Konuralp Journal of Mathematics
Volume 4 No. 1 pp. 282–290 (2016) ©KJM

# RELATIONS AMONG HIGHER ORDER CROSSED MODULES OVER GROUPOIDS

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Abstract. In this work, we explored the categorical relationships between crossed modules, 2-and 3- crossed modules of groupoids

## Introduction

Crossed modules were introduced by Whitehead in [11] as algebraic models for homotopy 2-types. They can be seem as 2- dimensional version of groups. The category of crossed modules is equivalent to that of [9] and simplicial groups with Moore complex of length 1. Conduche [8] in 1984 described the notion of 2-crossed module as a model of connected 3-types. He proved that the category of 2-crossed modules over groups is equivalent to the category of simplicial groups with Moore complex of length 2.The relations among algebraic models for homotopy 3-types were explored by Arvasi and Ulualan in [1] explicitly.The notion of 3- crossed module was defined in [2] as model for homotopy connected 4-types. It can be said that the category of 3- crossed modules is equivalent to that of crossed 3-cubes and simplicial groups with Moore complex of length 3. This equivalence has been constructed in [2].In this paper, we will consider these notion over groupoids. By using the definition of crossed modules over groupoids [6],the definition of 2-crossed modules can be extended to the notion of groupoids.Thus we can explore the relationship between 3-crossed modules over groups and 2- crossed modules over groupoids.Therefore, the results of this paper can be summarized by the following diagram

$$\begin{tikzcd}
 & \mathbf{XMod} \arrow[r] \arrow[dl] & \mathbf{Grpoid} \arrow[dr] & \\
\mathbf{X_2Mod} \arrow[r] & \mathbf{X_3Mod} \arrow[r] & \mathbf{2CMod} & \mathbf{CMod} \arrow[l]
\end{tikzcd}$$

with the arrows given below.

## 1. Preliminaries

In this section, we recall some basic definitions. The following definition is due to Whitehead [11].

**Definition 1.1.** [11] **(Crossed Modules Over Groups)**: A crossed module is a group homomorphism $\partial : M \to P$ together with an action of $P$ on $M$, written ${}^{p}m$ for $p \in P$ and $m \in M$, satisfying the conditions.

*Date*:
2000 *Mathematics Subject Classification.* 18D35,18D20,18G55.
*Key words and phrases.* Crossed modules, 2-crossed modules, Groupoids.

**CM1)** $\partial$ is $P$-equivariant, i.e, for all $p \in P$ , $m \in M$

$$\partial({}^{p}m) = p\partial(m)p^{-1}$$

**CM2)** (Peiffer Identity) for all $m, m' \in M$

$$^{\partial m}m' = mm'm^{-1}$$

We will denote such a crossed module by $(M, P, \partial)$. A *morphism of crossed module* from $(M, P, \partial)$ to $(M', P', \partial')$ is a pair of group homomorphisms

$$\phi : M \longrightarrow M' \ , \ \ \psi : P \longrightarrow P'$$

such that $\phi({}^{p}m) =^{\psi(p)} \phi(m)$ and $\partial'\phi(m) = \psi\partial(m)$.

We thus get a category **XMod** of crossed modules.

*Examples of Crossed Modules*

1) Any normal subgroup $N \trianglelefteq P$ gives rise to a crossed module namely the inclusion map, $i : N \hookrightarrow P$. Conversely, given any crossed module $\partial : M \longrightarrow P$ , $Im\partial$ is a normal subgroup of $P$.

2) Given any $P$-module $M$, the trivial map

$$1 : M \longrightarrow P$$

that maps everything to 1 in $P$, is a crossed module. Conversely, if $\partial : M \to P$ is a crossed module, $\ker \partial$ is central in $M$ and inherits a natural $P$-module structure from the $P$-action on $M$.

Conduche in [8] has defined 2-crossed module. The following definition of 2-crossed module is equivalent to that given by Conduche in [8];

**Definition 1.2. (2-Crossed Modules Over Groups)** A 2-crossed module of groups consists of a complex of groups

$$L \xrightarrow{\partial_2} M \xrightarrow{\partial_1} N$$

together with (a) actions of $N$ on $M$ and $L$ so that $\partial_2, \partial_1$ are morphisms of $N$-groups, and (b) an $N$-equivariant function

$$\{\quad , \quad\} : M \times M \longrightarrow L$$

called a *Peiffer lifting*. This data must satisfy the following axioms:

$$\begin{array}{llcl}
\textbf{2CM1}) & \partial_2\{m, m'\} & = & \left({}^{\partial_1 m}m'\right) mm'^{-1}m^{-1} \\
\textbf{2CM2}) & \{\partial_2 l, \partial_2 l'\} & = & [l', l] \\
\textbf{2CM3}) & (i) \ \ \{mm', m''\} & = & {}^{\partial_1 m}\{m', m''\}\{m, m'm''m'^{-1}\} \\
 & (ii) \ \ \{m, m'm''\} & = & \{m, m'\}^{mm'm^{-1}}\{m, m''\} \\
\textbf{2CM4}) & \{m, \partial_2 l\}\{\partial_2 l, m\} & = & {}^{\partial_1 m}ll^{-1} \\
\textbf{2CM5}) & {}^{n}\{m, m'\} & = & \{{}^{n}m, {}^{n}m'\}
\end{array}$$

for all $l, l' \in L$, $m, m', m'' \in M$ and $n \in N$.

We can give definition of 3-crossed module from [2] as follows;

**Definition 1.3.** [2] A *3-crossed module* consists of a complex of groups

$$K \xrightarrow{\partial_3} L \xrightarrow{\partial_2} M \xrightarrow{\partial_1} N$$

together with an action of $N$ on $K, L, M$ and an action of $M$ on $K, L$ and an action of $L$ on $K$ so that $\partial_3$, $\partial_2$,$\partial_1$ are morphisms of $N, M$-groups and the $M, N$-equivariant liftings

$$\{\ ,\ \}_{(1)(0)} : L \times L \longrightarrow K, \qquad \{\ ,\ \}_{(0)(2)} : L \times L \longrightarrow K, \qquad \{\ ,\ \}_{(2)(1)} : L \times L \longrightarrow K,$$

$$\{\ ,\ \}_{(1,0)(2)} : M \times L \longrightarrow K, \quad \{\ ,\ \}_{(2,0)(1)} : M \times L \longrightarrow K,$$

$$\{\ ,\ \}_{(0)(2,1)} : L \times M \longrightarrow K, \quad \{\ ,\ \} : M \times M \longrightarrow L$$

called 3-*dimensional Peiffer liftings*. This data must satisfy the following axioms:

**3CM1**) $K \xrightarrow{\partial_3} L \xrightarrow{\partial_2} M$ is a 2-crossed module with the Peiffer lifting $\{\ ,\ \}_{(2,1)}$

**3CM2**) $\{m,\partial_3 k\}_{(1,0)(2)} = \{m,\partial_3 k\}_{(2,0)(1)}\ {}^{m}(k)^{\partial_1 m}(k^{-1})$

**3CM3**) $\{\partial_3 k, m\}_{(0)(2,1)} = {}^{m}(k)k^{-1}$

**3CM4**) $\{m,\partial_3 k\}_{(1,0)(2)} = \{m,\partial_3 k\}_{(2,0)(1)}\ \{\partial_3 k,m\}_{(0)(2,1)}\ k^{\partial_1 m}(k^{-1})$

**3CM5**) $\{l',\partial_2 l\}_{(0)(2,1)} = \{l,l'\}^{-1}_{(2)(1)}\,\{l',l\}_{(1)(0)}$

**3CM6**) $\{\partial_2 l, l'\}_{(2,0)(1)} = \{l,l'\}^{-1}_{(0)(2)}\ {}^{[l',\,l]}(\{l,l'\}_{(2)(1)})\,\{l,l'\}_{(1)(0)}$

**3CM7**) $\{\partial_2 l,l'\}_{(1,0)(2)} = (\{l,l'\}_{(0)(2)})^{-1}$

**3CM8**) $\partial_3(\{l,l'\}_{(1)(0)}) = [l,l']\,\{\partial_2 l,\partial_2 l'\}$

**3CM9**) $\partial_3(\{l,l'\}_{(0)(2)}) = \partial_3(\{\partial_2 l,l'\}_{(1,0)(2)})^{-1}$

**3CM10**) $\partial_3\,\{l,m\}_{(0)(2,1)} = {}^{m}l\,l^{-1}\,\{\partial_2 l,m\}$

**3CM11**) $\partial_3\,\{m,l\}_{(2,0)(1)} = \partial_3\,\{m,l\}_{(1,0)(2)}\ {}^{\partial_1 m}l\ {}^{m}(l^{-1})\,\{m,\partial_2 l\}$

**3CM12a**) $\{\partial_3 k,l\}_{(1)(0)} = ({}^{l}k)k^{-1}$

**3CM12b**) $\{l,\partial_3 k\}_{(1)(0)}$ $k({}^{l}k)^{-1}$

**3CM13**) $\{\partial_3 k,\partial_3 k'\}_{(1)(0)} = [k',k]$

**3CM14**) $\{\partial_3 k, l'\}_{(0)(2)} = 1$

**3CM15**) $\{\partial_2 l,\partial_3 k\}_{(1,0)(2)} = \{l,\partial_3 k\}^{-1}_{(0)(2)}$

**3CM16**) $\{\partial_2 l,\partial_3 k\}_{(2,0)(1)} = \{l,\partial_3 k\}_{(0)(2)}\,k\left({}^{\partial_2 l}(k^{-1})\right)$

**3CM17**) $\{\partial_3 k,\partial_2 l\}_{(0)(2,1)} = {}^{\partial_2 l}k\ k^{-1}$

**3CM18**) $\partial_2\,\{m,m'\} = mm'm^{-1}({}^{\partial_1 m}m')^{-1}$

We denote such a 3-crossed module by $(K,L,M,N,\partial_3,\partial_2,\partial_1)$.

A *morphism of 3-crossed modules* of groups may be pictured by the diagram

$$\begin{array}{ccccccc}
L_3 & \xrightarrow{\partial_3} & L_2 & \xrightarrow{\partial_2} & L_1 & \xrightarrow{\partial_1} & L_0 \\
\downarrow{\scriptstyle f_3} & & \downarrow{\scriptstyle f_2} & & \downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f_0} \\
L'_3 & \xrightarrow[\partial'_3]{} & L'_2 & \xrightarrow[\partial'_2]{} & L'_1 & \xrightarrow[\partial'_1]{} & L'_0
\end{array}$$

where

$$f_1({}^{n}m) = {}^{(f_0(n))}f_1(m),\ f_2({}^{n}l) = {}^{(f_0(n))}f_2(l),\ f_3({}^{n}k) = {}^{(f_0(n))}f_3(k)$$

for $\{\ ,\ \}_{(0)(2)}, \{\ ,\ \}_{(2)(1)}, \{\ ,\ \}_{(1)(0)}$

$$\{\ ,\ \}\,f_2 \times f_2 = f_3\,\{\ ,\ \}$$

for $\{\ ,\ \}_{(1,0)(2)}, \{\ ,\ \}_{(2,0)(1)}$

$$\{\ ,\ \}\,f_1 \times f_2 = f_3\,\{\ ,\ \}$$

for $\{\ ,\ \}_{(0)(2,1)}$

$$\{\ ,\ \}\,f_2 \times f_1 = f_3\,\{\ ,\ \}$$

and for $\{\ ,\ \}$

$$\{\ ,\ \}\,f_1 \times f_1 = f_2\,\{\ ,\ \}$$

for all $k \in K, l \in L, m \in M, n \in N$. These compose in an obvious way. We thus can define the category of 3-crossed modules, denoting it by $\mathbf{X_3Mod}$.

## 2. Groupoids, Crossed Modules and 2-Crossed Modules over Groupoids

Recall that a*groupoid* is a small category in which every arrow is an isomorphisms. That is, for any morphisms $\alpha$ there exists a morphisms $\alpha^{-1}$ such that $\alpha \circ \alpha^{-1} = e_{t(\alpha)}$ and $\alpha^{-1} \circ \alpha = e_{s(\alpha)}$ where $e : C_0 \to C_1$ gives the identity morphism at an object. We write a groupoid as $(C_1, C_0)$, where $C_0$ is the set of object and $C_1$ is the set of morphisms. For any groupoid $\mathbf{C}$, if $C_1(x,y)$ is empty whenever x,y are distinct (that is, $s = t$), than $\mathbf{C}$ is called totaly disconnected. You can find detailed description of groupoids in [3],[4],[5],[7],[10]. A morphism of groupoid is $(f_1, f_0)$ in the following diagram

$$\begin{tikzcd} C_1 \arrow[r, "f_1"] \arrow[d, shift right, "s_1"'] \arrow[d, shift left, "t_1"] & C_1' \arrow[d, shift right, "s_1'"'] \arrow[d, shift left, "t_1'"] \\ C_0 \arrow[r, "f_0"'] & C_0' \end{tikzcd}$$

and we denote the category of groupoids by **Grpoid**. Now we recall the definition of crossed modules of groupoids. The basic reference is Brown-Higgins [6].

**Definition 2.1.** Let $\mathbf{G}$,$\mathbf{C}$ be groupoids over same object set and let $\mathbf{C}$ be totally disconnected. Then an action of $\mathbf{G}$ on $\mathbf{C}$ is partially defined function

$$\mathbf{C} \times \mathbf{G} \to \mathbf{C}$$

written $(c, a) \mapsto c^a$, which satisfies

I. $c^a$ is defined if only if $t(c) = s(a)$ and then $t(c^a) = t(a)$

II. $(c_1 \circ c_2)^a = c_1^a \circ c_2^a$

III. $c_1^{a \circ b} = (c_1^a)^b$ and $c_1^{e_x} = c_1$

for all $c_1, c_2 \in \mathbf{C}(x, x)$ and $a \in \mathbf{G}(x, y)$, $b \in \mathbf{G}(y, z)$.

**Definition 2.2.** *A crossed module of groupoids* consists of a morphisms $\delta : \mathbf{C} \to \mathbf{G}$ of groupoids $\mathbf{C}$ and $\mathbf{G}$ which is the identity on the object sets that $\mathbf{C}$ is totally disconnected, together with an action of $\mathbf{G}$ on $\mathbf{C}$ which satisfies

I. $\delta(c^a) = a^{-1} \circ \delta(c) \circ a$

II. $c^{\delta c_1} = c^{-1} \circ c \circ c_1$

for all $c_1, c \in \mathbf{C}(x, x)$ and $a \in \mathbf{G}(x, y)$. A crossed module of groupoids will be denoted by $(C, G, \delta)$. A morphism of crossed modules of groupoids from $(C, G, \delta)$ to $(C', G', \delta')$ is a pair of groupoid homomorphisms

$$\alpha : C \to C', \beta : G \to G'$$

such that $\alpha(c^a) = \alpha(c)^{\beta(a)}$ and $\delta'\alpha(c) = \beta\delta(c)$ for all $c \in \mathbf{C}(x, x)$ and $a \in \mathbf{G}(x, y)$. We thus get a category **CMod** of crossed modules of groupoids.

Using the definition of crossed modules of groupoids, we can define 2-crossed modules of groupoid similarly.

**Definition 2.3.** $L$,$M$ and $N$ are groupoids, having the same set of objects,O, and let $L$,$M$ be totally disconnected groupoids.

*A crossed modules of groupoids* consists of a complex of groupoids

$$\begin{tikzcd} L \arrow[r, "\partial_2"] \arrow[d, shift left] \arrow[d, shift left=2] & M \arrow[r, "\partial_1"] \arrow[d, shift left] \arrow[d, shift left=2] & N \arrow[d, shift left] \arrow[d, shift left=2] \\ O \arrow[u, shift left=2] \arrow[r, equal, "id"'] & O \arrow[u, shift left=2] \arrow[r, equal, "id"'] & O \arrow[u, shift left=2] \end{tikzcd}$$

together with an action of $N$ on $L$ and $M$, so that both $\partial_2$ and $\partial_1$ are morphisms of groupoid. Also

$$\{-, -\} : M \times M \to L$$

called a Peiffer lifting, which satisfies the following axioms:

$$\begin{array}{llrcl}
\mathbf{2CM1)} & & \partial_2\{m, m'\} & = & \left({}^{\partial_1 m}m'\right) \circ m \circ m'^{-1} \circ m^{-1} \\
\mathbf{2CM2)} & & \{\partial_2 l, \partial_2 l'\} & = & [l', l] \\
\mathbf{2CM3)} & (i) & \{m \circ m', m''\} & = & {}^{\partial_1 m}\{m', m''\}\{m, m' \circ m'' \circ m'^{-1}\} \\
 & (ii) & \{m, m' \circ m''\} & = & \{m, m'\}^{m \circ m' \circ m^{-1}}\{m, m''\} \\
\mathbf{2CM4)} & & \{m, \partial_2 l\}\{\partial_2 l, m\} & = & {}^{\partial_1 m}l \circ l^{-1} \\
\mathbf{2CM5)} & & {}^{n}\{m, m'\} & = & \{{}^{n}m, {}^{n}m'\}
\end{array}$$

for all $m, m', m" \in M$, $l, l' \in L$ and $n \in N$. A morphism of 2-crossed modules of groupoids is groupoid homomorphisms from $(L, M, N, \partial_2, \partial_1)$ to $(L', M', N', \partial_2', \partial_1')$

$$f_2 : L \to L', f_1 : M \to M', f_0 : N \to N'$$

such that

$$f_0\partial_1 = \partial_1' f_1, f_1\partial_2 = \partial_2' f_2,$$
$$f_1\left({}^n m\right) =^{f_0(n)} f_1(m), f_2\left({}^n l\right) =^{f_0(n)} f_2(l)$$

and

$$\{-,-\}f_1 \times f_1 = f_2\{-,-\}$$

for all $l \in L$, $m \in M$ and $n \in N$. We thus can define the category of 2-crossed modules of groupoids, denoting it by **2CMod**.

## 3. Relations Among Higher Order Crossed Modules

In this section, we would have given the following diagram by defining the functors between categories.

$$\begin{tikzcd}
\mathbf{XMod} \arrow[r, "F_1"] \arrow[d, "F_2"'] & \mathbf{Grpoid} \arrow[d, "F_6"] \\
\mathbf{X_2Mod} \arrow[r, "F_4"'] \arrow[d, "F_3"'] & \mathbf{CMod} \arrow[d, "F_7"] \\
\mathbf{X_3Mod} \arrow[r, "F_5"'] & \mathbf{2CMod}
\end{tikzcd}$$

3.1. **From crossed modules of groups to groupoids.** For this functor see also [4]. Let $\partial : M \to P$ be crossed module. In this case, there is an action of $P$ on $M$ so that we can define $M \rtimes P$, which is called semi direct product group. The operation in $M \rtimes P$ as follows:

$$(m,p).(m',p') = (mm'^p, pp')$$

Let $C_0 = P$ and $C_1 = M \rtimes P$. Now we must define $s, t, e$ and $\circ$ in $(C_1, C_0)$. The source, target maps and identity morphism are given by respectively

$$s(m,p) = p,\ t(m,p)\partial m.p,\ e(p) = (1,p)$$

The composition of two morphisms is given by

$$(m',p') \circ (m,p) = (m'm, p)$$

if $S(m',p') = p' = \partial m.p = t(m,p)$.
In this case, we obtain

$$\begin{aligned} s((m',p') \circ (m,p)) &= s(m'm,p) \\ &= p \\ &= s(m,p) \end{aligned}$$

and

$$\begin{aligned} t((m',p') \circ (m,p)) &= t(m'm,p) \\ &= \partial(m)\partial(m')p \\ &= \partial(m')p' \quad (\because \partial(m)p = p') \\ &= t(m',p') \end{aligned}$$

Furthermore

$$se(p) = s(1,p) = p \texttt{ and } se = id : C_0 \to C_0$$
$$te(p) = t(1,p) = \partial(1)p = p \texttt{ and } te = id : C_0 \to C_0$$

and we can show inverse morphism of $(m,p) : p \to \partial(m)p$ as follows

$$(m^{-1}, \partial(m)p) : \partial(m)p \to p.$$

Then we obtain a groupoid $(C_1, C_0, s, t, e)$. Therefore, we can define a functor from **XMod** to **Grpoid**. We denote this functor by

$$F_1 : \mathbf{XMod} \to \mathbf{Grpoid}$$

3.2. **From 2 crossed modules of groups to crossed modules of groupoids.** Let

$$K \xrightarrow{\partial_2} L \xrightarrow{\partial_1} M$$

be a 2-crossed module. There are actions of $M$ on $L$ and $K$ so we can define $K \rtimes M$ and $L \rtimes M$. For $(l,m) \in L \rtimes M$, $s(l,m) = m$ and $t(l,m) = \partial_1(l)m$,thus $L \rtimes M$ is groupoid over $M$. Moreover, for $(k,m) \in K \rtimes M$, $s(k,m) = m$ and $t(k,m) = \partial_1\partial_2(k)m = m$, so $K \rtimes M$ is totally disconnected groupoid.
We now try to show that the structure

$$K \rtimes M \xrightarrow{\delta} L \rtimes M$$

is a crossed modules of groupoid, where $\delta(k,m) = (\partial_2(k), m)$. Then we will use the action given by

$${}^{l}k = \{\partial_2(k), l\}.k$$

**CM1**. For $(k,m),(k',m) \in K \rtimes M$ and $(l,m) \in L \rtimes M$, we have

$$\begin{aligned}
\delta({}^{(l,m)}(k,m)) &= \delta({}^{l}k, m) = (\partial_2({}^{l}k), m)\\
&= (\partial_2(\{\partial_2(k), l\}.k), m)\\
&= ({}^{\partial_1\partial_2(k)}l.\partial_2(k)l^{-1}.\partial_2(k)^{-1}.\partial_2(k), m)\\
&= (l.\partial_2(k).l^{-1}, m) \circ (l,m) \circ (\partial_2(k), m) \circ (l,m)^{-1}\\
&= (l,m) \circ \delta(k,m) \circ (l,m)^{-1}
\end{aligned}$$

**CM2**. For $(k,m),(k',m) \in K \rtimes M$, we have

$$\begin{aligned}
{}^{\delta(k,m)}(k',m) &= {}^{(\partial_2(k),m)}(k',m) = ({}^{\partial_2(k)}k', m)\\
&= (\{\partial_2(k'), \partial_2(k)\}.k', m)\\
&= ([k,k']k', m)\\
&= (k,m) \circ (k',m) \circ (k,m)^{-1}
\end{aligned}$$

Then we obtain a crossed module of groupoids from 2-crossed module of groups. So we can define a functor from $\mathbf{X_2Mod}$ to **CMod**. We show this functor by

$$F_4 : \mathbf{X_2Mod} \to \mathbf{CMod}$$

3.3. **From crossed modules to 2-crossed modules of groups.** Let $M \xrightarrow{\partial_1} N$ be crossed module.

$$\{-,-\} : M \times M \to \{1\}$$

This map clearly satisfies the required conditions. Therefore, we can define a functor from **XMod** to $\mathbf{X_2Mod}$. We denote this functor by

$$F_2 : \mathbf{XMod} \to \mathbf{X_2Mod}$$

3.4. **From 2-crossed modules to 3-crossed modules of groups.** Let

$$\{\ ,\ \}_{(1)(0)} : L \times L \longrightarrow \{1\}, \quad \{\ ,\ \}_{(0)(2)} : L \times L \longrightarrow \{1\}, \quad \{\ ,\ \}_{(2)(1)} : L \times L \longrightarrow \{1\},$$

$$\{\ ,\ \}_{(1,0)(2)} : M \times L \longrightarrow \{1\}, \quad \{\ ,\ \}_{(2,0)(1)} : M \times L \longrightarrow \{1\},$$

$$\{\ ,\ \}_{(0)(2,1)} : L \times M \longrightarrow \{1\}, \quad \{\ ,\ \} : M \times M \longrightarrow L$$

be 3-dimensional Peiffer liftings.
**3CM1**. We showed that the structure

$$\{1\} \xrightarrow{\partial_2} L \xrightarrow{\partial_2} M$$

is a 2-crossed module in the previous section.
**3CM2**.$\{m,\partial_3 1\}_{(1,0)(2)}=1$ and $\{m,\partial_3 1\}_{(2,0)(1)}.{}^{m}1.{}^{\partial_1 m}1=1$.
**3CM3**.$\{\partial_3 1,m\}_{(0)(2,1)}=1$ and ${}^{m}1.1=1$
**3CM4**.$\{m,\partial_3 1\}_{(1,0)(2)}=1$ and $\{m,\partial_3 1\}_{(2,0)(1)}.\{\partial_3 1,m\}_{(0)(2,1)}.1.{}^{\partial_1 m}1=1$.
The other axioms of 3-crossed module can be showed similarly. Therefore, we can define a functor from $\mathbf{X_2Mod}$ to $\mathbf{X_3Mod}$. We denote this functor by

$$F_3:\mathbf{X_2Mod}\to\mathbf{X_3Mod}$$

3.5. **From 3-crossed modules to 2-crossed modules of over groupoids.** Let

$$K\xrightarrow{\partial_3}L\xrightarrow{\partial_2}M\xrightarrow{\partial_1}N$$

be a 3- crossed module. We try to show that structure

$$\begin{tikzcd}
K\rtimes N \arrow[r, "\delta_2"] \arrow[d, shift left] \arrow[d, shift left=2] & L\rtimes N \arrow[r, "\delta_1"] \arrow[d, shift left] \arrow[d, shift left=2] & M\rtimes N \arrow[d, shift left] \arrow[d, shift left=2] \\
N \arrow[u, shift left=2] \arrow[r, equal, "id"'] & N \arrow[u, shift left=2] \arrow[r, equal, "id"'] & N \arrow[u, shift left=2]
\end{tikzcd}$$

is a 2-crossed module over groupoid N.
In this structure, we obtain

$$\delta_1(l,n)=(\partial_2(l),n)$$
$$\delta_2(k,n)=(\partial_3(k),n)$$

and $s(l,n)=n$, $t(l,n)=\partial_1\circ\partial_2(l)n=n$ and $s(k,n)=n$, $t(k,n)=\partial_1\circ\partial_2\circ\partial_3(k)n=n$ then

$$L\rtimes N \underset{e}{\overset{s,t}{\rightrightarrows}} N\ ,\ K\rtimes N \underset{e}{\overset{s,t}{\rightrightarrows}} N$$

are totally disconnected groupoids.
We can define Peiffer lifting map for the above structure by using $\{-,-\}_{(2)(1)}:L\times L\to K$, namely.

$$\{-,-\}:(l\rtimes N)\times(l\rtimes N)\to K\rtimes N$$

$$\begin{array}{rcl}\{-,-\}:(l\rtimes N)\times(l\rtimes N) & \longrightarrow & K\rtimes N\\ ((l,n),(l',n)) & \longmapsto & (\{l,l'\}_{(2)(1)},n)\end{array}$$

We can show that all axioms of Peiffer lifting map given in Definition 2.3 are satisfied.

**PL.1**. It must be that

$$\delta_2\{(l,n),(l',n)\}=(l,n)\circ(l',n)\circ(l,n)^{-1}\circ^{\delta_1(l,n)}(l'^{-1},n)$$

For $(l,n),(l',n)\in L\rtimes N$ we obtain the following equality;

$$\begin{array}{rcl}\delta_2\{(l,n),(l',n)\} & = & \delta_2(\{l,l'\}_{(2)(1)},n)\\ & = & (\partial_3\{l,l'\}_{(2)(1)},n)\\ & = & (ll'l^{-1\partial_2(l)}(l')^{-1},n)\\ & = & (l,n)\circ(l',n)\circ(l,n)^{-1}\circ^{\delta_1(l,n)}(l'^{-1},n)\end{array}$$

**PL.2**. For $(k,n),(k',n)\in K\rtimes N$. We have

$$\begin{array}{rcl}\{\delta_2(k,n),\delta_2(k',n)\} & = & \{(\partial_3(k),n),(\partial_3(k'),n)\}\\ & = & (\{\partial_3(k),\partial_3(k')\}_{(2)(1)},n)\\ & = & ([k,k'],n)\\ & = & (k,n)\circ(k',n)\circ(k^{-1},n)\circ(k'^{-1},n)\\ & = & [(k,n),(k',n)]\end{array}$$

and this shows that the axiom **PL.2** is satisfied.

**PL.3.**For the elements $(l,n),(l',n),(l'',n)$ in $L\rtimes N$, we calculate the following result:

**i.**

$$\begin{aligned}
\{(l,n),(l'l'',n)\} &= (\{l,l'l''\}_{(2)(1)},n)\\
&= (\{l,l'\}_{(2)(1)}^{ll'l^{-1}}\{l,l''\}_{(2)(1)},n)\\
&= (\{l,l'\}_{(2)(1)}^{ll'l^{-1}},n)\circ(\{l,l''\}_{(2)(1)},n)\\
&= \{(l,n),(l',n)\}^{(l,n)\circ(l',n)\circ(l^{-1},n)}\{(l,n),(l'',n)\}.
\end{aligned}$$

**ii.**

$$\begin{aligned}
\{(ll',n),(l'',n)\} &= (\{ll',l''\}_{(2)(1)},n)\\
&= ({}^{\partial_1(l)}\{l',l''\}_{(2)(1)}\{l,l'l''l'^{-1}\}_{(2)(1)},n)\\
&= ({}^{\partial_1(l)}\{l',l''\}_{(2)(1)},n)\circ(\{l,l'l''l'^{-1}\}_{(2)(1)},n)\\
&= {}^{(\partial_1(l),n)}\{(l',n),(l'',n)\}\{(l,n),(l'',n)\circ(l'',n)\circ(l'^{-1},n)\}\\
&= {}^{\delta_2(l,n)}\{(l',n),(l'',n)\}\{(l,n),(l',n)\circ(l'',n)\circ(l'^{-1},n)\}.
\end{aligned}$$

**PL4.** For $(l,n)\in L\rtimes N$ and $(k,n)\in K\rtimes N$, we have

$$\begin{aligned}
\{(l,n),\delta_2(k,n)\}\{\delta_2(k,n),(l,n)\} &= \{(l,n),(\partial_3(k),n)\}\{(\partial_3(k),n),(l,n)\}\\
&= (\{l,\partial_3(k)\}_{(2)(1)},n)(\{\partial_3(k),l\}_{(2)(1)},n)\\
&= {}^{\delta_1(l,n)}(k,n)\circ(k^{-1},n).
\end{aligned}$$

**PL5.** For $(l,n),(l',n)\in L\rtimes N$ and $(m,n)\in M\rtimes N$, we have

$$\begin{aligned}
{}^{(m,n)}\{(l,n),(l',n)\} &= {}^{(m,n)}(\{l,l'\}_{(2)(1)},n)\\
&= ({}^{m}\{l,l'\}_{(2)(1)},n)\\
&= (\{{}^{m}l,{}^{m}l'\}_{(2)(1)},n)\\
&= \{({}^{m}l,n),({}^{m}l',n)\}\\
&= \{{}^{(m,n)}(l,n),{}^{(m,n)}(l',n)\}.
\end{aligned}$$

Therefore, we can define a functor from the category of 3-crossed module to that of 2-crossed module of groupoids. We denote this functor by

$$F_5:\mathbf{X_3Mod}\to\mathbf{2CMod}$$

3.6. **From groupodis to crossed module of groupoids.** Let $N \underset{e}{\overset{s,t}{\rightrightarrows}} O$ be groupoid, so it can be clearly seen that

$$\{e\}\xrightarrow{i} N \underset{e}{\overset{s,t}{\rightrightarrows}} O$$

is crossed module over $O$, where $i$ is inclusion map. Therefore, we can define a functor from the category of groupois to **CMod**. We show this functor by

$$F_6:\mathbf{Grpoid}\to\mathbf{CMod}$$

3.7. **From crossed module of groupoids to 2-crossed module of groupoids.** We can obtain a 2-crossed module of groupoids from crossed module over groupoids by using smiler way in subsection 3.3.

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