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Soft representation of soft groups

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Abstract: In this paper, we introduce the notion of soft representation of a soft group and obtain basic properties of soft representation of soft groups using the definition of soft sets and soft group. Also we study the relationship between soft representation of soft groups and soft G-Modules. Moreover we examine irreducibility, reducibility and complete reducibility of soft representations.

Keywords: Soft set, soft group, soft representation, soft G-module.

1 Introduction

Dealing with uncertainties is a major problem in many areas such as engineering, medical science, environmental science, social science et al. These kinds of problem cannot be dealt with by classical methods, because classical methods have inherent difficulties. To overcome these kinds of difficulties, Molodtsov [1] proposed a completely new approach, which is called soft set theory, for modeling uncertainty. Then Maji et al. [2] introduced several operations on soft sets. Aktaş and Çağman [3] defined soft groups and obtained the main properties of these groups. Moreover, they compared soft sets with fuzzy sets and rough sets. Besides, Jun et al. [4] defined soft ideals on BCK/ BCI-algebras. Feng et al. [5] defined soft semiring, soft ideals on soft semiring and idealistic soft semiring. Sun et al. [6] defined the concept of soft modules and studied their basic properties. Acar et al. [7] defined soft rings and have introduced their initial basic properties such as soft ideals, soft homomorphisms etc. by using soft set theory. Aktaş [8] defined bijective soft groups, dependent and independent soft groups. Some of their properties are investigated. Also he investigate the relationships between bijective soft groups and classical groups. Aktaş [8] introduced order of the soft groups, power of the soft sets, power of the soft groups, and cyclic soft group on a group. They also investigate the relationship between cyclic soft groups and classical groups. Sezer et al. [10] introduced union soft subrings and union soft ideals of a ring and union soft submodules of a left module and investigate their related properties with respect to soft set operations, anti-image and lower α-inclusion of soft sets. Jun et al. [11] introduced further properties and characterizations of int-soft left (right) ideals are studied, and the notion of int-soft (generalized) bi-ideals. Şahin et al. [12] introduced the notions of soft G-module and obtain basic properties of such soft G-modules using Molodtsov's definition of the soft sets. They examine irreducibility, reducibility and complete reducibility of soft G-modules. In [13], Sun et al., introduced the definition of soft modules and constructed some basic properties of soft modules.

In this study, we introduce the notion of soft representation of a soft group and obtain basic properties of soft representation of soft groups using the definition of soft sets and definition of soft group. In section 2, well-known results of some preliminaries are given. In section 3, soft representation of a soft group is studied. Finally section 4, presents the conclusion of our work.

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2 Preliminaries

Let U be an initial universe and E be a set of parameters. Let $P(U)$ denote the power set of U and A be a non-empty subset of E. Molodtsov defined the soft set in the following manner.

Definition 1. [1]. *Let U be an initial universe set and E be a set of parameters. Consider $A \subseteq E$. Let $P(U)$ denote the set of all soft set of U. The collection(F,A) is termed to be the soft set over$U$ where F is a mapping given by $F : A \to P(U)$.*

Definition 2. [2]. *For two soft sets (F,A) and (G,B) over U, (F,A) is called a soft subset of (G,B) if*

(i) $A \subseteq B$
(ii) *for all* $\varepsilon \in B, G(\varepsilon) \subseteq F(\varepsilon)$

This relationship is denoted by $(F,A) \subseteq (G,B)$.In this case (G,B) is called a soft subset of (F,A).

Definition 3. [3]. *Let G be a group and (F,A) be a soft set over G. Then (F, A) is said to be a soft group over G if F(a) is a subgroup of G, for each $a \in A$.*

Definition 4. [15]. *Let (F_1,A), and (F_2,A), be two soft groups over G_1 and G_2, respectively, and then (F_1,A), is said to be soft homomorphic to (F_2,A), denoted by$(F_1,A)\curlyvee(F_2,A)$, if for each $x \in A$, there exists a homomorphism $\alpha_x : F_1(x) \to F_2(x)$ such that $\alpha_x(F_1(x)) = F_2(x)$.*

Definition 5. [12]. *Let M be a G–module (G is a finite group and M is a vector space over a field K). Let A and B be parameter sets, (F,A) be a soft group over G and (V,B) be soft vector space over M. (V,B) is called a soft G-module if*

(i) $V(b)$ *is a subspace* $\forall b \in B$
(ii) $F(a)$ *is a subgroup* $\forall a \in A$
(iii) $V(b)$ *is a submodule of M G-module* $\forall b \in B$ *(the action of (F,A) on (V,B)).*

Definition 6. [12]. *Let (V,A) is any soft G-module over M and let $(V\ ,B)$ is a soft $G-$submoduleover $\{0\}$ subspace of (V,A). Then two soft G-submodules (V,A) and (V,B) of(V,A) called trivial soft G-submodules.*

Definition 7. [12]. *Let (F,A) and (H,B) be two soft G-modules over M and N respectively.*

Let $f : M \to N$ and $g : A \to B$ be two functions. Then we say that (f,g) is a soft G-module homomorphisms the following conditions are satisfied:

(i) $f : M \to N$ is a G-module homomorphism.
(ii) $g : A \to B$ is a mapping;
(iii) $f(F(x)) = H(g(x));\ \forall x \in A.$

Definition 8. [14]. *Let G be a finite group, M be a vector space over K and $GL(M)$ be the group of all linear isomorphisms from Monto itself. Alinear representation of G with representation space M is a homomorphism $T : G \to GL(M)$.*

Example 1. [14]. Let F be a field, K be an extension field of $F$and $a \in K$. Let $F(a)$ be the field obtained by adjoining a to F. (i.e)

$$F(a) = \{ b_0 + b_1 a + b_2 a^2 \cdots b_i \in F\}.$$

Let G = (a), the cyclic group generated by 'a'. For j ε $\mathbb{Z}$, define T_j: M $\to$ M by

$$T_j(\sum \beta_i\, a^j) = \sum \beta_i\, a^{j+i}.$$

Therefore T_j is an isomorphism of M onto itself. Also the map $T: G \to GL(M)$ defined by

$$T(a^j) = T_j, \forall j \in Z$$

is a homomorphism and hence a linear representation of G.

Definition 9. [14]. *Let G be a finite group and K_n be the set of all nxn matrices over K. Let $GL(n, K)$ denote the group of all invertible elements of K_n. Then a matrix representation of degree n of G is a homomorphism $T: G \to GL(n, K)$.*

Remark. [14]. Linear representation, matrix representation and G-module are equivalent concepts.

Theorem 1. *(Maschke)* [14]. *If $T: G \to GL(M)$ is a representation of a finite group G and $p = char.(K) \Delta o(G)$, then T is completely reducible.*

3 Soft representation of a soft group

In this section we define soft representation of soft groups and study the relation between soft representation and soft G-Module. Also the reducibility, irreducibility and completely reducibility of soft representations are defined using soft G-Modules. Here, $GL(M)$ in Definition 7 is the group of all isomorphism from M to M, and $GL(n,K)$ in Definition 9 is the group of all invertible elements of K_n, where K_n is the set of all matrices of the form nxn. Let A be the set of parameters. $\forall x \in A$, f_x, the family of all group homomorphism is denoted by $\Im$.

Definition 10. *Let (F, A) be a soft group over G, (M, A) be a soft vector space over V and (N, A) be a soft group over $GL(V)$. Then $\Im$ (family of group homomorphism f_x) is said to be a soft linear representation of (F, A) with representation (M, A), if there is a group homomorphism $f_x: F(x) \to N(x)$, $\forall x \in A$.*

Example 2. Take the vector space $M = (Z_5, +, .)$, the field $K = M$ and the group $G = M - 0$.

The mapping $\mu_k: M \to M$, $\mu_k(m) = k.m$ is a linear isomorphism $\forall k \in \{1, 2, 3\}$ and we can take $GL(M) = \{\mu_1, \mu_2, \mu_3\}$. For $A = \{1, 3\}$ the mapping $F: A \to G$,

$$F(x) = \{y \in G : y = x^n; n \in N\}$$

is a soft group on G and for $A = \{1, 3\}$ the mapping $N: A \to GL(M)$,

$$N(x) = \{y \in GL(M) : N(1) = \mu_1, N(3) = GL(M)\}$$

is a soft group on G and for $A = \{1, 3\}$ the mapping $N: A \to GL(M)$, is a soft group on $GL(M)$. $f_x: F(x) \to N(x)$, $\forall x \in A$,

$$f_x(g) = g.\mu_x$$

is a group homomorphism. So $\Im$ is a soft linear representation of (F, A).

Corollary 1. *Let (F, A) be a soft group over G, (M, A) be a soft G- module over V, (N, A) be a soft group over $GL(V)$ and f_x be a mapping such that $f_x: F(x) \to N(y)$, $\forall x \in A$;*

$$f_x(g) = f_g$$

where $f_g(m) = g.m$ Then $\Im$ is said to be a soft representation of G afforded by the soft G-module (M, A).

Proof. The mapping $f_g(m) = g.m$ is a group homomorphism. So $\forall\ x \in A$, each f_x is a representation of $F(x)$. $\forall\ x \in A, F(x) \leq G$ and $N(y) \leq GL(V)$, and therefore for some $x_i \in A$, by the soft G-module definition, for $g_1 \in F(x_i)$, $m_1 \in M(x_i)$, $g_1.m_1 \in M(x_i)$. It is easy to see that this operation is valid for all $x \in A$. By above Remark, linear representation, matrix representation and G-module are equivalent concepts and then the representation f_{x_i} is equivalent to the G- module $M(x_i)$. Again this is true for all $x \in A$. As a result, $\Im$ is said to be a soft matrix representation of $(F,\ A)$ afforded by the soft G -module $(M,\ A)$.

Example 3. As in Example 2, $f_x:\ F(x) \to N(x),\ \forall\ x\ \in\ A,$

$$f_x(g) = g.\mu_x$$

is a group homomorphism. Then $\Im$ is a soft representation of $(F,\ A)$. Also in Example 2, for $A = \{1,\ 3\}$, the mapping $W:\ A \to Z_p$,

$$W(x) = \{y \in Z_p\ :\ y = x^n;\ n \in\ N\}$$

is a soft G-module on M.

Hence by Corollary 1, $\Im$ is said to be a soft matrix representation of $(F,\ A)$ afforded by the soft G-module $(W,\ A)$.

Definition 11. *Let $(F,\ A)$ be a soft group over G, $(M,\ A)$ be a soft vector space over V and $(N,\ A)$ be a soft group over $GL(n,\ K)$. Then $\Im$ is said to be a soft matrix representation of $(F,\ A)$ with the representation $(M,\ A)$, if there is a group homomorphism $f_x:\ F(x) \to N(y),\ \forall\, x \in\ A.$*

Example 4. Let the vector space $M = (Z_5,\ +,\ .)$, the field $K = (Z_5,\ +,\ .)$ and the group $G = M - 0$. Let $A = \{E_1, E_2\}$ and

$$E_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, E_2 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

$$F:\ A \to G,$$

$$F(E_1) = \{y \in G\ :\ y = |E_1|^n;\ n \in\ N\},$$

$$F(E_2) = \{y \in G\ :\ y = (3|E_2|)^n;\ n \in N\}.$$

The mapping F is a soft group on G and for $A = \{1,\ 3\}$, the mapping N: A $\to GL(n, \mathbb{R})$,

$$N(x) = \{y \in GL(n,\ R)\ :\ y = x^n\ ;\ n \in\ N\ \}$$

is a soft group on $GL(\ n,\ R)$. $f_x:\ F(x) \to N(x),\quad \forall\, x \in\ A,$

$$f_{E_1}(g^k) = I_n\ (\text{Identitymatrix})$$

and

$$f_{E_2}(g^k) = X^k$$

is a group homomorphism. Hence, $\Im$ is the soft matrix representation of $(F,\ A)$.

Corollary 2. *If $\Im$ is a soft linear representation or a soft matrix representation of (F, A) with representation (M, A), then (F, A) is soft homomorphic to (M, A) under f_x.*

Proof. In Definition 4. Let (F, A) and (M, A) be two soft groups over G_1 and G_1, respectively. (F, A) is said to be soft homomorphic to (M, A), denoted by $(F, A)\tilde{}(M, A)$, if for each $x \in A$, there exists a homomorphism $f_x : F(x) \to M(x)$. If we take $f_x(F(x)) = M(x)$, by the definition of the soft representation there is a group homomorphism $f_x : F(x) \to M(y)$, $\forall x \in A$ such that (F, A) is soft homomorphic to (M, A). Therefore (F, A) is soft homomorphic to (M, A).

Corollary 3. *Let T be a linear representation or matrix representation of G. If $T = f_x$, $\forall x \in A$, then T is a soft representation of (F, A).*

Proof. As T is a representation of G, there exists a group homomorphism such that $T : G \to GL(V)$. $\forall x \in A$, as $F(x) \leq G$ and $N(y) \leq GL(V)$, if we take $T = f_x$, $\forall x \in A$ for $T : F(x) \to N(y)$ is a group homomorphism. Then, T is a soft representation of (F, A).

Example 5. In Example 1, the map $T : G \to GL(M)$ defined by $T(a^j) = T_j$, $\forall j \in Z$ is a group homomorphism and hence a linear representation of G. $\forall x \in A$, if we take $T = f_x$ then by Corollary 3.8, T is a soft linear representation of (F, A).

Corollary 4. *Let (F, A) be a soft group over G, (M, A) be a soft G- module over V, (N, A) be a soft group over $GL(n, K)$ and f_x be a mapping such that $f_x : F(x) \to N(y)$, $\forall x \in A$;*

$$f_x(g) = f_g,$$

where $f_g(m) = g.m$, $g \in G$, $m \in M$. Then $\Im$ is said to be a soft matrix representation of G afforded by the soft G-module (M, A).

Proof. Using the proof of Corollary 3, for some $x_i \in A$ from the definition of soft G– modules, for $g_1 \in F(x_i)$, $m_1 \in M(x_i)$, we have $g_1.m_1 \in M(x_i)$.

Here, only one $M(x_i)$ G-Module can be constructed on M. It is easy to see that this operation is valid $\forall$ x ε A. Then, for any soft matrix representation of (F, A) afforded by the soft G-module (M,A), $f_x : F(x) \to N(y)$, $\forall x \in A$;

$$f_x(g) = f_g,$$

where $f_g(m) = g.m$, $g \in G$, $m \in M$, and hence $\Im$ is said to be soft the matrix representation of G afforded by the soft G-module (M, A).

Example 6. In Example 6, let the vector space $M = (Z_5, +, .)$, the field $K = (Z_5, +, .)$ and the group G=M–$\{0\}$. As, $A = \{E_1, E_2\}$ and

$$E_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, E_2 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

the mapping W: A$\to$ G is a soft G-Module on M, such that

$$W(E_1) = \{y \in M : y = |E_1|^n; n \in N\},$$

$$W(E_2) = \{y \in M : y = (3|E_2|)^n; n \in N\}.$$

So, as $f_x:\ F(x)\to N(x),\ \forall\, x\in\ A,$

$$f_{E_1}(g^k) =\ I_n \text{ (Identity matrix)}$$

and $f_{E_1}(g^k)$= X^k is a group homomorphism, by Corollary 4. $\Im$ is said to be the soft matrix representation of G afforded by the soft G -module (W, A).

Definition 12. *Let $(M,\ A)$ be a soft G– module over M. If $\Im$ is a soft linear representation or soft matrix representation afforded by $(M,\ A)$ and if $(M,\ A)$ itself is soft irreducible, then we say that $\Im$ is soft irreducible.*

Example 7. In Example 4, $\Im$ is a soft linear representation afforded by (W, A). On the other hand (W, A) has two trivial soft G-submodules, namely $W(1)\ =\ W_0$ and $W(3)\ =\ M$. By the irreducible soft G-module definition (W, A) is an irreducible soft G-module. Therefore by Definition 3.12 $\Im$ is an irreducible soft linear representation.

Definition 13. *Let $(W,\ A)$ be a soft G – module over M and $\Im$ be a soft linear representation or soft matrix representation afforded by (W, A). If f_x is a completely reducible representation or reducible representation $\forall\, x\in\ A$, then we say that $\Im$ is a soft completely reducible representation or soft reducible representation.*

Now, in soft set theory we prove the Maschke Theorem, which holds an important place in representation theory.

Theorem 2. *(Soft Maschke Theorem). If $\Im$ is a soft representation of a soft group $(F,\ A)$ and $char(K)\ o(G)$, then $\Im$ is a soft completely reducible representation.*

Proof. As $\Im$ is a soft representation for $(F,\ A)$, $\forall\, x\ \in A$ each f_x is a linear representation of $F(x)$. $char(K)$ does not divide $o(G)$ and $\forall\ x\in A$ *as* $F(x)\leq\ G,\ char(K)$ does not divide o(F(x)). So, by the Maschke Theorem, ∀ x ∈ A each f_x is a completely reducible representation. Hence by Definition 3.14, $\Im$ is a soft completely reducible representation.

Example 8. In Example 6, let the vector space $M\ =\ (Z_5,\ +,\ .)$, the field $K\ =\ (Z_5,\ +,\ .)$ and the group $G=M-0$. For $A\ =\ \{E_1,E_2\}$

$$E_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, E_2 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

the mapping $F:\ A\to G,\, F\,(E_1) =\ \{y\in G\ :\ y\ =\ |E_1|^n;\ n\in\ N\},$

$$F(E_2)\ =\ \{y\in G\ :\ y\ =\ (3|E_2|)^n;\ n\in\ N\}$$

is a soft group on G and for A = {1, 3}, the mapping $N:\ A\to GL(n,\ R)$,

$$N(x)\ =\ \{y\in GL(n,\ R)\ :\ y\ =\ x^n\ ;\ n\in\ N\ \}$$

is a soft group on GL(M).

Let $f_x:\ F(x)\to N(x)$, $f_{E_1}(g^k) =\ I_n$ (Identity matrix) and $f_{E_2}(g^k)$=X^k are group homomorphism, ∀ x ε A. So, $\Im$ is a soft matrix representation of (F, A) and as

$$char(K)\ =\ char(Z_5)\ o(G)\ =\ o(M-\{0\}),$$

by Theorem 2 (Soft Maschke theorem), $\Im$ is a soft completely reducible representation.

4 Conclusions

In this paper, we defined the notion of soft representation of a soft group and obtain basic properties of soft representation of soft groups using the definition of soft sets and definition of soft group. Also we investigated some the relationship between soft representation of soft groups and soft G-Modules. Moreover we examine irreducibility, reducibility and complete reducibility of soft representations.

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