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A Sequence Bounded Above by the Lucas Numbers

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Abstract

In this work, we consider the sequence whose n^{th} term is the number of h-vectors of length n. The set of integer vectors $E(n)$ is introduced. For $n \geq 2$, the cardinality of $E(n)$ is the n^{th} Lucas number L_n is showed. The relation between the set of h-vectors $L(n)$ and the set of integer vectors $E(n)$ is given.

Keywords: Cardinality, h-vectors, Hilbert function, Lucas numbers

1. Introduction

Firstly, we give the well-known definitions of the Fibonacci and Lucas numbers. The Fibonacci numbers F_n are the terms of the sequence 1,1,2,3,5,8,13,21,34,55,89,… . Every Fibonacci number, except the first two, is the sum of the two previous Fibonacci numbers. The numbers F_n satisfy the second order linear recurrence relation.

$$F_n = F_{n-1} + F_{n-2}\,,\ n = 2,3,4, \qquad (1)$$

with initial values $F_0 = 0$, $F_1 = 1$.

The Lucas numbers L_n are defined

$$L_n = L_{n-1} + L_{n-2}\,,\ n = 2,3,4,\dots \qquad (2)$$

with initial conditions $L_0 = 2$, $L_1 = 1$. The first a few Lucas numbers are 2,1,3,4,7,11,18,29,47,76,… .

Hilbert functions of graded rings are more convenient for many applications and are known to relate to many different subjects such as dimensions, multiplicity and Betti numbers (see: Bruns and Herzog, [1]). In [2], Enkoskoy and Stone introduced recursion formulas related to Hilbert functions. They showed the n^{th} term of sequence, whose n^{th} term is the number of h-vectors of length n, is bounded above by the n^{th} Fibonacci number. Ozkan et al. [4] introduced the cardinality of the M-sequence of length n is bounded above by the n^{th} Lucas number.

The aim of this paper is to show the sequence defined by the number of h-vectors of length n is bounded above by the sequence of Lucas numbers. This paper is organized as follows. In Section 2 we give some concepts of h-vectors. Section 3 presents main results of this paper.

2. Materials and Methods

We first give some necessary background on Hilbert functions and h −vectors.

Let $R = k[x_1, x_2, \dots x_n]$ be a polynomial ring over a field k with the standard grading. In particular, $degx_i = 1$ for $1 \leq i \leq n$. If I is a graded ideal, the quotient ring $R/_I$ is also graded and we denote by $(R/_I)_t$ the k vector space of all degree t homogeneous elements of $R/_I$. The Hilbert function $H_{R/_I}: \mathbb{Z}_{\geq 0} \to \mathbb{Z}_{\geq 0}$ is defined to be the k vector space dimension of each graded

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component, i.e. $H_{R/I}(t) := dim_k(R/I)_t$. If the Krull dimension of the graded quotient ring is zero, there exists an $s \geq 0$ such that $H_{R/I}(s) \neq 0$ but $H_{R/I}(t) = 0$ for all $t > s$. In this case the h-vector of R/I is defined as

$$h(R/I) = \left(H_{R/I}(0), H_{R/I}(1), H_{R/I}(2), \ldots, H_{R/I}(s)\right) \quad (3)$$

Thus the h-vector of R/I has finitely many non-zero entries. The length of R/I is the k vector space dimension of R/I, denoted $\lambda(R/I)$. In particular $\lambda(R/I) = \sum_{i=0}^{s} H_{R/I}(s)$. Throughout this paper we will refer to $\lambda(R/I)$ as the length of $h(R/I)$.

The sequence $\{l(n)\}_{n\geq 1}$ is defined by the number of h-vectors of length n. In particular, for $n \geq 1$ we define

$$L(n) = \{h = (h_0, h_1, \ldots) \mid h\ is\ an\ h-vector\ and\ \textstyle\sum_i h_i = n\} \quad (4)$$

and set $l(n) = |L(n)|$.

Using Macaulay's Theorem, the authors of [2] constructed the h-vectors of length at most 7. The h-vectors of length at most 6 is given in Table 1. We write $t_0 t_1 \ldots t_s$ for the h-vector $(t_0, t_1, \ldots, t_s)$.

Table 1

λ	1	2	3	4	5	6
	1	11	111	1111	11111	111111
			12	121	1211	12111
				13	122	1221
					131	123
					14	1311
						132
						141
						15
Total	1	1	2	3	5	8

Definition 2.1. [3] For $n \geq 1$, the set of integer vectors $B(n)$ is defined recursively as follows:

1. $B(1) = \{(1)\}$,
2. $B(2) = \{(1,1)\}$,
3. For $n \geq 3$ define $B(n) := C(n) \cup D(n)$ where

$$C(n) := \{(1, t_1, \ldots, t_s, 1) \mid (1, t_1, \ldots, t_s) \in B(n-1)\}$$

$$D(n) := \{(1, t_1, \ldots t_s + 1) \mid (1, t_1, \ldots, t_s) \in B(n-1), with\ t_s - 1 > 1\ or\ s = 1\}.$$

Theorem 2.2. [3] The cardinality of $B(n)$ is the n^{th} Fibonacci number F_n.

Theorem 2.3. [2] For all $n \geq 1$, $L(n) \subseteq B(n)$. In particular the sequence of the cardinality of $L(n)$ is bounded above by the Fibonacci sequence.

Definition 2.4. For $n \geq 1$, the set of integer vectors $E(n)$ is defined recursively as follows:

1. $E(1) = \{(1)\}$,
2. $E(2) = \{(1,1,1), (1), (1,2)\}$,
3. For $n \geq 3$ define $E(n) := R(n) \cup S(n)$ where

$$R(n) := \{(1, t_1, \ldots, t_s, 1) \mid (1, t_1, \ldots, t_s) \in E(n-1)\},$$

$$S(n) := \{(1, t_1, \ldots, t_s + 1) \mid (1, t_1, \ldots, t_s) \in E(n-1), \text{with } t_{s-1} > 1 \text{ or } s = 1\}.$$

We set $e(n) = |E(n)|$.

Remark 2.5. It is worth noticing that the sets $R(n)$ and $S(n)$ of Definition 2.4 form a set partition of $E(n)$.

The first few sets $E(n)$ are

$E(1) = \{(1)\}$,

$E(2) = \{(1,1,1), (1), (1,2)\}$,

$E(3) = \{(1,1,1,1), (1,1), (1,2,1), (1,3)\}$,

$E(4) = \{(1,1,1,1,1), (1,1,1), (1,2,1,1), (1,2,2), (1,3,1), (1,2), (1,4)\}$,

$E(5) = \{(1,1,1,1,1,1), (1,1,1,1), (1,2,1,1,1), (1,2,2,1), (1,3,1,1), (1,2,1), (1,4,1) (1,2,3), (1,3,2), (1,3), (1,5)\}$.

In Table 2, the integer vectors of length at most 6 and cardinality of integer sets is given. We write $t_0 t_1 \ldots t_s$ for the h-vector $(t_0, t_1, \ldots, t_s)$.

Table 2

n	1	2	3	4	5	6
	1	111	1111	11111	111111	1111111
		1	11	111	1111	11111
		12	121	1211	12111	121111
			13	122	1221	12211
				12	1311	1231
				14	121	1321
					141	1411
					123	1211
					132	131
					13	151
					15	1222
						124
						133
						142
						122
						14
						16
Total	1	3	4	7	11	18

3. Main Results

Theorem 3.1. The $e(n)$ is the n^{th} Lucas number L_n, for $n \geq 2$.

Proof. We shall prove by induction that, for all $n \geq 1$. When $n = 1$, the claim is true, since $e(1) = L_1 = 1$. Since $e(2) = L_2 = 3$, the claim is true for $n = 2$.

Suppose the claim is true for all $n = s$, that is $e(s) = L_s$. Then

$$e(s) + e(s-1) = L_s + L_{s-1} = L_{s+1}. \quad (5)$$

Thus the claim holds for $n = s + 1$, that is $e(s+1) = |E(s+1)| = L_{s+1}$.

Theorem 3.2. For all $n \geq 2$, $L(n+1) \subseteq E(n)$. In particular, the sequence $l(n+1)$ is bounded from above by the Lucas sequence.

Proof. Note that $L(n)$ is the set of all integer vectors $(1, t_1, \ldots, t_s)$ with $1 + t_1 + t_2 + \cdots + t_s = n + 1$ and the property that if $t_i = 1$ then $t_j = 1$ for all $j \geq i$. We will prove this by induction for all $n \geq 2$. For $n = 2$, the claim is true, since $L(3) \subseteq E(2)$:

$L(3) = \{(1,1,1), (1,2)\}$ and $E(2) = \{(1,1,1), (1), (1,2)\}$.

When $n = 3$, the claim is true, since $L(4) \subseteq E(3)$:

$L(4) = \{(1,1,1,1), (1,2,1), (1,3)\}$ and $E(3) = \{(1,1,1,1), (1,1), (1,2,1), (1,3)\}$.

Suppose $L(k+1) \subseteq E(k)$, for $n = k$. We have to show that the claim is true for $n = k + 1$, that is, $L(k+2) \subseteq E(k+1)$.

Denote the number of element of a set A by $s(A)$. Then

$$L(k) \subseteq E(k-1) \Rightarrow s\big(L(k)\big) \leq s\big(E(k-1)\big),$$

$$L(k+1) \subseteq E(k) \Rightarrow s\big(L(k+1)\big) \leq s\big(E(k)\big),$$

Since $L(k) \cap L(k+1) = \emptyset$, this also gives

$$s\big(L(k) \cup L(k+1)\big) = s\big(L(k)\big) + s\big(L(k+1)\big).$$

Since $L(k) \subseteq E(k-1)$ and $L(k+1) \subseteq E(k)$, we set

$$L(k) \cup L(k+1) \subseteq E(k-1) \cup E(k). \quad (7)$$

Similarly, since $E(k-1) \cap E(k) = \emptyset$, we get

$$s\big(E(k-1) \cup E(k)\big) = s\big(E(k-1)\big) + s\big(E(k)\big).$$

We then get from (7)

$$s\big(L(k)\big) + s\big(L(k+1)\big) \leq s\big(E(k-1)\big) + s\big(E(k)\big). \quad (8)$$

Hence

$$s\big(L(k)\big) + s\big(L(k+1)\big) = s\big(L(k+2)\big),$$

$$s\big(E(k-1)\big) + s\big(E(k)\big) = s\big(E(k+1)\big).$$

We know $s\big(L(k+2)\big) \leq s\big(E(k+1)\big)$. Hence $L(k+2) \subseteq E(k+1)$.

Theorem 3.3. For all $n \geq 2$, we have the relation

$$E(n) \setminus L(n+1) = L(n-1).$$

Proof. We will prove this by induction for all $n \geq 2$. When $n = 2$, the claim is true, since $E(2) \setminus L(3) = L(1)$. For $n = 3$, the claim is true, since

$E(3) \setminus L(4) = L(2)$. Suppose that the claim is true for $n = s$, that is $E(s) \setminus L(s+1) = L(s-1)$.

We have to show that the claim is true for $n = s + 1$, that is, $E(s+1) \setminus L(s+2) = L(s)$.

The identity $E(s) \setminus L(s+1) = L(s-1)$ implies $E(s) = L(s-1) \cup L(s+1)$. From the last equality, it can be easily seen that

$$E(s+1) \setminus L(s+2) = L(s). \tag{9}$$

Example 3.4.

$$\begin{aligned} E(4)\backslash L(5) &= \{(1,1,1,1,1),(1,1,1),(1,2,1,1),\\ &\quad (1,2,2),(1,3,1),(1,2),(1,4)\}\backslash\{(1,1,1,1,1),\\ &\quad (1,2,1,1),(1,2,2),(1,3,1),(1,4)\}\\ &= \{(1,1,1),(1,2)\} = L(3) \end{aligned}$$

Corollary 3.5. For all $n \geq 2$, we have $|E(n)| - |B(n+1)| = |B(n-1)|$.

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