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MATRICES OF GENERALIZED DUAL QUATERNIONS

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Abstract. After a brief review of some algebraic properties of a generalized dual quaternion, we investigate properties of matrix associated with a generalized dual quaternion and examine De Moivre's formula for this matrix, from which the n-th power of such a matrix can be determined. We give the relation between the powers of these matrices.

1. Introduaction

Mathematically, quaternions represent the natural extension of complex numbers, forming an associative algebra under addition and multiplication. Dual numbers and dual quaternions were introduced in the 19th century by W.K. Clifford [5], as a tool for his geometrical investigation. Study [17] and Kotel'nikov [12] systematically applied the dual number and dual vector in their studies of line geometry and kinematics and independently discovered the transfer principle.

The use of dual numbers, dual numbers matrix and dual quaternions in instantaneous spatial kinematics are investigated in [15,18]. The Euler's and De-Moivre's formulas for the complex numbers are generalized for quaternions in [4]. These formulas are also investigated for the cases of split and dual quaternions in [11,14]. Some algebraic properties of Hamilton operators are considered in [1,2] where dual quaternions have been expressed in terms of 4×4 matrices by means of these operators. Properties of these matrices have applications in mechanics, quantum physics and computer-aided geometric design [3,20]. Recently, we have derived the De-Moivre's and Euler's formulas for matrices associated with real, dual quaternions and every power of these matrices are immediately obtained [9,10].

A generalization of real and dual quaternions are also investigated by author and et al. [6,7]. Here, after a review of some algebraic properties of generalized dual quaternions, we study the Euler's and De-Moivre's formulas for generalized dual quaternions and for the matrices associated with them. Also, the n-th roots of these matrices are obtained. Finally, we give some examples for more clarification.

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2. Preliminaries

In this section, we give a brief summary algebra of generalized dual quaternions. For detailed information about this concept, we refer the reader to [7, 8].

Definition 2.1. A generalized dual quaternion Q is written as

$$Q = A_{\circ}1 + A_1 i + A_2 j + A_3 k,$$

where $A_{\cdot}, A_1, A_2$ and A_3 are dual numbers and i, j, k are quaternionic units which satisfy the equalities

$$\begin{aligned} i^2 &= -\alpha,\ j^2 = -\beta,\ k^2 = -\alpha\beta, \\ ij &= k = -ji,\ jk = \beta i = -kj, \end{aligned}$$

and

$$ki = \alpha j = -ik, \quad \alpha, \beta \in \mathbb{R}.$$

As a consequence of this definition, a generalized dual quaternion Q can also be written as;

$$Q = q + \varepsilon q^*,\ q, q^* \in H_{\alpha\beta}$$

where q and q^*, real and pure generalized dual quaternion components, respectively. A quaternion $Q = A_0 1 + A_1 i + A_2 j + A_3 k$ is pieced into two parts with scalar piece $S_Q = A_{\cdot}$ and vectorial piece $\overrightarrow{V}_Q = A_1 i + A_2 j + A_3 k$. We also write $Q = S_Q + \overrightarrow{V}_Q$. The conjugate of $Q = S_Q + \overrightarrow{V}_Q$ is then defined as $\overline{Q} = S_Q - \overrightarrow{V}_Q$. If $S_Q = 0$, then Q is called pure generalized dual quaternion, we may be called its generalized dual vector. The set of all generalized dual vectors denoted by $D^3_{\alpha\beta}$ [15].

Dual quaternionic multiplication of two dual quaternions $Q = S_Q + \overrightarrow{V}_Q$ and $P = S_P + \overrightarrow{V}_P$ is defined;

$$\begin{aligned} QP &= S_Q S_P - g(\overrightarrow{V}_Q, \overrightarrow{V}_P) + S_P \overrightarrow{V}_Q + S_Q \overrightarrow{V}_P + \overrightarrow{V}_Q \wedge \overrightarrow{V}_P \\ &= A_{\circ}B_{\circ} - (\alpha A_1 B_1 + \beta A_2 B_2 + \alpha\beta A_3 B_3) + A_{\circ}(B_1, B_2, B_3) + B_{\circ}(A_1, A_2, A_3) \\ &\quad +(\beta(A_2 B_3 - A_3 B_2), \alpha(A_3 B_1 - A_1 B_3), (A_1 B_2 - A_2 B_1)). \end{aligned}$$

Also, It could be written

$$QP = \begin{bmatrix} A_{\circ} & -\alpha A_1 & -\beta A_2 & -\alpha\beta A_3 \\ A_1 & A_{\circ} & -\beta A_3 & \beta A_2 \\ A_2 & \alpha A_3 & A_{\circ} & -\alpha\ A_1 \\ A_3 & -A_2 & A_1 & A_{\circ} \end{bmatrix} \begin{bmatrix} B_{\circ} \\ B_1 \\ B_2 \\ B_3 \end{bmatrix}.$$

So, the multiplication of dual quaternions as matrix-by-vector product. The norm of Q is defined as $N_Q = Q\overline{Q} = \overline{Q}Q = A_0^2 + \alpha A_1^2 + \beta A_2^2 + \alpha\beta A_3^2$. If $N_Q = 1$, then Q is called a unit generalized dual quaternion.The set of all generalized dual quaternions (abbreviated GDQ) are denoted by $\widetilde{H}_{\alpha\beta}$.

Theorem 2.1. *Every unit generalized dual quaternion is a screw operator* [8].

We investigate the properties of the generalized dual quaternions in two different cases.

Case 1: Let α, β be positive numbers.

Definition 2.2. Let $\widehat{S}^3_D$ be the set of all unit generalized dual quaternions and $\widehat{S}^2_D$ the set of unit generalized dual vector, that is,

$$\begin{aligned}\widehat{S}^3_D &= \left\{Q \in \widetilde{H}_{\alpha\beta} : N_Q = 1\right\} \subset \widetilde{H}_{\alpha\beta},\\ \widehat{S}^2_D &= \left\{\overrightarrow{V}_Q = (A_1, A_2, A_3) : g(\overrightarrow{V}_Q, \overrightarrow{V}_Q) = \alpha A_1^2 + \beta A_2^2 + \alpha\beta A_3^2 = 1\right\}.\end{aligned}$$

Definition 2.3. Every nonzero unit generalized dual quaternion can be written in the polar form

$$\begin{aligned}Q &= A_0 + A_1 i + A_2 j + A_3 k\\ &= \cos\phi + \overrightarrow{W}\sin\phi,\end{aligned}$$

where $\cos\phi = A_0$, $\sin\phi = \sqrt{\alpha A_1^2 + \beta A_2^2 + \alpha\beta A_3^2}$. $\phi = \varphi + \varepsilon\varphi^*$ is a dual angle and the unit generalized dual vector $\overrightarrow{W}$ is given by

$$\overrightarrow{W} = \frac{A_1 i + A_2 j + A_3 j}{\sqrt{\alpha A_1^2 + \beta A_2^2 + \alpha\beta A_3^2}} = \frac{A_1 i + A_2 j + A_3 j}{\sqrt{1 - A_0^2}},$$

with $\alpha A_1^2 + \beta A_2^2 + \alpha\beta A_3^2 \neq 0$.

Note that $\overrightarrow{W}$ is a unit generalized dual vector to which a directed line in $\mathbb{R}^3_{\alpha\beta}$ corresponds by means of the generalized E. Study map [16].

Theorem 2.2. *(De-Moivre's formula) Let* $Q = e^{\overrightarrow{W}\phi} = \cos\phi + \overrightarrow{W}\sin\phi \in \widehat{S}^3_D$, *where* $\phi = \varphi + \varepsilon\varphi^*$ *is dual angle and* $\overrightarrow{W} \in \widehat{S}^2_D$. *Then for every integer* n;

$$Q^n = \cos n\phi + \overrightarrow{W}\sin n\phi.$$

Proof. The proof follows immediately from the induction (see [13]). □

Every generalized dual qauetrnion can be separated into two cases:
1) Generalized dual quaternions with dual angles ($\phi = \varphi + \varepsilon\varphi^*$); *i.e.*

$$Q = \sqrt{N_Q}(\cos\phi + \overrightarrow{W}\sin\phi).$$

2) Generalized dual quaternions with real angles ($\phi = \varphi$, $\varphi^* = 0$); *i.e.*

$$Q = \sqrt{N_Q}(\cos\varphi + \overrightarrow{W}\sin\varphi).$$

Theorem 2.3. *Let* $Q = \cos\varphi + \overrightarrow{W}\sin\varphi \in \widehat{S}^3_D$.*De-Moivre's formula implies that there are uncountably many unit dual generalized quaternions* Q *satisfying* $Q^n = 1$ *for* $n > 2$ [13].

Case 2: Let α be a positive and β a negative numbers.

In this case, for a generalized dual quaternion $Q = A_0 + A_1 i + A_2 j + A_3 k$, we can consider three different subcases.

Subcase (i): The norm of generalized dual quaternion is negative, *i.e.*

$$N_Q = A_0^2 + \alpha A_1^2 + \beta A_2^2 + \alpha\beta A_3^2 < 0,$$

since $0 < A_0^2 < -\alpha A_1^2 - \beta A_2^2 - \alpha\beta A_3^2$ thus $\alpha A_1^2 + \beta A_2^2 + \alpha\beta A_3^2 < 0$. In this case, the polar form of Q is defined as

$$Q = r(\sinh\Psi + \overrightarrow{W}\cosh\Psi)$$

where we assume

$$\begin{aligned} r &= \sqrt{|N_Q|} = \sqrt{|A_\circ^2 + \alpha A_1^2 + \beta A_2^2 + \alpha\beta A_3^2|}, \\ \sinh\Psi &= \frac{A_0}{\sqrt{|N_Q|}}, \quad \cosh\Psi = \frac{\sqrt{-\alpha A_1^2 - \beta A_2^2 - \alpha\beta A_3^2}}{\sqrt{|N_Q|}}. \end{aligned}$$

The unit dual vector $\overrightarrow{W}$ (axis of quaternion) is defined as

$$\overrightarrow{W} = (w_1, w_2, w_3) = \frac{1}{\sqrt{-\alpha A_1^2 - \beta A_2^2 - \alpha\beta A_3^2}}(A_1, A_2, A_3).$$

Theorem 2.4. *(De-Moivre's formula) Let $Q = \sinh\Psi + \overrightarrow{W}\cosh\Psi$ be a unit generalized dual quaternion with $N_Q < 0$. Then for every integer n;*

$$Q^n = \sinh n\Psi + \overrightarrow{W}\cosh n\Psi.$$

Proof. The proof follows immediately from the induction [13]. □

Subcase (ii): The norm of generalized dual quaternion is positive and the norm of its vector part to be negative, *i.e.*

$$N_Q > 0, \quad N_{\overrightarrow{V}_Q} = \alpha A_1^2 + \beta A_2^2 + \alpha\beta A_3^2 < 0,$$

In this case, the polar form of Q is defined as

$$Q = r(\cosh\Phi + \overrightarrow{W}\sinh\Phi)$$

where we assume

$$\begin{aligned} r &= \sqrt{N_Q} = \sqrt{A_0^2 + \alpha A_1^2 + \beta A_2^2 + \alpha\beta A_3^2}, \\ \cosh\Phi &= \frac{A_0}{\sqrt{N_Q}}, \quad \sinh\Phi = \frac{\sqrt{-\alpha A_1^2 - \beta A_2^2 - \alpha\beta A_3^2}}{\sqrt{N_Q}}. \end{aligned}$$

The unit dual vector $\overrightarrow{W}$ (axis of quaternion) is defined as

$$\overrightarrow{W} = (w_1, w_2, w_3) = \frac{1}{\sqrt{-\alpha A_1^2 - \beta A_2^2 - \alpha\beta A_3^2}}(A_1, A_2, A_3).$$

Theorem 2.5. *Let* $Q = \cosh\Phi + \overrightarrow{W}\sinh\Phi$ *be a unit generalized dual quaternion with* $N_Q > 0$ *and* $N_{\overrightarrow{V}_Q} < 0$. *Then for every integer* n;

$$Q^n = \cosh n\Phi + \overrightarrow{W}\sinh n\Phi.$$

Proof. The proof follows immediately from the induction [13]. □

Subcase (iii): The norm of generalized dual quaternion is positive and the norm of its vector part to be positive, *i.e.*

$$N_Q > 0, \quad N_{\overrightarrow{V}_Q} = \alpha A_1^2 + \beta A_2^2 + \alpha\beta A_3^2 > 0,$$

In this case, the polar form of Q is defined as

$$Q = r(\cos\Theta + \overrightarrow{W}\sin\Theta)$$

where we assume

$$\begin{aligned} r &= \sqrt{N_Q} = \sqrt{A_\circ^2 + \alpha A_1^2 + \beta A_2^2 + \alpha\beta A_3^2}, \\ \cos\Theta &= \frac{A_0}{\sqrt{N_Q}}, \quad \sin\Theta = \frac{\sqrt{\alpha A_1^2 + \beta A_2^2 + \alpha\beta A_3^2}}{\sqrt{N_Q}}. \end{aligned}$$

The unit dual vector $\overrightarrow{W}$ (axis of quaternion) is defined as

$$\overrightarrow{W} = (w_1, w_2, w_3) = \frac{1}{\sqrt{\alpha A_1^2 + \beta A_2^2 + \alpha\beta A_3^2}}(A_1, A_2, A_3).$$

Theorem 2.6. *Let* $Q = \cos\Theta + \overrightarrow{W}\sin\Phi$ *be a unit generalized dual quaternion with* $N_Q > 0$ *and* $N_{\overrightarrow{V}_Q} > 0$. *Then for every integer* n;

$$Q^n = \cos n\Theta + \overrightarrow{W}\sin n\Theta.$$

Proof. The proof follows immediately from the induction. □

2.1. 4×4 Dual Matrix representation of GDQ.

In this section, we introduce the $\mathbb{R}$-linear transformations representing left multiplication in $\widetilde{H}_{\alpha\beta}$ and look for also the De-Moiver's formula for corresponding matrix representation. Let Q be a generalized dual quaternion, then the linear map $\overset{+}{h}_Q : \widetilde{H}_{\alpha\beta} \to \widetilde{H}_{\alpha\beta}$ defined as follows;

$$\overset{+}{h}_Q(P) = QP, \quad P \in \widetilde{H}_{\alpha\beta}.$$

The Hamilton's operator $\overset{+}{H}$, could be represented as the matrix

$$\overset{+}{H}(Q) = \begin{bmatrix} A_0 & -\alpha A_1 & -\beta A_2 & -\alpha\beta A_3 \\ A_1 & A_0 & -\beta A_3 & \beta A_2 \\ A_2 & \alpha A_3 & A_0 & -\alpha A_1 \\ A_3 & -A_2 & A_1 & A_0 \end{bmatrix}.$$

Theorem 2.7. *If Q and P are two generalized dual quaternions, λ is a real number, then the following identities hold;*

i. $Q = P \Leftrightarrow \overset{+}{H}(Q) = \overset{+}{H}(P)$

ii. $\overset{+}{H}(Q+P) = \overset{+}{H}(Q) + \overset{+}{H}(P)$

iii. $\overset{+}{H}(\lambda Q) = \lambda \overset{+}{H}(Q)$

iv. $\overset{+}{H}(QP) = \overset{+}{H}(Q)\overset{+}{H}(P)$

v. $\overset{+}{H}(Q^{-1}) = [\overset{+}{H}(Q)]^{-1}$, $N_Q \neq 0$.

vi. $\overset{+}{H}(\overline{Q}) = [\overset{+}{H}(Q)]^{T}$

vii. $\det[\overset{+}{H}(Q)] = (N_Q)^2$

viii. $tr[\overset{+}{H}(Q)] = 4A_\circ$

Proof. The proof can be found in [7]. □

Following the usual matrix nomenclature, a matrix $\hat{A}$ is called a dual quasi-orthogonal matrix if $\hat{A}^T \epsilon \hat{A} = A\epsilon$, where A is a dual number and ϵ is a 4×4 diagonal matrix. A matrix $\hat{A}$ is called dual quasi-orthonormal matrix if $A = 1$ [8].

Theorem 2.8. *Matrices generated by operators by $\overset{+}{H}$ is a dual quasi-orthogonal matrices; i.e.* $[\overset{+}{H}(Q)]^T \epsilon \overset{+}{H}(Q) = N_Q \epsilon$ *where*

$$\epsilon = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \alpha & 0 & 0 \\ 0 & 0 & \beta & 0 \\ 0 & 0 & 0 & \alpha\beta \end{bmatrix}.$$

Also, $\overset{+}{H}(Q)$ is a dual quasi-orthonormal matrices if Q is a unit generalized dual quaternion [8].

Theorem 2.9. *The ϕ map defined as*

$$\phi : (\widetilde{H}_{\alpha\beta}, +, .) \rightarrow (M_{(4,D)}, \oplus, \otimes)$$

$$\phi(A_0 + A_1 i + A_2 j + A_3 k) \rightarrow \begin{bmatrix} A_0 & -\alpha A_1 & -\beta A_2 & -\alpha\beta A_3 \\ A_1 & A_0 & -\beta A_3 & \beta A_2 \\ A_2 & \alpha A_3 & A_0 & -\alpha A_1 \\ A_3 & -A_2 & A_1 & A_0 \end{bmatrix},$$

is an isomorphism of algebras.

Proof. We first demonstrate its homomorphic properties. If $Q = A_0 1 + A_1 i + A_2 j + A_3 k$ and $P = B_0 1 + B_1 i + B_2 j + B_3 k$ are any two GDQ, then

$$
\begin{aligned}
\phi\{Q+P\} &= \phi\{(A_0+B_0)+(A_1+B_1)i+(A_2+B_2)j+(A_3+B_3)k\} \\
&= \begin{bmatrix} A_0+B_0 & -\alpha(A_1+B_1) & -\beta(A_2+B_2) & -\alpha\beta(A_3+B_3) \\ (A_1+B_1) & A_0+B_0 & -\beta(A_3+B_3) & \beta(A_2+B_2) \\ (A_2+B_2) & \alpha(A_3+B_3) & A_0+B_0 & -\alpha(A_1+B_1) \\ (A_3+B_3) & -(A_2+B_2) & (A_1+B_1) & A_0+B_0 \end{bmatrix} \\
&= \begin{bmatrix} A_0 & -\alpha A_1 & -\beta A_2 & -\alpha\beta A_3 \\ A_1 & A_0 & -\beta A_3 & \beta A_2 \\ A_2 & \alpha A_3 & A_0 & -\alpha A_1 \\ A_3 & -A_2 & A_1 & A_0 \end{bmatrix} + \begin{bmatrix} B_0 & -\alpha B_1 & -\beta B_2 & -\alpha\beta B_3 \\ B_1 & B_0 & -\beta B_3 & \beta B_2 \\ B_2 & \alpha B_3 & B_0 & -\alpha B_1 \\ B_3 & -B_2 & B_1 & B_0 \end{bmatrix} \\
&= \phi\{Q\} \oplus \phi\{P\}, \\
\phi\{QP\} &= \phi\{A_0B_0-(\alpha A_1B_1+\beta A_2B_2+\alpha\beta A_3B_3)+A_\circ(B_1,B_2,B_3)+B_\circ(A_1,A_2,A_3) \\
&\quad (\beta(A_2B_3-A_3B_2),\alpha(A_3B_1-A_1B_3),(A_1B_2-A_2B_1))\} \\
&= \begin{bmatrix} A_0 & -\alpha A_1 & -\beta A_2 & -\alpha\beta A_3 \\ A_1 & A_0 & -\beta A_3 & \beta A_2 \\ A_2 & \alpha A_3 & A_0 & -\alpha A_1 \\ A_3 & -A_2 & A_1 & A_0 \end{bmatrix} \otimes \begin{bmatrix} B_0 & -\alpha B_1 & -\beta B_2 & -\alpha\beta B_3 \\ B_1 & B_0 & -\beta B_3 & \beta B_2 \\ B_2 & \alpha B_3 & B_0 & -\alpha B_1 \\ B_3 & -B_2 & B_1 & B_0 \end{bmatrix}
\end{aligned}
$$

□

We can express the matrix $\overset{+}{H}(Q)$ in polar form. Let Q be a unit generalized dual quaternion and $\alpha,\beta>0$. Since

$$
\begin{aligned}
Q &= A_0+A_1e_1+A_2e_2+A_3e_3 \\
&= \cos\phi+\overrightarrow{W}\sin\phi \\
&= \cos\phi+(w_1,w_2,w_3)\sin\phi
\end{aligned}
$$

so we have

$$
\begin{bmatrix} A_0 & -\alpha A_1 & -\beta A_2 & -\alpha\beta A_3 \\ A_1 & A_0 & -\beta A_3 & \beta A_2 \\ A_2 & \alpha A_3 & A_0 & -\alpha A_1 \\ A_3 & -A_2 & A_1 & A_0 \end{bmatrix} = \begin{bmatrix} \cos\phi & -\alpha w_1\sin\phi & -\beta w_2\sin\phi & -\alpha\beta w_3\sin\phi \\ w_1\sin\phi & \cos\phi & -\beta w_3\sin\phi & \beta w_2\sin\phi \\ w_2\sin\phi & \alpha w_3\sin\phi & \cos\phi & -\alpha w_1\sin\phi \\ w_3\sin\phi & -w_2\sin\phi & w_1\sin\phi & \cos\phi \end{bmatrix}.
$$

Theorem 2.10. *(De-Moivre's formula) For an integer n and matrix*

$$
A = \begin{bmatrix} \cos\phi & -\alpha w_1\sin\phi & -\beta w_2\sin\phi & -\alpha\beta w_3\sin\phi \\ w_1\sin\phi & \cos\phi & -\beta w_3\sin\phi & \beta w_2\sin\phi \\ w_2\sin\phi & \alpha w_3\sin\phi & \cos\phi & -\alpha w_1\sin\phi \\ w_3\sin\phi & -w_2\sin\phi & w_1\sin\phi & \cos\phi \end{bmatrix} \quad (1.1)
$$

the n-th power of the matrix A reads

$$
A^n = \begin{bmatrix} \cos n\phi & -\alpha w_1\sin n\phi & -\beta w_2\sin n\phi & -\alpha\beta w_3\sin n\phi \\ w_1\sin n\phi & \cos n\phi & -\beta w_3\sin n\phi & \beta w_2\sin n\phi \\ w_2\sin n\phi & \alpha w_3\sin n\phi & \cos n\phi & -\alpha w_1\sin n\phi \\ w_3\sin n\phi & -w_2\sin n\phi & w_1\sin n\phi & \cos n\phi \end{bmatrix}.
$$

Proof. The proof follows immediately from the induction. □

Special cases:

1) If ϕ, w_1, w_2 and w_3 be real numbers, then Theorem 3.4 holds for real quaternions (see [10]).
2) If $\alpha = \beta = 1$, then Theorem 3.4 holds for dual quaternions (see [9]).

Example 2.1. Let $Q = \frac{1}{\sqrt{2}} + \frac{1}{2}(\frac{1}{\sqrt{\alpha}}, \frac{1}{\sqrt{\beta}}, \varepsilon)$ be a unit generalized dual quaternion. The matrix corresponding to this quaternion is

$$
\begin{aligned}
A &= \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{\sqrt{\alpha}}{2} & -\frac{\sqrt{\beta}}{2} & -\frac{\alpha\beta\varepsilon}{2} \\ \frac{1}{\sqrt{2\alpha}} & \frac{1}{\sqrt{2}} & -\frac{\beta\varepsilon}{2} & \frac{\sqrt{\beta}}{2} \\ \frac{1}{\sqrt{2\beta}} & \frac{\alpha\varepsilon}{2} & \frac{1}{\sqrt{2}} & -\frac{\sqrt{\alpha}}{2} \\ \frac{\varepsilon}{2} & -\frac{1}{\sqrt{2\beta}} & \frac{1}{\sqrt{2\alpha}} & \frac{1}{\sqrt{2}} \end{bmatrix}, \\
&= \begin{bmatrix} \cos\frac{\pi}{4} & -\alpha w_1 \sin\frac{\pi}{4} & -\beta w_2 \sin\frac{\pi}{4} & -\alpha\beta w_3 \sin\frac{\pi}{4} \\ w_1 \sin\frac{\pi}{4} & \cos\frac{\pi}{4} & -\beta w_3 \sin\frac{\pi}{4} & \beta w_2 \sin\frac{\pi}{4} \\ w_2 \sin\frac{\pi}{4} & \alpha w_3 \sin\frac{\pi}{4} & \cos\frac{\pi}{4} & -\alpha w_1 \sin\frac{\pi}{4} \\ w_3 \sin\frac{\pi}{4} & -w_2 \sin\frac{\pi}{4} & w_1 \sin\frac{\pi}{4} & \cos\frac{\pi}{4} \end{bmatrix}
\end{aligned}
$$

every powers of this matix are found to be with the aid of Theorem *3.4* , for example, 6−th and 15-th power is

$$
\begin{aligned}
A^6 &= \begin{bmatrix} 0 & \sqrt{\frac{\alpha}{2}} & \sqrt{\frac{\beta}{2}} & \frac{1}{\sqrt{2}}\varepsilon\alpha\beta \\ -\frac{1}{\sqrt{2\alpha}} & 0 & \frac{1}{\sqrt{2}}\varepsilon\beta & -\sqrt{\frac{\beta}{2}} \\ -\frac{1}{\sqrt{2\beta}} & -\frac{1}{\sqrt{2}}\varepsilon\alpha & 0 & \sqrt{\frac{\alpha}{2}} \\ -\frac{1}{\sqrt{2}}\varepsilon & \frac{1}{\sqrt{2\beta}} & -\frac{1}{\sqrt{2\alpha}} & 0 \end{bmatrix}, \\
A^{15} &= \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{\sqrt{\alpha}}{2} & \frac{\sqrt{\beta}}{2} & \alpha\beta\frac{\varepsilon}{2} \\ -\frac{1}{\sqrt{2\alpha}} & \frac{1}{\sqrt{2}} & \beta\frac{\varepsilon}{2} & -\frac{\sqrt{\beta}}{2} \\ \frac{1}{2\sqrt{\beta}} & -\alpha\frac{\varepsilon}{2} & \frac{1}{\sqrt{2}} & \frac{\sqrt{\alpha}}{2} \\ -\frac{\varepsilon}{2} & \frac{1}{2\sqrt{\beta}} & -\frac{1}{\sqrt{2\alpha}} & \frac{1}{\sqrt{2}} \end{bmatrix}.
\end{aligned}
$$

2.2. **Euler's Formula for Matrices of GDQ.**

Definition 2.4. Let A be a dual matrix. We choose

$$
A = \begin{bmatrix} 0 & -\alpha w_1 & -\beta w_2 & -\alpha\beta w_3 \\ w_1 & 0 & -\beta w_3 & \beta w_2 \\ w_2 & \alpha w_3 & 0 & -\alpha w_1 \\ w_3 & -w_2 & w_1 & 0 \end{bmatrix}
$$

then one immediately finds $A^2 = -I_4$. We have a netural generalization of Euler's formula for matrix A;

$$\begin{aligned}
e^{A\phi} &= I_4 + A\phi + \frac{(A\phi)^2}{2!} + \frac{(A\phi)^3}{3!} + \frac{(A\phi)^4}{4!} + ... \\
&= I_4(1 - \frac{\phi^2}{2!} + \frac{\phi^4}{4!} -)... + A(\phi - \frac{\phi^3}{3!} + \frac{\phi^5}{5!} - ...) \\
&= \cos\phi + A\sin\phi, \\
&= \begin{bmatrix} \cos\phi & -\alpha w_1 \sin\phi & -\beta w_2 \sin\phi & -\alpha\beta w_3 \sin\phi \\ w_1 \sin\phi & \cos\phi & -\beta w_3 \sin\phi & \beta w_2 \sin\phi \\ w_2 \sin\phi & \alpha w_3 \sin\phi & \cos\phi & -\alpha w_1 \sin\phi \\ w_3 \sin\phi & -w_2 \sin\phi & w_1 \sin\phi & \cos\phi \end{bmatrix}.
\end{aligned}$$

2.3. *n*-th Roots of Matrices of GDQ.

Let Q be a unit generalized dual quaternion with real angle, *i.e.* $\phi = \varphi$ and $\varphi^* = 0$. The matrix associated with the quaternion Q is of the form (*1.1*). In a more general case, we assume for the matrix of (*1.1*)

$$A = \begin{bmatrix} \cos(\varphi + 2k\pi) & -\alpha w_1 \sin(\varphi + 2k\pi) & -\beta w_2 \sin(\varphi + 2k\pi) & -\alpha\beta w_3 \sin(\varphi + 2k\pi) \\ w_1 \sin(\varphi + 2k\pi) & \cos(\varphi + 2k\pi) & -\beta w_3 \sin(\varphi + 2k\pi) & \beta w_2 \sin(\varphi + 2k\pi) \\ w_2 \sin(\varphi + 2k\pi) & \alpha w_3 \sin(\varphi + 2k\pi) & \cos(\varphi + 2k\pi) & -\alpha w_1 \sin(\varphi + 2k\pi) \\ w_3 \sin(\varphi + 2k\pi) & -w_2 \sin(\varphi + 2k\pi) & w_1 \sin(\varphi + 2k\pi) & \cos(\varphi + 2k\pi) \end{bmatrix},$$

where $k \in \mathbb{Z}$.

The equation $X^n = A$ has n roots. Thus

$$A_k^{\frac{1}{n}} = \begin{bmatrix} \cos(\frac{\varphi+2k\pi}{n}) & -\alpha w_1 \sin(\frac{\varphi+2k\pi}{n}) & -\beta w_2 \sin(\frac{\varphi+2k\pi}{n}) & -\alpha\beta w_3 \sin(\frac{\varphi+2k\pi}{n}) \\ w_1 \sin(\frac{\varphi+2k\pi}{n}) & \cos(\frac{\varphi+2k\pi}{n}) & -\beta w_3 \sin(\frac{\varphi+2k\pi}{n}) & \beta w_2 \sin(\frac{\varphi+2k\pi}{n}) \\ w_2 \sin(\frac{\varphi+2k\pi}{n}) & \alpha w_3 \sin(\frac{\varphi+2k\pi}{n}) & \cos(\frac{\varphi+2k\pi}{n}) & -\alpha w_1 \sin(\frac{\varphi+2k\pi}{n}) \\ w_3 \sin(\frac{\varphi+2k\pi}{n}) & -w_2 \sin(\frac{\varphi+2k\pi}{n}) & w_1 \sin(\frac{\varphi+2k\pi}{n}) & \cos(\frac{\varphi+2k\pi}{n}) \end{bmatrix}.$$

For $k = 0$, the first root is

$$A_0^{\frac{1}{n}} = \begin{bmatrix} \cos(\frac{\varphi}{n}) & -\alpha w_1 \sin(\frac{\varphi}{n}) & -\beta w_2 \sin(\frac{\varphi}{n}) & -\alpha\beta w_3 \sin(\frac{\varphi}{n}) \\ w_1 \sin(\frac{\varphi}{n}) & \cos(\frac{\varphi}{n}) & -\beta w_3 \sin(\frac{\varphi}{n}) & \beta w_2 \sin(\frac{\varphi}{n}) \\ w_2 \sin(\frac{\varphi}{n}) & \alpha w_3 \sin(\frac{\varphi}{n}) & \cos(\frac{\varphi}{n}) & -\alpha w_1 \sin(\frac{\theta}{n}) \\ w_3 \sin(\frac{\varphi}{n}) & -w_2 \sin(\frac{\varphi}{n}) & w_1 \sin(\frac{\varphi}{n}) & \cos(\frac{\varphi}{n}) \end{bmatrix},$$

for $k = 1$, the second root is

$$A_1^{\frac{1}{n}} = \begin{bmatrix} \cos(\frac{\varphi+2\pi}{n}) & -\alpha w_1 \sin(\frac{\varphi+2\pi}{n}) & -\beta w_2 \sin(\frac{\varphi+2\pi}{n}) & -\alpha\beta w_3 \sin(\frac{\varphi+2\pi}{n}) \\ w_1 \sin(\frac{\varphi+2\pi}{n}) & \cos(\frac{\varphi+2\pi}{n}) & -\beta w_3 \sin(\frac{\varphi+2\pi}{n}) & \beta w_2 \sin(\frac{\varphi+2\pi}{n}) \\ w_2 \sin(\frac{\varphi+2\pi}{n}) & \alpha w_3 \sin(\frac{\varphi+2\pi}{n}) & \cos(\frac{\varphi+2\pi}{n}) & -\alpha w_1 \sin(\frac{\varphi+2\pi}{n}) \\ w_3 \sin(\frac{\varphi+2\pi}{n}) & -w_2 \sin(\frac{\varphi+2\pi}{n}) & w_1 \sin(\frac{\varphi+2\pi}{n}) & \cos(\frac{\varphi+2\pi}{n}) \end{bmatrix}.$$

Similarly, for $k = n - 1$, we obtain the n-th root.

2.4. **Relation Between Power of Matrices.** The relations between the powers of matrices associated with a generalized dual quaternion can be realized by the following Theorem.

Theorem 2.11. *Q be a unit generalized dual quaternion with the polar form $Q = \cos\varphi + \overrightarrow{W}\sin\varphi$. If $m = \frac{2\pi}{\varphi} \in \mathbb{Z}^+ - \{1\}$, then $n \equiv p(mod\ m)$ is possible if and only if $Q^n = Q^p$.*

Proof. Let $n \equiv p(mod\ m)$. Then we have $n = a.m + p$, where $a \in \mathbb{Z}$.

$$\begin{aligned}
Q^n &= \cos n\varphi + \overrightarrow{W}\sin n\varphi \\
&= \cos(am+p)\varphi + \overrightarrow{W}\sin(am+p)\varphi \\
&= \cos(a\frac{2\pi}{\varphi}+p)\varphi + \overrightarrow{W}\sin(a\frac{2\pi}{\varphi}+p)\varphi \\
&= \cos(p\varphi + a2\pi) + \overrightarrow{W}\sin(p\varphi + a2\pi) \\
&= \cos(p\varphi) + \overrightarrow{W}\sin(p\varphi) \\
&= Q^p.
\end{aligned}$$

Now suppose $Q^n = \cos n\varphi + \overrightarrow{W}\sin n\varphi$ and $Q^p = \cos p\varphi + \overrightarrow{W}\sin p\varphi$. Since $Q^n = Q^p$, we have $\cos n\varphi = \cos p\varphi$ and $\sin n\varphi = \sin p\varphi$, which means $n\varphi = p\varphi + 2\pi a$, $a \in \mathbb{Z}$. Thus $n = a\frac{2\pi}{\varphi} + p$, $n \equiv p(mod\ m)$.

□

Example 2.2. Let $Q = \frac{1}{\sqrt{2}} + \frac{1}{2}(\frac{1}{\sqrt{\alpha}}, \frac{1}{\sqrt{\beta}}, \varepsilon)$ be a unit generalized dual quaternion. From the theorem 6.1, $m = \frac{2\pi}{\pi/4} = 8$, we have

$$\begin{aligned}
Q &= Q^9 = Q^{17} = ... \\
Q^2 &= Q^{10} = Q^{18} = ... \\
Q^3 &= Q^{11} = Q^{19} = ... \\
Q^4 &= Q^{12} = Q^{20} = ... = -1 \\
&\quad ... \\
Q^8 &= Q^{16} = Q^{24} = ... = 1.
\end{aligned}$$

Theorem 2.12. *Q be a unit dual quaternion with the polar form $Q = \cos\varphi + \overrightarrow{W}\sin\varphi$. Let $m = \frac{2\pi}{\varphi} \in \mathbb{Z}^+ - \{1\}$ and the matrix A corresponds to Q. Then $n \equiv p(mod\ m)$ is possible if and only if $A^n = A^p$.*

Proof. Proof is same as above. □

Example 2.3. Let $Q = -\frac{1}{2} + (\varepsilon, \frac{1}{\sqrt{2\beta}}, \frac{1}{2\sqrt{\alpha\beta}}) = \cos\frac{2\pi}{3} + \overrightarrow{W}\sin\frac{2\pi}{3}$ be a unit generalized dual quaternion. The matrix corresponding to this dual quaternion is

$$A = \begin{bmatrix}
-\frac{1}{2} & -\alpha\varepsilon & -\sqrt{\frac{\beta}{2}} & -\frac{1}{2}\sqrt{\alpha\beta} \\
\varepsilon & -\frac{1}{2} & -\frac{1}{2}\sqrt{\frac{\beta}{\alpha}} & \frac{\sqrt{\beta}}{2} \\
\frac{1}{\sqrt{2\beta}} & \frac{1}{2}\sqrt{\frac{\alpha}{\beta}} & -\frac{1}{2} & -\alpha\varepsilon \\
\frac{1}{2\sqrt{\alpha\beta}} & -\frac{1}{\sqrt{2\beta}} & \varepsilon & -\frac{1}{2}
\end{bmatrix},$$

From the Theorem 6.2, $m = \frac{2\pi}{2\pi/3} = 3$, we have

$$\begin{aligned} A &= A^4 = A^7 = ... \\ A^2 &= A^5 = A^8 = ... \\ A^3 &= A^6 = A^9 = ... = I_4 \end{aligned}$$

The square roots of the matrix A can be achieved too;

$$A_k^{\frac{1}{2}} = \begin{bmatrix} \cos\left(\frac{2k\pi+120^{\cdot}}{2}\right) & -\alpha w_1 \sin\left(\frac{2k\pi+120^{\cdot}}{2}\right) & -\beta w_2 \sin\left(\frac{2k\pi+120^{\cdot}}{2}\right) & -\alpha\beta w_3 \sin\left(\frac{2k\pi+120^{\cdot}}{2}\right) \\ w_1 \sin\left(\frac{2k\pi+120^{\cdot}}{2}\right) & \cos\left(\frac{2k\pi+120^{\cdot}}{2}\right) & -\beta w_3 \sin\left(\frac{2k\pi+120^{\cdot}}{2}\right) & \beta w_2 \sin\left(\frac{2k\pi+120^{\cdot}}{2}\right) \\ w_2 \sin\left(\frac{2k\pi+120^{\cdot}}{2}\right) & \alpha w_3 \sin\left(\frac{2k\pi+120^{\cdot}}{2}\right) & \cos\left(\frac{2k\pi+120^{\cdot}}{2}\right) & -\alpha w_1 \sin\left(\frac{2k\pi+120^{\cdot}}{2}\right) \\ w_3 \sin\left(\frac{2k\pi+120^{\cdot}}{2}\right) & -w_2 \sin\left(\frac{2k\pi+120^{\cdot}}{2}\right) & w_1 \sin\left(\frac{2k\pi+120^{\cdot}}{2}\right) & \cos\left(\frac{2k\pi+120^{\cdot}}{2}\right) \end{bmatrix}$$

The first root for $k = 0$ reads

$$A_0^{\frac{1}{2}} = \begin{bmatrix} \frac{1}{2} & -\alpha\varepsilon & -\sqrt{\frac{\beta}{2}} & -\frac{1}{2}\sqrt{\alpha\beta} \\ \varepsilon & \frac{1}{2} & -\frac{1}{2}\sqrt{\frac{\beta}{\alpha}} & \frac{\sqrt{\beta}}{2} \\ \frac{1}{\sqrt{2\beta}} & \frac{1}{2}\sqrt{\frac{\alpha}{\beta}} & \frac{1}{2} & -\alpha\varepsilon \\ \frac{1}{2\sqrt{\alpha\beta}} & -\frac{1}{\sqrt{2\beta}} & \varepsilon & \frac{1}{2} \end{bmatrix}$$

and the second one for $k = 1$ becomes

$$A_1^{\frac{1}{2}} = \begin{bmatrix} -\frac{1}{2} & \alpha\varepsilon & \sqrt{\frac{\beta}{2}} & \frac{1}{2}\sqrt{\alpha\beta} \\ -\varepsilon & -\frac{1}{2} & \frac{1}{2}\sqrt{\frac{\beta}{\alpha}} & -\frac{\sqrt{\beta}}{2} \\ -\frac{1}{\sqrt{2\beta}} & -\frac{1}{2}\sqrt{\frac{\alpha}{\beta}} & -\frac{1}{2} & \alpha\varepsilon \\ -\frac{1}{2\sqrt{\alpha\beta}} & \frac{1}{\sqrt{2\beta}} & -\varepsilon & -\frac{1}{2} \end{bmatrix}.$$

Also, it is easy to see that $A_0^{\frac{1}{2}} + A_1^{\frac{1}{2}} = 0$.

Remark 2.1. Let α be a positive number and β be a negative number, the Theorem 3.4 holds.

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