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A combinatorial approach to the classification of resolution graphs of weighted homogeneous plane curve singularities

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Abstract

In this article we describe the classification of the resolution graphs of weighted homogeneous plane curve singularities in terms of their weights by using the concepts of graph theory and combinatorics. The classification shows that the resolution graph of a weighted homogeneous plane curve singularity is always a caterpillar.

Keywords: Plane Curves, Weights, Caterpillar.

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1. Introduction

The history of resolution of plane curve singularities is very old. It started with Newton in 1676, who showed the existence of Puiseux series. The resolution of plane curve singularities is an easy consequence. There is a large group of mathematicians who introduced new methods to resolve a plane curve singularity and they found deep and important applications of resolution of plane curve singularities. János Kollár lists about 20 ways of resolution of plane curve singularities (cf. [5]). Moreover an algebraic and combinatorial information about plane curve singularities can be found in [7], [9].
A *graph* is an ordered pair $G = (V, E)$, where V is called vertex set and E is called edge set. $|V|$ and $|E|$ denote the order and the size of a graph, respectively. A *tree* is an acyclic connected graph with n vertices and $n - 1$ edges and a *caterpillar* is a special type of tree with the property that a path remains if all leaves are deleted. A *vertex labeling* is

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a bijection from the set of vertices V to the set of labels $\{1, 2, \ldots, |V|\}$.
In [2], S. Dale Cutkosky and H. Srinivasan compute the resolution graph combinatorically by using the characteristic pairs of an irreducible plane curve singularity. Also Y. Jingen in [4], associates each singularity of a curve on a surface to a tree called S-tree which is some kind of "structured graph" and obtained by using the procedure of blow-ups.
In this article we introduce a combinatorial approach to compute the resolution graph of a weighted homogeneous plane curve singularity. As a first step we compute the order and the size of the resolution graph of a weighted homogeneous plane curve singularity. Then we define a vertex labeling on the set of vertices obtained from the weights of a weighted homogeneous plane curve singularity and finally construct the resolution graph.

2. Basic Definitions

In this section, we give some basic definitions related to the resolution of plane curve singularities. Definitions can be found in [3].

2.1. Definition. Let $Bl(0)_{\mathbb{C}^2}$ denote the $blowing-up$ of $0 \in \mathbb{C}^2$, which is the subset of $\mathbb{C}^2 \times P$, where P is the projective line and it is defined as

$$Bl(0)_{\mathbb{C}^2} = \{(a, l_a) : a \in l_a\},$$

where a is a point and l_a is a line in $\mathbb{C}^2$ on which point a lies.
Then there is a projection map

$$\pi : Bl(0)_{\mathbb{C}^2} \to \mathbb{C}^2$$

which is called the blowing-up map.

We denote $E := \pi^{-1}(0)$ the exceptional divisor of π.
$Bl(0)_{\mathbb{C}^2}$ can also be defined in coordinates as follows

$$Bl(0)_{\mathbb{C}^2} = \{(x, y, u : v) : xv = yu\} \subset \mathbb{C}^2 \times P.$$

here $(x, y) \in \mathbb{C}^2$ and $(u : v)$ are the homogeneous coordinates of P.
Let

$$V_1 = \{(x, xv, 1 : v) : x, v \in \mathbb{C}^2\} \cong \mathbb{C}^2$$
$$V_2 = \{(yu, u, u : 1) : y, u \in \mathbb{C}^2\} \cong \mathbb{C}^2$$

then $V_1 \cup V_2 = Bl(0)_{\mathbb{C}^2}$ is an affine covering.

2.2. Definition. Let $(V(f), 0)$ be a curve singularity. Then the closure of $\pi^{-1}(f \setminus 0)$ is called the strict transform of f, and the inverse image $\pi^{-1}(f)$ is called the total transform of f.

Let $f = \bigcup\limits_{i=1}^{r} f_i \subset \mathbb{C}^2$ be a small representative of a reducible plane curve singularity with branches $f_1, ..., f_r$ $\;r \geq 1$. Assume that $X_i \overset{\pi_i}{\to} ... \overset{\pi_2}{\to} X_1 \overset{\pi_1}{\to} \mathbb{C}^2$ is a sequence of blowing up points. Denote by $E^{(i)} = (\pi_1 \circ ... \circ \pi_i)^{-1}(0)$ the exceptional divisor, $f^{(i)} = \overline{(\pi_1 \circ ... \circ \pi_i)^{-1}(f \setminus \{0\})}$ the strict transform and $(\pi_1 \circ ... \circ \pi_i)^{-1}(f)$ the total transform of f. Let $X_{i+1} \overset{\pi_{i+1}}{\to} X_i$ be the blowing up of X_i in all points of $f^{(i)} \cap E^{(i)}$ which are still singular on $f^{(i)}$ or non-transversal intersection of $f^{(i)}$ with $E^{(i)}$ that is the points with intersection multiplicity of $f^{(i)}$ and $E^{(i)}$ greater than one or where two exceptional divisors and $f^{(i)}$ meet.

2.3. Definition. (i) $X_k \overset{\pi_k}{\to} ... \overset{\pi_1}{\to} X_1 \to \mathbb{C}^2$ is called a standard resolution of $(V(f), 0)$ if all branches of $f^{(k)}$ are smooth, do not intersect each other, do intersect just one component of $E^{(k)}$ and do intersect this component transversally.

We consider the following weighted graph, the resolution graph of f.
(i) To each component of $E^{(k)}$ a point $\bullet$ is associated.
(ii) To each component of $f^{(k)}$ a point $*$ is associated.
(iii) Two points are connected by an edge if the corresponding components intersect.
(iv) The points of type(i) are weighted. Let E be a component of $E^{(k)}$.We give to the corresponding point the weight i if E is created in the $i-th$ level of the blowing ups that is i is minimal such that $\pi_{i+1} \circ ... \circ \pi_n(E)$ is not a point.

2.4. Definition. Let n, m be positive integers then by the Euclidean algorithm, we can expand $\frac{n}{m}$ in a continued fraction in nonnegative integers c_i,

$$\frac{n}{m} = c_1 + \cfrac{1}{c_2 + \cfrac{1}{c_3 + \cfrac{1}{\ddots + \cfrac{1}{c_n}}}} \tag{2.1}$$

We denote it by $\frac{n}{m} = [c_1, c_2, \dots, c_n]$.

3. Classification of Resolution Graphs of Weighted Homogeneous plane Curve Singularities

The type of nondegenerate quasihomogeneous polynomials in two variables according to V.I. Arnold, S. M. Gusein-Zade and A. N. Varchenko [1] is given in the following table.

Table 1

Type	Quasihomogeneous polynomial	Weighted Vector
I	$x^a + y^b \quad a, b \in \mathbb{Z}_{>0}$	(b, a, ab)
II	$x^a y + y^b \quad a \in \mathbb{Z}_{>0}, b \in \mathbb{Z}_{>1}$	$(b-1, a, ab)$
III	$x^a y + y^b x \quad a, b \in \mathbb{Z}_{>1}$	$(b-1, a-1, ab-1)$

3.1. Proposition. *Any weighted homogeneous polynomial $f \in \mathbb{C}[x, y]$ defining an isolated singularity is of the type $f = f_0 + h$, where f_0 is one of the form given in the table.*1 *and h consist on terms having the same degree as f_0.*

Proof. See [1]. □

3.2. Remark. The resolution of weighted homogeneous polynomial f defining an isolated singularity does not depend on h and depends only on weights and degree (see Theorem-3.5 in [6]).

3.3. Definition. $(V(f), 0) \subseteq (\mathbb{C}^2, 0)$ is called a quasihomogeneous plane curve singularity, if there exist an automorphism

$$\phi : \mathbb{C}[[x, y]] \to \mathbb{C}[[x, y]]$$

such that $\phi(f)$ is a weighted homogeneous plane curve singularity.

The following proposition gives us a combinatorial formula to compute the order and size of the resolution graph of a weighted homogeneous plane curve singularity.

3.4. Proposition. *Let G_f denote the resolution graph of a plane curve singularity $(V(f),0)$, where f is weighted homogeneous polynomial defining an isolated singularity is of the type $f = f_0 + h$, where f_0 is one of the form given in the table.1 and h consist on terms having the same degree as f_0. Then if*

- **i:** *f is of type-I then the order of G_f is $\sum_{i=1}^{n} c_i + gcd(a,b)$, where $\frac{b}{a} = [c_1, c_2, \ldots, c_n]$.*
- **ii:** *f is of type-II then the order of G_f is $\sum_{i=1}^{n} c_i + gcd(a, b-1) + 1$, where $\frac{b-1}{a} = [c_1, c_2, \ldots, c_n]$.*
- **iii:** *f is of type-III then the order of G_f is $\sum_{i=1}^{n} c_i + gcd(a-1, b-1) + 2$, where $\frac{b-1}{a-1} = [c_1, c_2, \ldots, c_n]$.*

Moreover G_f is always a caterpillar.

Proof. **i:** If $(V(f),0)$ be a weighted homogeneous plane curve singularity of type-I. Then it is noted that number of branches of the plane curve singularity is $gcd(a,b)$ and resolution graphs G_f and G_{f_0} are same (see remark 3.2). We consider $a \leq b$. The case for $a > b$ can be treated in a similar way.
We start the resolution of singularity by the following blow up

$$x \to xy, y \to y$$

(This chart is only considered since the exceptional divisor does not intersect the curve in the other chart.)
Then we have the strict transformation $x^a + y^{b-a} = 0$ and exceptional divisor $E_1 : y = 0$. After $[\frac{b}{a}] = c_1$ blow ups we have the strict transformation $x^a + y^{b-c_1a} = 0$ such that $b - c_1a < a$. Then multiplicity of strict transformation dropped and is equal to $b - c_1a$ and exceptional divisor $E_{c_1} : y = 0$. Then make the blow up

$$x \to x, y \to xy$$

we get the strict transformation $x^{a-(b-c_1a)} + y^{b-c_1a} = 0$ and exceptional divisor $E_{c_1+1} : x = 0$. After $[\frac{a}{c}] = c_2$ blow ups we have the strict transformation $x^{a-c_2(b-c_1a)} + y^{b-c_1a} = 0$ such that $a - c_2(b - c_1a) < b - c_1a$. Then multiplicity of strict transformation dropped and becomes equal to $a - c_2(b - c_1a)$ and exceptional divisor $E_{c_1+c_2} : y = 0$. Continue in this way, after $c_1 + c_2 + \cdots + c_n$ blow ups we get the standard resolution. So the number of vertices of G_f is $\sum_{i=1}^{n} c_i + gcd(a,b)$ and if we construct the dual graph of this resolution as described in section-2 then we find G_f is a caterpillar.
ii and **iii** can be proved similarly to **i**. □

3.5. Remark. In the above proposition $\sum_{i=1}^{n} c_i$ is the number of dot vertices which represents the number of blow-ups required to make the standard resolution and $gcd(a,b)$ is the number of star vertices of the resolution graph which denote the number of branches of weighted homogeneous plane curve singularity of type-I.

3.6. Remark. If $\frac{n}{m} = [c_1, c_2, \ldots, c_n]$ then in the following proposition, the integers e_i, a_t and b_s denote the sum $\sum_{k=1}^{i} c_k$, the weight of the vertex v_t and the weight of the vertex v_s respectively.

In the following proposition we describe a combinatorial construction to compute the resolution graph of a weighted homogeneous plane curve singularity of type-I.

3.7. Proposition. *Let $(V(f),0)$ be a weighted homogeneous plane curve singularity of type-I then its resolution grpah G_f can be obtained by using the following construction.*

Proof. Since $(V(f),0)$ be a weighted homogeneous plane curve singularity of type-I, then from Proposition-3.4 we have the set of vertices

$$V = \{v_i : 1 \leq i \leq \sum_{i=1}^{n} c_i + gcd(a,b)\},$$

where $\frac{b}{a} = [c_1, c_2, \ldots, c_n]$. Then we can define the integers such that $e_1 < e_2 < \cdots < e_n$, where $e_i = \sum_{k=1}^{i} c_k$ and a partion on the set of vertices V such that $V = V_\bullet^{(1)} \cup V_\bullet^{(2)} \cup V_\bullet^{(3)} \cup V_*$, where

$$V_\bullet^{(1)} = \{v_i : 1 \leq i \leq e_1\},$$

$$V_\bullet^{(2)} = \{v_i : e_1 + 1 \leq i \leq e_1 + l\},$$

$$V_\bullet^{(3)} = \{v_i : e_1 + l + 1 \leq i \leq e_n\},$$

where

$$l = \begin{cases} \sum_{j=1}^{\frac{n-2}{2}} (e_{2j+1} - e_{2j}), & \text{if n is even;} \\ \sum_{j=1}^{\frac{n-1}{2}} (e_{2j+1} - e_{2j}), & \text{if n is odd;} \end{cases}$$

$$V_* = \{v_i^* : 1 \leq i \leq gcd(a,b)\}.$$

Now define

$$A := \{e_2+1, \ldots, e_3, e_4+1, \ldots, e_5, \ldots, e_{2[\frac{n-1}{2}]}+1, \ldots, e_{2[\frac{n-1}{2}]+1}\} = \{a_{e_1+1}, \ldots, a_{e_1+l}\}$$

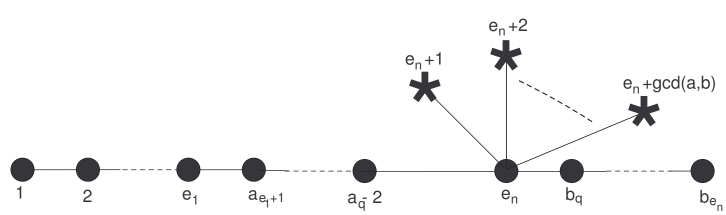
and

$$B := \{e_1+1, \ldots, e_2, e_3+1, \ldots, e_4, \ldots, e_{2[\frac{n-1}{2}]+1}+1, \ldots, e_{2[\frac{n}{2}]}\} = \{b_{e_n}, b_{e_n-1}, \ldots, b_{e_1+l+1}\}.$$

Note that $|V_\bullet^{(2)}| = |A|$ and $|V_\bullet^{(3)}| = |B|$.
Let

$$s = \begin{cases} 1, & \text{if n is even;} \\ 0, & \text{if n is odd;} \end{cases}$$

and $q = e_1 + l + 1 + s$ then we obtain the following resolution graph



$\square$

3.8. Example. Consider $(V(f),0)$ be a weighted homogeneous plane curve singularity with weights $(230, 1055)$.
Then $V = \{v_1, v_2, \ldots, v_{13}, v_1^*, v_2^*, v_3^*, v_4^*, v_5^*\}$ (see Proposition-3.4). Now by using the Proposition-3.7 we can construct the following data:
Step − 1 : Construct

$$e_1 = 4, e_2 = 5, e_3 = 6, e_4 = 8, e_5 = 10, e_6 = 11, e_7 = 13.$$

Step − 2 : Since $n = 7$ which is odd therefore $V = V_\bullet^{(1)} \cup V_\bullet^{(2)} \cup V_\bullet^{(3)} \cup V_*$, where

$$V_\bullet^{(1)} = \{v_1, v_2, v_3, v_4\},$$
$$V_\bullet^{(2)} = \{v_5, v_6, v_7, v_8, v_9\},$$
$$V_\bullet^{(3)} = \{v_{10}, v_{11}, v_{12}, v_{13}\},$$

and

$$V_* = \{v_1^*, v_2^*, v_3^*, v_4^*, v_5^*\}.$$

Step − 3 :

$$A = \{6, 9, 10, 12, 13\} = \{a_5, a_6, a_7, a_8, a_9\}$$

and

$$B = \{5, 7, 8, 11\} = \{b_{13}, b_{12}, b_{11}, b_{10}\}.$$

Step − 4 : Attach all $*$ vertices with v_9 then we get the resolution graph as given in Figure 1.

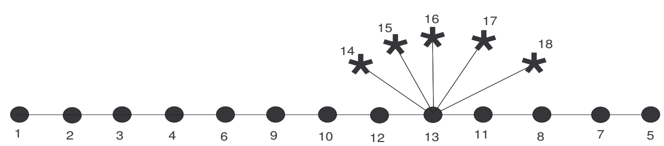


Figure 1. Resolution graph of a weighted homogeneous plane curve singularity with weights $(230, 1055)$.

In the following two propositions we describe a combinatorial construction to compute the resolution graph of weighted homogeneous plane curve singularities of type II and III.

3.9. Proposition. *Let $(V(f),0)$ be a weighted homogeneous plane curve singularity of type-II then its resolution graph G_f is one of the graphs given in Table 2.*

Proof. Since $(V(f),0)$ be a weighted homogeneous plane curve singularity of type-II, then we have

$$V = \{v_i : 1 \leq i \leq \sum_{i=1}^{n} c_i + (gcd(a, b-1) + 1)\},$$

Make a partition of V such that $V = V^{'} \cup V_*^{(1)}$, where

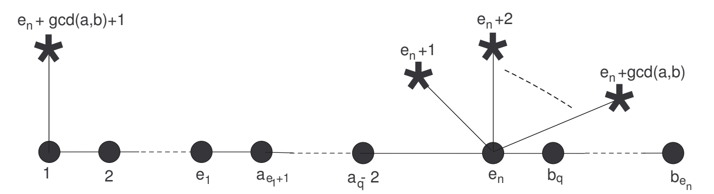
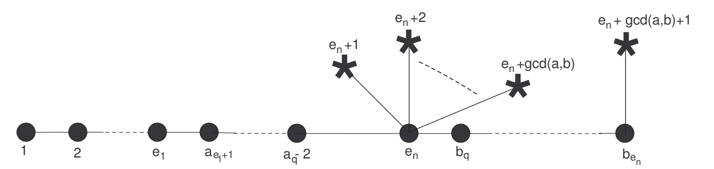
$$V^{'} = \{v_i : 1 \leq i \leq \sum_{i=1}^{n} c_i + gcd(a, b-1)\}$$

and

$$V_*^{(1)} = \{v_i^* : i = gcd(a, b-1) + 1\}.$$

For $V^{'}$ follow the construction as explained in Proposition-3.7. Assign the weight $e_n + gcd(a, b-1) + 1$ to the $*$ vertex v_i^* for $i = gcd(a, b-1) + 1$. If $a \leq b - 1$ then attach

Table 2

<table>
<tr><th>Type</th><th>Condition</th><th>Resolution Graph</th></tr>
<tr><td>II</td><td>$a \leq b-1$</td><td>
</td></tr>
<tr><td>II</td><td>$a > b-1$</td><td>
</td></tr>
</table>

v_i^* with the $\bullet$ vertex of weight 1. If $a > b-1$ then attach v_i^* with the vertex of $V_\bullet^{'(3)}$ with smallest weight. $\square$

3.10. Proposition. *Let $(V(f), 0)$ be a weighted homogeneous plane curve singularity of type-III then its resolution grpah G_f is the following:*

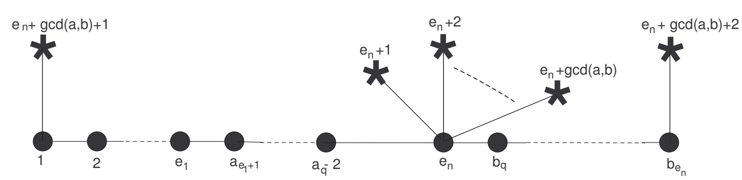


Figure 2. Resolution graph of $(V(f), 0)$

Proof. Since $(V(f), 0)$ be a weighted homogeneous plane curve singularity of type-III, then we have

$$V = \{v_i : 1 \leq i \leq \sum_{i=1}^{n} c_i + (gcd(a-1, b-1) + 2)\},$$

Make a partition of V such that $V = V^{'} \cup V_*^{(1)}$, where

$$V^{'} = \{v_i : 1 \leq i \leq \sum_{i=1}^{n} c_i + gcd(a-1, b-1)\}$$

and

$$V_*^{(1)} = \{v_i^* : i = gcd(a-1, b-1) + 1, gcd(a-1, b-1) + 2\},$$

For $V^{'}$ follow the construction as explained in Proposition-3.7. Assign the weights $e_n + gcd(a-1, b-1) + 1$, $e_n + gcd(a-1, b-1) + 2$, to the $*$ vertices v_i^* for $i = gcd(a-1, b-1) + 1$ and $i = gcd(a-1, b-1) + 2$ respectively. Attach v_i^* for $i = gcd(a-1, b-1) + 1$ with the $\bullet$ vertex of weight 1 and attach v_i^* for $i = gcd(a-1, b-1) + 2$ with the $\bullet$ vertex of $V_\bullet^{'(3)}$ with smallest weight. $\square$

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