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VISUALISATION OF CAUCHY PROBLEM SOLUTION FOR LINEAR T-HYPERBOLIC PDE

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Abstract. Graphics Constructor and Cauchy Solver computer dialogue programs were created by A.Bulgak and D.Eminov [1, 2]. These programs use the one dimensional spline functions for visualisation of graphics of real functions. The study generalises this approach to the Cauchy problem for linear one dimensional t-hyperbolic PDE[4, 5, 7].

1. Introduction

Two-dimensional spline functions are important at applied mathematics and computer applications of mathematics. It offers approaches surface creation and approximate value search on over surface.

For one-dimensional spline functions "Graphics Constructor" interactive computer software was created by Bulgak and Eminov in 2003[1]. This computer program provides opportunities to graphically display the first, second and third-order one-dimensional spline functions. The algorithms which are based on "Graphics Constructor" were used a Cauchy problem in another study and "Cauchy Solver" [2] software were obtained.

Let $t_0, t_1 \in R$, A is a square N dimensional real matrix, y_0 is a real N dimensional real vector. Takes Cauchy problem,

$$y^{'}(t) = Ay(t), \qquad t_0 \leq t \leq t_1, \qquad y(t_0) = y_0,$$

"Cauchy Solver" solves this problem and shows each component of the solution obtained as graphs by using the approximate one dimensional cubic spline functions.

This study discusses the two dimensional spline functions. Based on existing background it develops similar programs and algorithms. The results of this study give us new algorithms and software which have abilities for visualisation.

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A Cauchy problem solution for first-order linear homogeneous constant coefficients partial differential equation is displayed with two dimensional spline functions.

Let a, α, β and $T > 0$ be real numbers, $\phi \in C^1(\alpha, \beta) \cup C([\alpha, \beta])$ is a derivable real function. G be a parallelogram as $G = \{(t, x) : at+\alpha \leq x \leq at+\beta, 0 \leq t \leq T\}$.

$$\begin{cases} \hat{u}_t(t,x) + a\hat{u}_x(t,x) = 0, & t, x \in G \\ \hat{u}(0,x) = \phi(x), & \alpha \leq x \leq \beta \end{cases}$$

It is known, there exists the solution of this problem and it is unique[7]. The aim of this study is to show the solution graphically in the mentioned G parallelogram.

2. Linear t-hyperbolic partial differential equations

a is a real number. Let us consider the following PDE,

$$\hat{u}_t(t,x) + a\hat{u}_x(t,x) = 0, \qquad t, x \in R \tag{2.1}$$

It is known as t-hyperbolic equation in literature. For example,[4, 5] mentioned this type of equations. Let us give some basic information about this equation from the literature. The line sets as

$$x - at = c, \qquad t, c \in R$$

which provides the condition;

$$\frac{dt}{t} = \frac{dx}{a}, \qquad a \neq 0$$

is known as the characteristic set of equation (2.1). Every element of this set is known as characteristic of the equation (2.1).

Let's give well-known theorems in the literature[4, 5]

Theorem 2.1. *The general solution of* (2.1) *is as follow*

$$\hat{u}(t,x) = f(x - at), \qquad t, x \in R. \quad \textit{Here} \quad f \in C^1$$

Theorem 2.2. *$a \in R$ and $\phi \in C^1$ then Cauchy problem*

$$\begin{cases} \hat{u}_t(t,x) + a\hat{u}_x(t,x) = 0, & t, x \in R \\ \hat{u}(0,x) = \phi(x), & x \in R \end{cases}$$

has a unique solution as follows $\hat{u}(t,x) = \phi(x - at)$.

Let $a, \alpha, \beta, T > 0$ are real numbers, $\phi : [\alpha, \beta] \to R$ is a derived function and

$$G = \{(t, x) : at + \alpha \leq x \leq at + \beta, 0 \leq t \leq T\}.$$

In this case;

$$\begin{cases} \hat{u}_t(t,x) + a\hat{u}_x(t,x) = 0; & t, x \in G \\ \hat{u}(0,x) = \phi(x), x \in [\alpha, \beta] \end{cases} \tag{2.2}$$

There exists the solution of this Cauchy problem and it is unique. The solution is $\hat{u}(t,x) = \phi(x - at), t, x \in G$. If desired the solution is until T, the solution zone is a parallelogram. For example, if it is $a > 0$, solution zone is shown in figure 1.

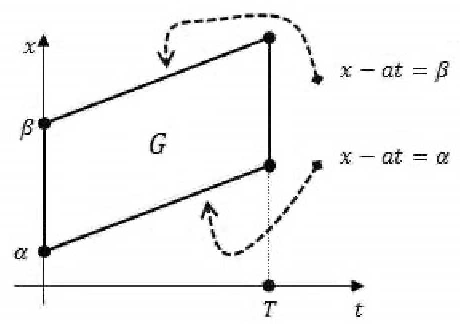


FIGURE 1. G parallelogram.

T	t_0	t_1	t_2	t_3	$\dots$	t_{n-3}	t_{n-2}	t_{n-1}
Y	y_0	y_1	y_2	y_3	$\dots$	y_{n-3}	y_{n-2}	y_{n-1}

TABLE 1

3. SPLINE FUNCTIONS AND CUBIC SPLINE

Here, $t_0 < t_1 < \cdots < t_{n-1}$ are distinct ordered real numbers and $y_0, y_1, \dots, y_{n-1}$ are real numbers that represent each node as figure 2. It describes a spline function f_{sp} according to the table 1.

$$f_{sp}(t) = \begin{cases} f_0(t), & t_0 \le t \le t_1 \\ f_1(t), & t_1 < t \le t_2 \\ \vdots & \vdots \\ f_{n-3}(t), & t_{n-3} \le t \le t_{n-2} \\ f_{n-2}(t), & t_{n-2} \le t \le t_{n-1} \end{cases}$$

$f_j(t_j) = y_j$ and $f_j(t_{j+1}) = y_{j+1}$ seems for each $j = 0, 1, \cdots, n-2$. Let $a, b \in R$ and $a = t_0 < t_1 < \cdots < t_{n-2} < t_{n-1} = b$ under this circumstances $f_j : [t_j, t_{j+1}] \to R$ and $f_{sp} : [a, b] \to R$. Each f_j function may have any degree that is polynomial function. Often the first, second and third order polynomial functions are used in practice.

3.1. **Cubic Spline.** Take table 2 with a real sequence $F_0,\ F_1,\ \cdots,\ F_{n-1}$ a cubic spline function $f_{sp} : [t_0, t_{n-1}] \to R, \quad y = f_{sp}(t), \quad t \in [t_0, t_{n-1}]$. For each

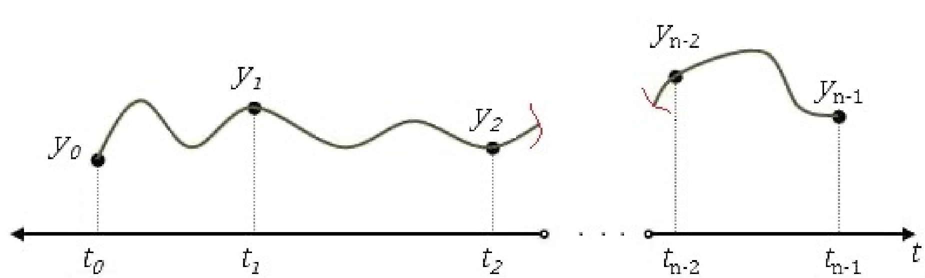


FIGURE 2

T	t_0	t_1	t_2	t_3	$\dots$	t_{n-3}	t_{n-2}	t_{n-1}
Y	y_0	y_1	y_2	y_3	$\dots$	y_{n-3}	y_{n-2}	y_{n-1}
F	F_0	F_1	F_2	F_3	$\dots$	F_{n-3}	F_{n-2}	F_{n-1}

Table 2

sequential nodes interval, every polynomial functions

$$f_j : [t_j, t_{j+1}] \to R, \quad f_j\,(t) = a_j t^3 + b_j t^2 + c_j t + d_j \quad j = 0, 1, \cdots, n-2$$

which satisfied table 2 as $f_i^{'}\,(t_i) = F_i,\ f_i\,(t_i) = y_i\ f_i^{'}\,(t_{i+1}) = F_{i+1},\ f_i\,(t_{i+1}) = y_{i+1}$ and the condition $f_{sp}^{'}\,(t_i) = F_i$ for $i = 0, 1, \cdots, n-1$ is unique. This condition can provides, at least third degree spline functions[7].

This situation is important for us in this study. Now let us remember the Cauchy problem for linear t- hyperbolic PDE, presented in section 2. There exists a unique solution of (2.2). Here; if it is $a \neq 0$, $\hat{u}_x\,(t,x)$ partial derivative must be there. In this case, ϕ function must be selected derived. $\phi^{'}\,(x)$ would not have been, hence $\hat{u}_t\,(t,x) + a\hat{u}_x\,(t,x) = 0$ equation would not have been.

An Algorithm. To calculation for any t , $t \in R$ according the table 3.2, process steps created algorithm are on following lines[1].

Input :	$t_0, t_1, t_2, \dots, t_{n-1}; \quad y_0, y_1, y_2, \dots, y_{n-1}; \quad F_0, F_1, F_2, \dots, F_{n-1}; \quad t$
Output :	y

if $((n < 2)$ *or* $(t < t_0)$ *or* $(t > t_{n-1}))$ *then*
{*get out of processing steps that make up the algorithm*};
for $(j = 1$ *to* $n-1)$ *do begin*
if $((t \geq t_{j-1})$ *and* $(t < t_j))$ *then begin*
$w = [(y_j - y_{j-1})/(t_j - t_{j-1}) - F_{j-1}]/(t_j - t_{j-1})$
$a = [(F_j - F_{j-1})/(t_j - t_{j-1}) - 2w]/(t_j - t_{j-1})$
$b = -\,(t_j + 2t_{j-1})\,a + w$
$c = F_{j-1} - 3a(t_{j-1})^2 - 2b(t_{j-1})$
$d = y_{j-1} - a(t_{j-1})^3 - b(t_{j-1})^2 - c(t_{j-1})$
end if;
end for;
Output $at^3 + bt^2 + ct + d.$

4. The use of cubic spline functions for the problem of two dimensional interpolations

An interpolation problem the brief analysis of on one dimensional cubic spline functions showed on section 3. Now we can expand this approach to the two dimensional functions. $\hat{R} = [a,b] \times [c,d]$, consider the rectangle on tOx plane.

$$a = t_0 < t_1 < \cdots < t_{m-1} = b, \quad m \geq 1$$

$$c = x_0 < x_1 < \cdots < x_{n-1} = d, \quad n \geq 1$$

$n \times m$ points are located on tOx plane and these points are identifying a grid.

$$u = \left\{u_{(0,0)}, u_{(0,1)}, \dots, u_{(0,n-1)}, u_{(1,0)}, \dots, u_{(m-1,n-1)}\right\},$$

$$u_{(i,j)} \in R,\ i = 0, 1, \dots, m-1, \qquad j = 0, 1, \dots, n-1$$

each $u_{(i,j)}$ values are defined be on this grid. However, let come two sets as

$$f_t = \left\{ f_{t(0,0)}, f_{t(0,1)}, \dots, f_{t(0,n-1)}, f_{t(1,0)}, \dots, f_{t(m-1,n-1)} \right\},$$
$$f_{t(i,j)} \in R,\ i = 0, 1, \dots, m-1, j = 0, 1, 2, \dots, n-1$$

and

$$f_x = \left\{ f_{x(0,0)}, f_{x(0,1)}, \dots, f_{x(0,n-1)}, f_{x(1,0)}, \dots, f_{x(m-1,n-1)} \right\},$$
$$f_{x(i,j)} \in R,\ i = 0, 1, \dots, m-1; j = 0, 1, 2, \dots, n-1$$

both of them have $n \times m$ elements.

The aim is to find a derived function $f(t,x)$, which was defined on $\hat{R}$. Let it provide the condition:$f(t_i, x_j) = u_{(i,j)}$, $f_t^{'}(t_i, x_j) = f_{t(i,j)}$ and $f_x^{'}(t_i, x_j) = f_{x(i,j)}$ for $i = 0, 1, \dots, m-1$ and $j = 0, 1, \dots, n-1$.

As a first, table 3 is created with the help of aforesaid information. $H(t_0, x)$, $x_0 \le x \le x_{m-1}$, cubic spline function, is calculated according to the table 3. Then table 4 is created. Basing on this table calculated the $H(t_1, x)$, $x_0 \le x \le x_{m-1}$ cubic spline function. Similarly $H(t_2, x), \dots, H(t_{n-1}, x)$, $x_0 \le x \le x_{m-1}$ functions are calculated on the basis of the other data tables. So, n units one dimensional cubic spline functions are acquired.

Tt_0	t_0	t_1	t_2	$\dots$	t_{m-1}
Xt_0	x_0	x_0	x_0	$\dots$	x_0
Ft_0	$f_{t(0,0)}$	$f_{t(1,0)}$	$f_{t(2,0)}$	$\dots$	$f_{t(m-1,0)}$
Ut_0	$u_{0,0}$	$u_{1,0}$	$u_{2,0}$	$\dots$	$u_{m-1,0}$

TABLE 3

Tt_1	t_0	t_1	t_2	$\dots$	t_{m-1}
Xt_1	x_1	x_1	x_1	$\dots$	x_1
Ft_1	$f_{t(0,1)}$	$f_{t(1,1)}$	$f_{t(2,1)}$	$\dots$	$f_{t(m-1,1)}$
Ut_1	$u_{0,1}$	$u_{1,1}$	$u_{2,1}$	$\dots$	$u_{m-1,1}$

TABLE 4

Tx_0	t_0	t_0	t_0	$\dots$	t_0
Xx_0	x_0	x_1	x_2	$\dots$	x_{n-1}
Fx_0	$f_{x(0,0)}$	$f_{x(0,1)}$	$f_{x(0,2)}$	$\dots$	$f_{x(0,n-1)}$
Ux_0	$u_{0,0}$	$u_{0,1}$	$u_{0,2}$	$\dots$	$u_{0,n-1}$

TABLE 5

Tx_1	t_1	t_1	t_1	$\dots$	t_1
Xx_1	x_0	x_1	x_2	$\dots$	x_{n-1}
Fx_1	$f_{x(1,0)}$	$f_{x(1,1)}$	$f_{x(1,2)}$	$\dots$	$f_{x(1,n-1)}$
Ux_1	$u_{1,0}$	$u_{1,1}$	$u_{1,2}$	$\dots$	$u_{1,n-1}$

TABLE 6

The same process is repeated for each t_i, for $i = 0, 1, 2, \dots, m-1$. Table 5 is created with the help of aforesaid information.$S(t, x_0)$, $t_0 \le t \le t_{n-1}$, cubic spline function, is calculated according to the table 5. In addition table 6 is created. Basing on this table calculated the $S(t, x_1)$, $t_0 \le t \le t_{n-1}$ cubic spline function. Similarly $S(t, x_2), \dots, S(t, x_{n-1})$, $t_0 \le t \le t_{n-1}$ functions are calculated on the basis of the other data tables. So, m units one dimensional cubic spline functions are acquired.

Placement of Spline functions on Three-Dimensional Coordinate System.

$$H(t_0,x), H(t_1,x), H(t_2,x), \dots, H(t_{m-1},x), \qquad x_0 \le x \le x_{m-1}$$

$$S(t,x_0), S(t,x_1), S(t,x_2), \dots, S(t,x_{n-1}), \qquad t_0 \le t \le t_{n-1}$$

include totally $n+m$ spline functions.

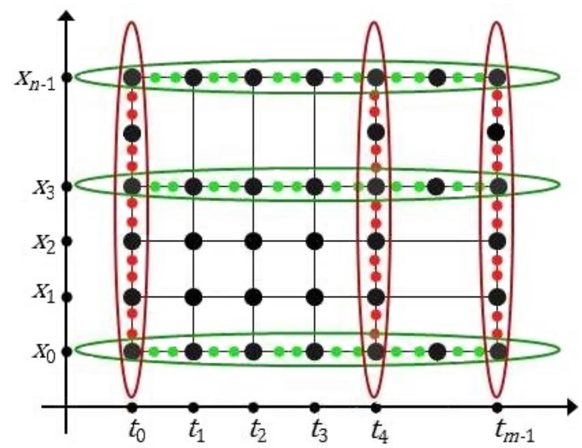


FIGURE 3

$$a = t_0 < t_1 < \cdots < t_{m-1} = b, \qquad m \ge 1$$

$$c = x_0 < x_1 < \cdots < x_{n-1} = d, \qquad n \ge 1$$

on $\hat{R} = [a,b] \times [c,d]$ define a grid.

The values of step size corresponding successive pixels and functions are calculated for each spline functions in the x and t directions. 3.

5. THE GRID AND SPLINE FUNCTIONS RELATED TO LINEAR T-HYPERBOLIC PDE

It is important to identify the following factors for visualisation of Cauchy solutions. It is chosen n points representing "well-chosen" ϕ function given on $[\alpha,\beta]$ interval.

$$x_0 = \alpha < x_1 < x_2 < \cdots < x_{n-2} < x_{n-1} = \beta$$

The value of $\phi'(x_i)$ function for each x_i is the height of spline function in the direction x . By considering $\phi(x_i)$ a cubic spline function, $g(x)$, is obtained and the graphic of $z(t,x) = g(x-at)$, $t,x \in G$, $0 = t_0 < t_1 < \cdots < t_{m-1} = T$ functions are visualised.

$$z(t,x) = g(x - at), \quad t = t_0, t_1, \ldots, t_{m-1}, \quad x \in [\alpha + at_j, \beta + at_j]$$
$$z(t,x) = g(x - at), \quad x = x_0, x_1, \ldots, x_{n-1}, \quad t \in [0, T]$$

this information identifies a grid as figure 4.

A cubic spline function representing $\phi(x)$ is taken instead of $\phi(x)$ function.

Between $x - at = \beta$ and $x - at = \alpha$, $t \in [0,T]$ lines, left edging is $(0,x)$, $x \in [\alpha, \beta]$ and right edging is (T,x), $x \in [\alpha + aT, \beta + aT]$ belonging to G parallelogram. A grid on G parallelogram and nodes are shown in figure 4. $\zeta_{i,j}$ nodes

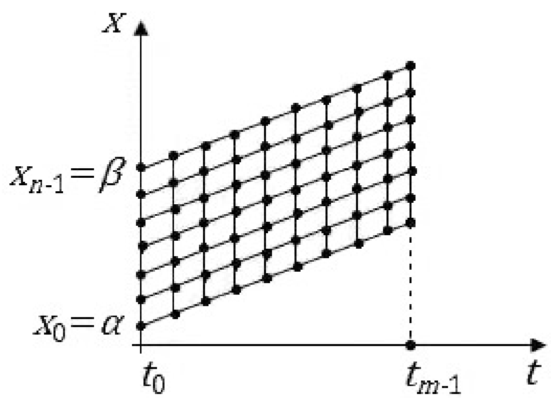


FIGURE 4

are defined as

$$\zeta_{i,j} = (t_i, a(t_i - t_0) + x_j), \quad i = 0, 1, \ldots, m-1, \quad j = 0, 1, \ldots, n-1$$

Tx_0	t_0	t_0	t_0	$\ldots$	t_0
Xx_0	x_0	x_1	x_2	$\ldots$	x_{n-1}
Fx_0	$\phi'(x_0)$	$\phi'(x_1)$	$\phi'(x_2)$	$\ldots$	$\phi'(x_{n-1})$
Ux_0	$\phi(x_0)$	$\phi(x_1)$	$\phi(x_2)$	$\ldots$	$\phi(x_{n-1})$

TABLE 7. This table is used for setting aforesaid $g(x)$ spline function.

Example 5.1. $G = \{(t,x) : 0 \le t \le 100, \quad 0 + 0.5t \le x \le 100 + 0.5t\}$ is a parallelogram and $\phi(x) = 2x^{1.45} - 2.5x^{1.4}$, $0 \le x \le 100$ is an initial function. Consider this Cauchy problem

$$\hat{u}_t(t,x) + 0.5\hat{u}_x(t,x), \quad t, x \in G$$
$$\hat{u}(0,x) = \phi(x), \quad 0 \le x \le 100$$

and let it be $x_0 = 0$, $x_1 = 10$, $x_2 = 20$, $x_3 = 30$, $x_4 = 40$, $x_5 = 50$, $x_6 = 60$, $x_7 = 70$, $x_8 = 80$, $x_9 = 90$, $x_{10} = 100$. Take $g(x)$ cubic spline function in approach to $\phi(x)$ instead of $\phi(x)$ initial function. $g(x)$ cubic spline function is given in table 8.

In this case;

$$v_t(t,x) + 0.5v_x(t,x), t, x \in G, \qquad v(0,x) = g(x); 0 \le x \le 100.$$

The Cauchy problem must be visualised on G parallelogram. For this, the following t values are chosen; $t_0 = 0$, $t_1 = 10$, $t_2 = 20$, $t_3 = 30$, $t_4 = 40$, $t_5 = 50$, $t_6 = 60$, $t_7 = 70$, $t_8 = 80$, $t_9 = 90$, $t_{10} = T = 100$; Generated surface visualisation is shown in figure 5.

X	0	10	20	30	40	50
$\phi(x)$	0	-6.43	-11.72	-15.11	-16.60	-16.24
$\phi'(x)$	0	-0.62	-0.43	-0.24	-0.05	0.12
X	60	70	80	90	100	
$\phi(x)$	-14.08	-10.20	-4.65	2.51	11.26	
$\phi'(x)$	0.30	0.47	0.63	0.79	0.95	

TABLE 8

X	-2.5	-2.0	-1.5	-1.0	-0.5	0
$\phi(x)$	-0.598	-0.909	-0.997	-0.842	-0.479	0
$\phi'(x)$	-0.801	-0.416	0.071	0.540	0.877	1
X	0.5	1.0	1.5	2.0	2.5	
$\phi(x)$	0.479	0.842	-0.997	0.909	0.598	
$\phi'(x)$	0.877	0.540	0.071	-0.416	-0.801	

TABLE 9

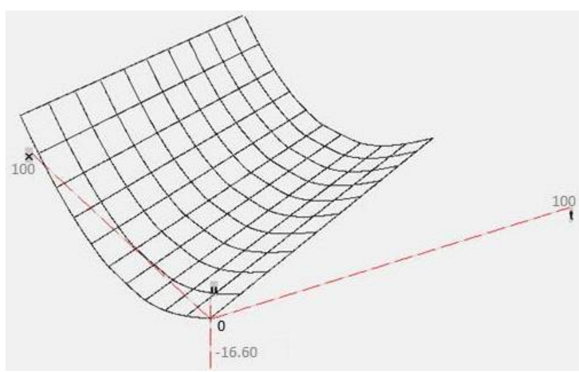


FIGURE 5. Output screen of computer program.

Example 5.2. Consider this Cauchy problem

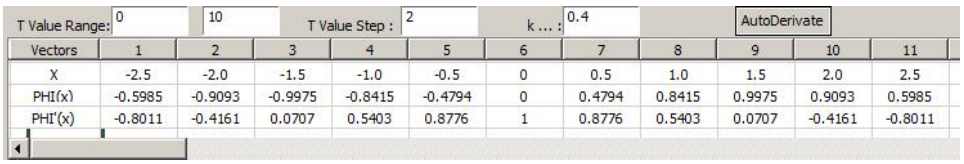
$$\begin{cases} \hat{u}_t(t,x) + 0.4\hat{u}_x(t,x), & t,x \in G \\ \hat{u}(0,x) = \phi(x), & -2.5 \le x \le 2.5 \end{cases}$$

and let it be $x_0 = -2.5$, $x_1 = -2$, $x_2 = -1.5$, $x_3 = -1.0$, $x_4 = -0.5$, $x_5 = 0$, $x_6 = 0.5$, $x_7 = 1.0$, $x_8 = 1.5$, $x_9 = 2.0$, $x_{10} = 2.5$.

$$G = \{(t,x) : 0 \le t \le 10 \quad and - 2.5 + 0.4t \le x \le 2.5 + 0.4t\}$$

is a parallelogram and $\phi(x) = \sin(x)$, $-2.5 \le x \le 2.5$ is an initial function. The following t values are chosen. $t_0 = 0$, $t_1 = 2$, $t_2 = 4$, $t_3 = 6$, $t_4 = 8$, $t_5 = 10$. The cubic spline function is given in table 9. Figure 6 shows input panel for cubic spline function. This panel is related on the computer program. This computer program was developed based on result of this study. Generated surface visualisation is shown in figure 7.

This computer program is used perspective projection method. There have many kinds of three dimensional projection methods. Generally three dimensional projection methods is any method of mapping three dimensional points to a two dimensional plane.



Vectors	1	2	3	4	5	6	7	8	9	10	11
X	-2.5	-2.0	-1.5	-1.0	-0.5	0	0.5	1.0	1.5	2.0	2.5
PHI(x)	-0.5985	-0.9093	-0.9975	-0.8415	-0.4794	0	0.4794	0.8415	0.9975	0.9093	0.5985
PHI'(x)	-0.8011	-0.4161	0.0707	0.5403	0.8776	1	0.8776	0.5403	0.0707	-0.4161	-0.8011

FIGURE 6. Computer program input panel.

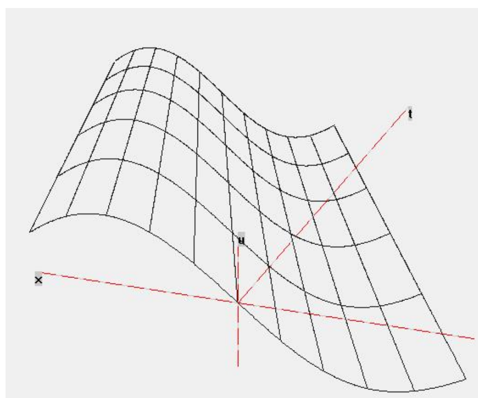


FIGURE 7

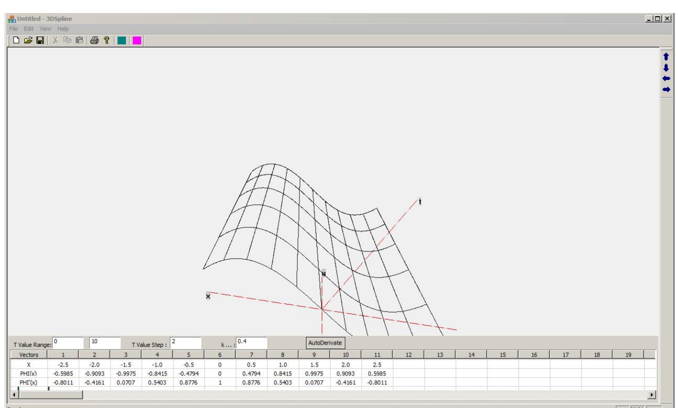


FIGURE 8. General view of the computer program.

Conclusion

This software togetter with “MVC” Matrix Vector Calculator programs[6] allows to give visualisation of Cauchy problem selection for $AU_t + BU_t = 0, \quad A = A^T > 0$, $B = B^T$ t-hyperbolic PDE.

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