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FIBONACCI AND LUCAS SEQUENCES AT NEGATIVE INDICES

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Abstract. This study investigate the Fibonacci and Lucas sequences at negative indices. In this paper we give the formulas of $F_{-(nk+r)}$ and $L_{-(nk+r)}$ depending on whether the indices are odd or even. For this purpose we consider a special matrix and we give various combinatorial identities related with the Fibonacci and Lucas sequences by using the matrix method. Some of the resulting identities are well known identities in the literature, but some of these are new.

1. Introduction

Horadam sequences $\{W_n\}_{n\in\mathbb{N}}$, $W_n = W_n(a, b; p, q)$, are defined as follows,

$$W_n = pW_{n-1} - qW_{n-2};\ \ W_0 = a,\ \ W_1 = b.$$

Where a, b, p and q are arbitrary complex numbers, with $q \geq 0$. The sequences $\{W_n\}_{n\in\mathbb{N}}$ have several famous number sequences as special cases. For example, E. Lucas investigated the special cases $\{U_n\}$ and $\{V_n\}$ of this sequence such as

$$U_n = W_n(0, 1; p, q),\ \ V_n = W_n(2, p; p, q).$$

Further and detailed knowledge can be found in the references [2, 5, 6, 9]. If α and β are assumed distinct roots of the characteristic equation $\lambda^2 - p\lambda + q = 0$, then the terms W_n of this sequence can be computed by the Binet formula.

$$W_n = (A\alpha^n - B\beta^n)/(\alpha - \beta),$$

where $A = b - a\beta$, $B = b - a\alpha$. Note that the Binet formula of the Horadam sequence at negative indices can be given as

$$W_{-n} = \frac{pW_{-n+1} - W_{-n+2}}{q}.$$

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From [1, 3, 4, 10, 11, 12] we know that the matrix M, $M = \begin{pmatrix} p & -q \\ 1 & 0 \end{pmatrix}$, is reduces to the Fibonacci $Q-$ matrix if $p = 1$ and $q = -1$.

In [1, 3], using the powers of the matrix $M = \begin{pmatrix} p & -q \\ 1 & 0 \end{pmatrix}$, we derived various combinatorial identities involving the terms of the sequence $\{W_n\}_{n\in\mathbb{N}}$. Furthermore, we consider a special matrix A as follows[3],

$$A = \begin{pmatrix} p^2 - 2q & p \\ -qp & -2q \end{pmatrix}.$$

And for $n \geq 1$, we calculated the power of it as follows.

$$A^n = (p^2 - 4q)^{(n-1)/2} \begin{pmatrix} V_{n+1} & V_n \\ -qV_n & -qV_{n-1} \end{pmatrix}; \ \textit{if } n \textit{ is odd number},$$

and

$$A^n = (p^2 - 4q)^{n/2} \begin{pmatrix} U_{n+1} & U_n \\ -qU_n & -qU_{n-1} \end{pmatrix}; \ \textit{if } n \textit{ is even number}.$$

In [7, 8, 9] for any matrix $B = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, the authors investigated the nth power of this matrix B;

$$B^n = \begin{pmatrix} y_n - dy_{n-1} & by_{n-1} \\ cy_{n-1} & y_n - ay_{n-1} \end{pmatrix},$$

where

$$y_n = \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} \begin{pmatrix} n-i \\ i \end{pmatrix} T^{n-2i}(-D)^i.$$

Note that T and D are the trace and determinant of the matrix B, respectively.

In this study, we derive various identities involving the generalized Fibonacci and Lucas sequences at negative index. For this purpose, we consider a new matrix E as follows,

$$E = \begin{pmatrix} 3 & -1 \\ -1 & 2 \end{pmatrix}.$$

Now, using the powers of the matrix E we will give the following theorem.

Theorem 1.1. *For integers n, we have the following matrices.*

$$E^n = 5^{(n-1)/2} \begin{pmatrix} L_{-(n+1)} & L_{-n} \\ L_{-n} & L_{-(n-1)} \end{pmatrix}; \ \textit{if } n \textit{ is odd number}$$

and

$$E^n = 5^{n/2} \begin{pmatrix} F_{-(n+1)} & F_{-n} \\ F_{-n} & F_{-(n-1)} \end{pmatrix}; \; if \;\; n \;\; is \;\; even \;\; number,$$

where $F_n = W_n(0,1;1,1)$ *and* $L_n = W_n(2,1;1,1)$

Proof. We use induction on n. First, we consider even numbers n. For $n = 0$, the claim is obvious. We suppose that it is true for $n = k$ and k is even number, then by using the equations, $L_{-n} = (-1)^n L_n$ and $F_{-n} = (-1)^{n+1} F_n$ we get

$$E^{k+1} = 5^{k/2} \begin{pmatrix} L_{-(k+2)} & L_{-(k+1)} \\ L_{-(k+1)} & L_{-k} \end{pmatrix}.$$

When $n = k$ is an odd number if we use the identities $L_{-n} = (-1)^n$ and $L_{n-1} + L_{n+1} = 5F_n$, then we obtain

$$E^{k+1} = E^k E = 5^{(k+1)/2} \begin{pmatrix} F_{-(k+2)} & F_{-(k+1)} \\ F_{-(k+1)} & F_{-k} \end{pmatrix}.$$

So, using the equations, $L_{-n} = (-1)^n L_n$ and $F_{n-1} + F_{n+1} = L_n$, we have

$$E^n = 5^{n/2} \begin{pmatrix} F_{-(n+1)} & F_{-n} \\ F_{-n} & F_{-(n-1)} \end{pmatrix}.$$

In a similar way, for odd number n

$$E^n = 5^{(n-1)/2} \begin{pmatrix} L_{-(n+1)} & L_{-n} \\ L_{-n} & L_{-(n-1)} \end{pmatrix}$$

can be written. Thus, the proof is completed. □

Consequently, by the aid of the matrix E we can get the following equations.

$$det(E^n) = 5^n, \;\; F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}, \;\; n \geq 0$$

and

$$F_{n-1}F_{n+1} - F_n^2 = (-1)^n, \quad L_{n-1}L_{n+1} - L_n^2 = 5(-1)^{n-1}.$$

Thus, the various identities are well known in the literature and involving the terms of Fibonacci and Lucas numbers at negative indices can be easily obtain by the matrix E.

In the following theorem, we give the binomial expansion of the Binet formula for the golden ratio at negative indices.

Theorem 1.2. *For* $n \geq 1$, *we have*

$$F_{-n} = \sum_{i=0}^{\lfloor \frac{n-1}{2} \rfloor} \begin{pmatrix} n-1-i \\ i \end{pmatrix} 5^{\frac{n-2i-2}{2}} (-1)^{(i+1)}, \;\; if \;\; n \;\; is \;\; even \;\; number$$

and

$$L_{-n} = 5^{-\frac{n-1}{2}} \sum_{i=0}^{\lfloor \frac{n-1}{2} \rfloor} \begin{pmatrix} n-1-i \\ i \end{pmatrix} 5^{n-1-i} (-1)^{(i+1)}, \;\; if \;\; n \;\; is \;\; odd \;\; number.$$

Proof. From the following matrices B^n and E^n we can write

$$E^n = \begin{pmatrix} y_n - 2y_{n-1} & -y_{n-1} \\ -y_{n-1} & y_n - 3y_{n-1} \end{pmatrix},$$

where n is an even number and $y_{n-1} = \sum_{i=0}^{\lfloor \frac{n-1}{2} \rfloor} \begin{pmatrix} n-1-i \\ i \end{pmatrix} 5^{n-1-2i}(-5)^i$.

If we equal the corresponding elements of the matrices E^n and B^n in the Theorem 1.2, then we have

$$5^{\frac{-n}{2}} F_{-n} = \sum_{i=0}^{\lfloor \frac{n-1}{2} \rfloor} \begin{pmatrix} n-1-i \\ i \end{pmatrix} 5^{\frac{n-2-2i}{2}} (-5)^i.$$

If we make necessary arrangements, then we get the following formula.

$$F_{-n} = \sum_{i=0}^{\lfloor \frac{n-1}{2} \rfloor} \begin{pmatrix} n-1-i \\ i \end{pmatrix} 5^{\frac{n-2-2i}{2}} (-1)^i.$$

A similar proof can be written down for the odd numbers n. □

Note that, for $n = 5$ we can write

$$L_{-5} = 5^{-2} \sum_{i=0}^{2} \begin{pmatrix} 4-i \\ i \end{pmatrix} 5^{4-i}(-1)^{i+1} = -11.$$

Theorem 1.3. *For all integers n and $k \geq 1$, we have the following combinatorial identities,*

i) $F_{-nk} = F_{-n} L_n^{k-1} \sum_{i=0}^{\lfloor \frac{k-1}{2} \rfloor} \begin{pmatrix} k-1-i \\ i \end{pmatrix} L_n^{-2i}(-1)^i$, *$n$ and k are even numbers.*

ii) $L_{-nk} = 5^{-\frac{1-k}{2}} L_{-n} \sum_{i=0}^{\lfloor \frac{k-1}{2} \rfloor} \begin{pmatrix} k-1-i \\ i \end{pmatrix} 5^{-i} F_n^{k-1-2i}(-1)^i$, *$n$ and k are odd numbers.*

Proof. When n is even number, if we calculate the kth power of the matrices E^n, then

$$E^{nk} = \begin{pmatrix} y_k - 5^{\frac{n}{2}} F_{-(n-1)} y_{k-1} & 5^{\frac{n}{2}} F_{-n} y_{k-1} \\ 5^{\frac{n}{2}} F_{-n} y_{k-1} & y_k - 5^{\frac{n}{2}} F_{-(n+1)} y_{k-1} \end{pmatrix},$$

where

$$y_{k-1} = \sum_{i=0}^{\lfloor \frac{k-1}{2} \rfloor} \begin{pmatrix} k-1-i \\ i \end{pmatrix} (5^{\frac{n}{2}} L_n)^{k-1-2i}((-5)^n)^i.$$

On the other hand, we know that

$$E^{nk} = 5^{\frac{nk}{2}} \begin{pmatrix} F_{-(nk+1)} & F_{-nk} \\ F_{-nk} & F_{-(nk-1)} \end{pmatrix}, \text{ } nk \text{ is even number.}$$

So, if we equal reciprocal elements of these matrices, we have

$$5^{\frac{nk}{2}} F_{-n} = 5^{\frac{n}{2}} F_{-n} y_{k-1}.$$

Thus, we can get

$$F_{-nk}=F_{-n}L_n^{k-1}\sum_{i=0}^{\lfloor\frac{k-1}{2}\rfloor}\left(\begin{array}{c}k-1-i\\ i\end{array}\right)L_n^{-2i}(-1)^i.$$

Note that, for $n=2$, $k=4$ we get

$$F_{-8}=F_{-2}L_2^3\sum_{i=0}^{1}\left(\begin{array}{c}3-i\\ i\end{array}\right)L_2^{-2i}(-1)^i=-21.$$

When n and k numbers are odd, the proof can be seen alike for even numbers. □

In addition using the multiplication matrix, for m and n even numbers, we can get the following equations.

$$F_{-(m+n)}=F_{-(m+1)}F_{-n}+F_{-m}F_{-(n-1)},\ \ m\ \ and\ \ n\ \ even,$$

and

$$F_{-(m-n)}=5^{\frac{nk}{2}}F_{-(m-1)}F_{-n}-F_{-m}F_{-(n-1)},\ \ m\ \ and\ \ n\ \ even.$$

Lemma 1.1. *For n, r integers we have*

$$i)\ F_{-(n-r)}=F_{-n}F_{-(r-1)}-F_{-(n-1)}F_{-r},\ \ r\ \ and\ \ n\ \ even,$$

$$ii)\ 5F_{-(n-r)}=L_{-n}L_{-(r-1)}-L_{-(n-1)}L_{-r},\ \ r\ \ and\ \ n\ \ even,$$

$$iii)\ L_{-(n-r)}=L_{-n}F_{-(r-1)}-L_{-(n-1)}F_{-r},\ \ r\ \ even\ \ and\ \ n\ \ odd,$$

$$iv)\ L_{-(n-r)}=F_{-n}L_{-(r-1)}-F_{-(n-1)}L_{-r},\ \ r\ \ odd\ \ and\ \ n\ \ even.$$

Proof. From the powers of the matrices E and B the proof can be easily seen. □

Theorem 1.4. *For all integers n, r and $k\geq 1$ we have the following equations.*

$$i)\ F_{-(nk+r)}=\sum_{i=0}^{\lfloor\frac{k}{2}\rfloor}\left(\begin{array}{c}k-i\\ i\end{array}\right)L_n^{k-2i}(-1)^i(F_{-r}+\frac{k-2i}{k-i}\frac{F_{-(n-r)}}{L_n}),\ \ n,\ \ k\ \ and\ \ r\ \ even,$$

$$ii)\ F_{-(nk+r)}=\sum_{i=0}^{\lfloor\frac{k}{2}\rfloor}\left(\begin{array}{c}k-i\\ i\end{array}\right)5^{\frac{k-1-2i}{2}}F_n^{k-2i}(-1)^i(L_{-r}+\frac{k-2i}{k-i}\frac{F_{-(n-r)}}{F_n}),\ \ n,\ \ k\ \ and\ \ r\ \ odd.$$

Proof. If n, k, r are even numbers, then $nk+r$ is also even number. So, using Theorem 1.2

$$E^{nk+r}=5^{\frac{nk+r}{2}}\left(\begin{array}{cc}F_{-(nk+r+1)} & F_{-(nk+r)}\\ F_{-(nk+r)} & F_{-(nk+r-1)}\end{array}\right)\tag{1.1}$$

can be written. On the other hand,

$$(E^n)^kE^r=5^{\frac{r}{2}}\left(\begin{array}{cc}y_k-5^{\frac{n}{2}}F_{-(n-1)}y_{k-1} & 5^{\frac{n}{2}}F_{-n}y_{k-1}\\ 5^{\frac{n}{2}}F_{-n}y_{k-1} & y_k-5^{\frac{n}{2}}F_{-(n+1)}y_{k-1}\end{array}\right)\left(\begin{array}{cc}F_{-(r+1)} & F_{-r}\\ F_{-r} & F_{-(r-1)}\end{array}\right)\tag{1.2}$$

can be written.

So, if we equal the reciprocal elements of matrices in the equations (1.1) and (1.2), then we get

$$5^{\frac{nk+r}{2}}F_{-(nk+r)}=5^{\frac{r}{2}}F_{-r}(y_k-5^{\frac{n}{2}}F_{-(n-1)}y_{k-1})+5^{\frac{r}{2}}F_{-(r-1)}5^{\frac{n}{2}}F_{-n}y_{k-1}.$$

Since, $T = 5^{\frac{n}{2}} L_n$, $D = 5^n$, $y_k = \sum_{i=0}^{\lfloor \frac{k}{2} \rfloor} \begin{pmatrix} k-i \\ i \end{pmatrix} T^{k-2i}(-D)^i$, n even number

$$5^{\frac{nk+r}{2}} F_{-(nk+r)} = (\sum_{i=0}^{\lfloor \frac{k}{2} \rfloor} \begin{pmatrix} k-i \\ i \end{pmatrix} T^{k-2i}(-D)^i)(5^{\frac{r}{2}} F_{-r} + 5^{\frac{n+r}{2}} F_{-(n-r)} \frac{k-2i}{k-i} T^{-1})$$

and

$$5^{\frac{nk+r}{2}} F_{-(nk+r)} = (\sum_{i=0}^{\lfloor \frac{k}{2} \rfloor} \begin{pmatrix} k-i \\ i \end{pmatrix} 5^{\frac{nk+r}{2}} (-1)^i L_n^{k-2i})(F_{-r} + \frac{k-2i}{k-i} \frac{F_{-(n-r)}}{L_n}).$$

can be written.
Thus, the proof of $i)$ is completed. In case $ii)$ the proof can be seen in a similar way. □

If we want to give an example, then for $n = 4$, $k = 2$, $r = 6$ we get

$$F_{-14} = (\sum_{i=0}^{1} \begin{pmatrix} 2-i \\ i \end{pmatrix} L_4^{2-2i}(-1)^i)(F_{-6} + \frac{2-2i}{2-i} \frac{F_2}{L_4}) = -377.$$

And, for $n = 5$, $k = 3$, $r = 7$ we get $F_{-22} = -17711$.

Theorem 1.5. *For $k \geq 1$, the odd numbers n, k and even number r we have*

$$i) \;\; L_{-(nk+r)} = \sum_{i=0}^{\lfloor \frac{k}{2} \rfloor} \begin{pmatrix} k-i \\ i \end{pmatrix} 5^{\frac{k-2i+1}{2}} F_n^{k-2i}(-1)^i(F_{-r} + \frac{k-2i}{k-i} \frac{L_{-(n-r)}}{5F_n}).$$

And when n, k are even numbers and r is odd number we have

$$ii) \;\; L_{-(nk+r)} = \sum_{i=0}^{\lfloor \frac{k}{2} \rfloor} \begin{pmatrix} k-i \\ i \end{pmatrix} L_n^{k-2i}(-1)^i(L_{-r} + \frac{k-2i}{k-i} \frac{L_{-(n-r)}}{L_n}).$$

Now, we will give the following theorems without proof.

Theorem 1.6. *For $k \geq 1$, we have*

$$i) \;\; F_{-(nk+r)} = \sum_{i=0}^{\lfloor \frac{k}{2} \rfloor} \begin{pmatrix} k-i \\ i \end{pmatrix} 5^{\frac{k-2i}{2}} F_n^{k-2i}(-1)^i(F_{-r} + \frac{k-2i}{k-i} \frac{L_{-(n-r)}}{5F_n}), \;\; n \text{ odd}, \;\; k \text{ and } r \text{ even},$$

$$ii) \;\; F_{-(nk+r)} = \sum_{i=0}^{\lfloor \frac{k}{2} \rfloor} \begin{pmatrix} k-i \\ i \end{pmatrix} L_n^{k-2i}(-1)^i(F_{-r} + \frac{k-2i}{k-i} \frac{F_{-(n-r)}}{L_n}), \;\; n, \;\; r \text{ even}, \;\; k \text{ odd}.$$

Theorem 1.7. *For $k \geq 1$, the odd numbers n, r and number even k we have*

$$i) \;\; L_{-(nk+r)} = \sum_{i=0}^{\lfloor \frac{k}{2} \rfloor} \begin{pmatrix} k-i \\ i \end{pmatrix} 5^{\frac{k-2i}{2}} F_n^{k-2i}(-1)^i(L_{-r} + \frac{k-2i}{k-i} \frac{F_{-(n-r)}}{F_n}).$$

And when n is even number and k, r are odd numbers we have

$$ii) \;\; L_{-(nk+r)} = \sum_{i=0}^{\lfloor \frac{k}{2} \rfloor} \begin{pmatrix} k-i \\ i \end{pmatrix} L_n^{k-2i}(-1)^i(L_{-r} + \frac{k-2i}{k-i} \frac{L_{-(n-r)}}{L_n}).$$

Thus, we derive various identities for the Fibonacci and Lucas sequences at negative index.

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