j(x_{n+1}-\rho) > \\
& = & \|x_n-\rho\|^2+2\alpha_n < T^n x_{n+1}-x_{n+1}, j(x_{n+1}-\rho) > \\
& & + < x_{n+1}-x_n, j(x_{n+1}-\rho) > \\
& & + < T^n y_n^1-T^n x_{n+1}, j(x_{n+1}-\rho) > \\
& \leq & \|x_n-\rho\|^2+2\alpha_n(k_n\|x_{n+1}-\rho\|^2-\Phi(\|x_{n+1}-\rho\|)+\epsilon_n) \\
& & -2\alpha_n\|x_{n+1}-\rho\|^2+2\alpha_n L(\|y_n^1-x_{n+1}\|+a_n)\|x_{n+1}-\rho\| \\
& & +2\alpha_n\|x_{n+1}-x_n\|\|x_{n+1}-\rho\| \\
& \leq & \|x_n-\rho\|^2+2\alpha_n(k_n-1)\|x_{n+1}-\rho\|^2 \\
& & -2\alpha_n\Phi(\|x_{n+1}-\rho\|)+\epsilon_n \\
& & +2\alpha_n L(\|y_n^1-x_{n+1}\|+a_n)\|x_{n+1}-\rho\| \\
& & +2\alpha_n\|x_{n+1}-x_n\|\|x_{n+1}-\rho\| \\
& = & \|x_n-\rho\|^2-2\alpha_n\Phi(\|x_{n+1}-\rho\|)+o(\alpha_n),
\end{array}
$$

where

$$
\begin{array}{l}
2\alpha_n(k_n-1)\|x_{n+1}-\rho\|^2++2\alpha_n L(\|y_n^1-x_{n+1}\|+a_n)\|x_{n+1}-\rho\| \\
+2\alpha_n\|x_{n+1}-x_n\|\|x_{n+1}-\rho\|+2\alpha_n\epsilon_n \\
=o(\alpha_n).
\end{array}
$$

By Lemma 1.2, we obtain that

$$
\lim_{n\to\infty}\|x_n-\rho\|=0.
$$

This completes the proof.

Remarks 2.1. Theorem 2.1 improves and extends the corresponding results of [1-3, 6, 13, 17, 18] in some aspects.
(i) The method of proof of Theorem 2.1 is different from the method given in Chang [1], Ofoedu [13] and Chang et al. [3] .
(ii) The control conditions (ii)-(iv) in Theorem 2.1 of Chang [1] and that of Ofoedu[13] are replaced by weaker condition $\lim_{n\to\infty}\alpha_n=0$.
(iii) Under suitable conditions, sequence $\{x_n\}_{n=0}^{\infty}$ defined by (2.1) in Theorem 2.1 can also be generalized to multi-step iterative scheme with errors.
(iv) The assumption that there exists a strictly increasing function $\Phi:[0,\infty)\to$

$[0,\infty)$ with $\Phi(0) = 0$ such that

$$< T^n x - T^n \rho, j(x-\rho) >\leq k_n\|x-\rho\|^2 - \Phi(\|x-\rho\|)$$

for all $x \in K$ used by several authors in literature is extended to a more general assumption: there exists a strictly increasing function $\Phi : [0,\infty) \to [0,\infty)$ with $\Phi(0) = 0$ such that

$$< T^n x - T^n \rho, j(x-\rho) >\leq k_n\|x-\rho\|^2 - \Phi(\|x-\rho\|) + \epsilon_n$$

for all $x \in K$

(v) The iteration sequences used in Chang [1], Ofoedu [13] Chang et al. [3] and Mogbademu [10] are extended to (1.3).

(vi) The mappings in [1, 3, 9, 13, 15] are extended to a more general class of nearly Lipschitzian mappings.

The following reveals that Theorem 2.1 is applicable.

Application 2.1. Let $X = R, K = [0,1]$ and $T : K \to K$ be a map defined by

$$Tx = \frac{x}{1+x}, \forall x \in [0,1)$$

Clearly, T is nearly uniformly Lipschitzian ($a_n = \frac{1}{2^n}$) with $F(0) = 0$.

Define $\Phi : [0,\infty) \to [0,\infty)$ by

$$\Phi(t) = \frac{t^2}{1+nt}$$

then, Φ is a strictly increasing function with $\Phi(0) = 0$. For all $x \in K, \rho \in F(T)$, we have that operator T in Theorem 2.1 satisfies

$$< T^n x - T^n \rho, j(x-\rho) >\leq k_n\|x-\rho\|^2 - \Phi(\|x-\rho\|) + \epsilon_n$$

with the sequences $k_n = 1$ and $\epsilon_n = \frac{x^2}{1+nx}$. Set $\alpha_n = \frac{1}{2+n}$ and $\beta_n^i = \frac{1}{3+n}$,$(i = 1, 2, ..., p-1)$ $\forall n \geq 0$.

Remarks 2.2. Our results enrich and develop the theory of multi-step iterative sequence introduced by Rhoades and Soltuz [16].

Taking $p = 3$ in (1.3), Theorem 2.1 leads to the following corllaries.

Corollary 2.2. Let X be a real Banach space, K be a nonempty closed convex subset of X, $T : K \to K$ be a nearly uniformly L-Lipschitzian mapping with sequence $\{a_n\}$. Let $k_n \subset [1,\infty)$ and ϵ_n be sequences with $\lim_{n\to\infty} k_n = 1$, $\lim_{n\to\infty} \epsilon_n = 0$ and $F(T) = \{\rho \in K : T\rho = \rho\}$. Let $\{\alpha_n\}_{n\geq 0}$ and $\{\beta_n^i\}_{n\geq 0}, (i = 1, 2)$ be real sequences in $[0,1]$ satisfying the following conditions: (i)$\sum_{n\geq 0} \alpha_n = \infty$ (ii) $\lim_{n\to\infty} \alpha_n, \beta_n^i = 0, (i = 1, 2)$. For arbitrary $x_0 \in K$, let $\{x_n\}_{n\geq 0}$ be iteratively defined by (1.4). If there exists a strictly increasing function
$\Phi : [0,\infty) \to [0,\infty)$ with $\Phi(0) = 0$ such that

$$< T^n x - T^n \rho, j(x-\rho) >\leq k_n\|x-\rho\|^2 - \Phi(\|x-\rho\|) + \epsilon_n$$

for all $x \in K$. Then, $\{x_n\}_{n\geq 0}$ converges strongly to $\rho \in F(T)$.

Taking $p = 2$ in (1.3), Theorem 2.1 leads to the following results.

Corollary 2.3. Let X be a real Banach space, K be a nonempty closed convex subset of X, $T : K \to K$ be a nearly uniformly L-Lipschitzian mapping with sequence $\{a_n\}$. Let $k_n \subset [1, \infty)$ and ϵ_n be sequences with $\lim_{n\to\infty} k_n = 1,\ \lim_{n\to\infty} \epsilon_n = 0$ and $F(T) = \{\rho \in K : T\rho = \rho\}$. Let $\{\alpha_n\}_{n\geq 0}$ and $\{\beta_n^i\}_{n\geq 0}, (i = 1)$ be real sequences in $[0, 1]$ satisfying the following conditions: (i)$\sum_{n\geq 0} \alpha_n = \infty$ (ii) $\lim_{n\to\infty} \alpha_n, \beta_n^i = 0, (i = 1)$. For arbitrary $x_0 \in K$, let $\{x_n\}_{n\geq 0}$ be iteratively defined by (1.5). If there exists a strictly increasing function
$\Phi : [0, \infty) \to [0, \infty)$ with $\Phi(0) = 0$ such that

$$< T^n x - T^n \rho, j(x - \rho) > \leq k_n \|x - \rho\|^2 - \Phi(\|x - \rho\|) + \epsilon_n$$

for all $x \in K$. Then, $\{x_n\}_{n\geq 0}$ converges strongly to $\rho \in F(T)$.

Corollary 2.4. Let X be a real Banach space, K be a nonempty closed convex subset of X, $T : K \to K$ be a nearly uniformly L-Lipschitzian mapping with sequence $\{a_n\}$. Let $k_n \subset [1, \infty)$ and ϵ_n be sequences with $\lim_{n\to\infty} k_n = 1,\ \lim_{n\to\infty} \epsilon_n = 0$ and $F(T) = \{\rho \in K : T\rho = \rho\}$. Let $\{\alpha_n\}_{n\geq 0}$ be a real sequence in $[0, 1]$ satisfying the following conditions: (i)$\sum_{n\geq 0} \alpha_n = \infty$ (ii) $\lim_{n\to\infty} \alpha_n = 0$. For arbitrary $x_0 \in K$, let $\{x_n\}_{n\geq 0}$ be iteratively defined by (1.6). If there exists a strictly increasing function
$\Phi : [0, \infty) \to [0, \infty)$ with $\Phi(0) = 0$ such that

$$< T^n x - T^n \rho, j(x - \rho) > \leq k_n \|x - \rho\|^2 - \Phi(\|x - \rho\|) + \epsilon_n$$

for all $x \in K$. Then, $\{x_n\}_{n\geq 0}$ converges strongly to $\rho \in F(T)$.

Acknowlegment: The author would like to express his thanks to Professor Z. Xue and Professor Chika Moore for their unique style of mentoring and supports toward the completion of this work.

References

[1] Chang, S. S. *Some results for asymptotically pseudocontractive mappings and asymptotically nonexpansive mappings*, Proc. Amer. Math. Soc., 129 , **99** (2000), 845-853.

[2] Chang, S. S., Cho, Y. J., Lee, B. S. and Kang, S. H. *Iterative approximation of fixed points and solutions for strongly accretive and strongly pseudo-contractive mappings in Banach spaces* , J. Math. Anal. Appl., 224 (1998), 165-194.

[3] Chang, S. S., Cho, Y. J., and Kim, J. K. *Some results for uniformly L-Lipschitzian mappings in Banach spaces* , Appl. Math. Lett., 22, (2009), 121-125.

[4] Goebel, K. and Kirk, W. A. *A fixed point theorem for asymptotically nonexpansive mappings* , Proc. Amer. Math. Soc., Vol. 35 , (1972), 171-174.

[5] Ishikawa, S. *Fixed points by a new iteration method*, Proc. Amer. Math. Soc., 44 (1974), 147-150.

[6] Kim, J. K., Sahu, D. R. and Nam, Y. M.*Convergence theorem for fixed points of nearly uniformly $L-$ Lipschitzian asymptotically generalized $\Phi-$ hemicontractive mappings*, Nonl. Anal., 71, **99**(2009), e2833- e2838.

[7] Mann, W.R. *Mean value methods in iteration*, Proc. Amer. Math. Soc., 4, **99** (1953), 506-610 .

[8] Moore, C. and Nnoli, B. V. C. *Iterative solution of nonlinear equations involving set-valued uniformly accretive operators* , Comput. Math. Anal. Appl., 42, (2001), 131-140 .

[9] Mogbademu, A.A. and Xue, Z. *Some convergence results for nonlinear maps in Banach spaces*, Int. J. Open Problems Compt. Math., Vol. 6, (2013), 1- 10.

[10] Mogbademu, A.A. *Convergence theorem of modified Noor iteration for nonlinear maps in Banach spaces*, J. Adv. Math. Stud., Vol. 7 (2014), nos. 1,56-64.

[11] Noor, M.A. *Three-step iterative algorithms for multi-valued quasi variational inclusions* J. Math. Anal. Appl., 225 (2001), 589-604.

[12] Noor, M.A., Kassias, T. M. and Huang, Z. *Three-step iterations for nonlinear accretive operator equations* , J. Math. Anal. Appl., 274 (2002), 59-68.

[13] Ofoedu, E.U. *Strong convergence theorem for uniformly L-Lipschitzian asymptotically pseudocontractive mapping in real Banach space* , J. Math. Anal. Appl., 321 (2006), 722-728.

[14] Olaleru, J.O. and Mogbademu, A.A. *Modified Noor iterative procedure for uniformly continuous mappings in Banach spaces*, Boletin de la Asociacion Matematica Venezolana, Vol. XVIII, No. 2 (2011), 127- 135.

[15] Rafiq, A., Acu, A. M. and Sofonea, F. *An iterative algorithm for two asymptotically pseudocontractive mappings* , Int. J. Open Problems Compt. Math., Vol. 2 (2009), 371- 382.

[16] Rhoades, B.E. and Soltuz, S.M. *The equivalence between Mann-Ishikawa iterations and multistep iteration* , Nonl. Anal.: Theory, Methods and Applications, Vol. 58 (2004), 218-228.

[17] Schu, J. *Iterative construction of fixed points of asymptotically nonexpansive mappings* , J. Math. Anal. Appl., 158 (1999), 407-413.

[18] Sahu, D. R. *Fixed points of demicontinuous nearly Lipschitzian mappings in Banach spaces* , Comment. Math. Univ. Carolin 46 (4) (2005), 653-666.

[19] Xue, Z., Rafiq, A. and Zhou, H. *On the convergence of multi-step iteration for uniformly continuous* $\Phi-$ *Hemicontractive mappings*, Abstract and Applied Analysis, Vol. 2012, Article ID 386983, (2012), 1-9.

Department of Mathematics, Faculty of Science, University of Lagos, Yaba- Lagos, Nigeria

E-mail address: amogbademu@unilag.edu.ng