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Applications of k-Fibonacci numbers for the starlike analytic functions

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Abstract

The k-Fibonacci numbers $F_{k,n}$ $(k > 0)$, defined recursively by $F_{k,0} = 0, F_{k,1} = 1$ and $F_{k,n} = kF_{k,n} + F_{k,n-1}$ for $n \geq 1$ are used to define a new class $\mathcal{SL}^k$. The purpose of this paper is to apply properties of k-Fibonacci numbers to consider the classical problem of estimation of the Fekete–Szegö problem for the class $\mathcal{SL}^k$. An application for inverse functions is also given.

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1. Introduction

Let $\mathbb{D} = \{z : |z| < 1\}$ denote the unit disc on the complex plane. The class of all holomorphic functions f in the open unit disc $\mathbb{D}$ with normalization $f(0) = 0,\ f'(0) = 1$ is denoted by $\mathcal{A}$ and the class $\mathcal{S} \subset \mathcal{A}$ is the class which consists of univalent functions in $\mathbb{D}$. We say that f is subordinate to F in $\mathbb{D}$, written as $f \prec F$, if and only if $f(z) = F(\omega(z))$ for some $\omega \in \mathcal{A}$, $|\omega(z)| < 1$, $z \in \mathbb{D}$.

Recently, N. Yilmaz Özgür and J. Sokół [5] defined and introduced the class $\mathcal{SL}^k$ of shell-like functions as the set of functions $f \in \mathcal{A}$ which is described in the following definition.

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1.1. Definition. Let k be any positive real number. The function $f \in \mathcal{A}$ belongs to the class $\mathcal{SL}^k$ if it satisfies the condition that

$$\frac{zf'(z)}{f(z)} \prec \widetilde{p}_k(z),\ z \in \mathbb{D}, \tag{1.1}$$

where

$$\widetilde{p}_k(z) = \frac{1+\tau_k^2 z^2}{1-k\tau_k z-\tau_k^2 z^2},\ \tau_k = \frac{k-\sqrt{k^2+4}}{2},\ z \in \mathbb{D}. \tag{1.2}$$

For $k = 1$, the class $\mathcal{SL}^k$ becomes the class $\mathcal{SL}$ of shell-like functions defined in [3], see also [4].

It was proved in [5] that functions in the class $\mathcal{SL}^k$ are univalent in $\mathbb{D}$. Moreover, the class $\mathcal{SL}^k$ is a subclass of the class of starlike functions $\mathcal{S}^*$, even more, starlike of order $k(k^2+4)^{-1/2}/2$. The name attributed to the class $\mathcal{SL}^k$ is motivated by the shape of the curve

$$\mathcal{C} = \left\{\widetilde{p}_k(e^{it}) : t \in [0,2\pi) \setminus \{\pi\}\right\}.$$

The curve $\mathcal{C}$ has a shell-like shape and it is symmetric with respect to the real axis. Its graphic shape, for $k = 1$, is given below in Fig.1.

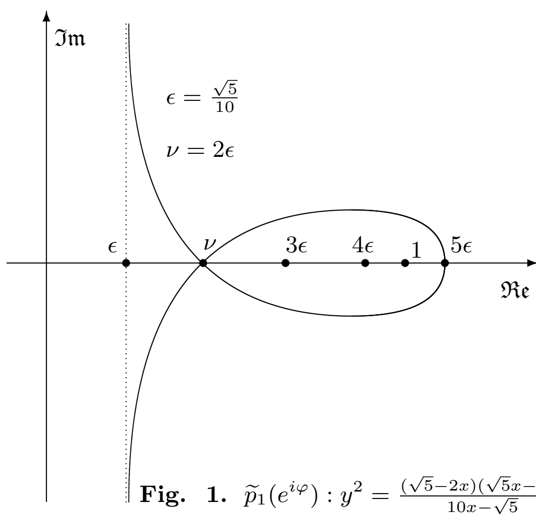

Fig. 1. $\widetilde{p}_1(e^{i\varphi}) : y^2 = \frac{(\sqrt{5}-2x)(\sqrt{5}x-1)^2}{10x-\sqrt{5}}$.

For $k \le 2$, note that we have

$$\widetilde{p}_k\left(e^{\pm i \arccos\left(k^2/4\right)}\right) = k(k^2+4)^{-1/2},$$

and so the curve $\mathcal{C}$ intersects itself on the real axis at the point $w_1 = k(k^2+4)^{-1/2}$. Thus $\mathcal{C}$ has a loop intersecting the real axis also at the point $w_2 = (k^2+4)/(2k)$. For $k > 2$, the curve $\mathcal{C}$ has no loops and it is like a conchoid, see for details [5]. Moreover, the coefficients of $\widetilde{p}_k$ are connected with k-Fibonacci numbers.

For any positive real number k, the k-Fibonacci number sequence $\{F_{k,n}\}_{n=0}^{\infty}$ is defined recursively by

$$F_{k,0} = 0,\ F_{k,1} = 1 \text{ and } F_{k,n} = kF_{k,n} + F_{k,n-1} \text{ for } n \ge 1. \tag{1.3}$$

When $k = 1$, we obtain the well-known Fibonacci numbers F_n. It is known that the n^{th} k-Fibonacci number is given by

$$F_{k,n} = \frac{(k - \tau_k)^n - \tau_k^n}{\sqrt{k^2 + 4}}, \tag{1.4}$$

where $\tau_k = (k - \sqrt{k^2 + 4})/2$. If $\widetilde{p}_k(z) = 1 + \sum_{n=1}^{\infty} \widetilde{p}_{k,n} z^n$, then we have

$$\widetilde{p}_{k,n} = (F_{k,n-1} + F_{k,n+1})\tau^n, \; n = 1, 2, 3, \ldots, \tag{1.5}$$

see also [5].

1.2. Lemma. [5] *If $f(z) = z + \sum\limits_{n=2}^{\infty} a_n z^n$ belongs to the class $\mathcal{SL}^k$, then we have*

$$|a_n| \leq |\tau_k|^{n-1} F_{k,n}, \tag{1.6}$$

where $\tau_k = (k - \sqrt{k^2 + 4})/2$. Equality holds in (1.6) for the function

$$\begin{aligned} f_k(z) &= \frac{z}{1 - k\tau_k z - \tau_k^2 z^2} \\ &= \sum_{n=1}^{\infty} \tau_k^{n-1} F_{k,n} z^n \\ &= z + \frac{(k - \sqrt{k^2 + 4})k}{2} z^2 + (k^2 + 1)\left(\frac{(k - \sqrt{k^2 + 4})k}{2} + 1\right) z^3 + \cdots . \end{aligned} \tag{1.7}$$

2. The classical Fekete–Szegö functional

A typical problem in geometric function theory is to study a functional made up of combinations of the coefficients of the original function. Let $\mathcal{S}$ be the class of univalent functions $f(z) = z + a_2 z^2 + a_3 z^3 + \cdots$ mapping $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ into $\mathbb{C}$ (the complex plane). The classical Fekete–Szegö functional is $\mathcal{L}_\lambda = |a_3 - \lambda a_2^2|$, $0 < \lambda \leq 1$. Over the years, many results have been found for the classical functional $\mathcal{L}_\lambda$. Fekete and Szegö [1] bounded $\mathcal{L}_\lambda$ by $1 + 2\exp(-2\lambda/(1 - \lambda))$, for $0 \leq \lambda < 1$ and $f \in \mathcal{S}$, where $\mathcal{S}$ denotes the subclass of $\mathcal{A}$ consisting of functions univalent in $\mathbb{D}$. This inequality is sharp for each λ. In particular, for $\lambda = 1$, one has $|a_3 - a_2^2| \leq 1$ if $f \in \mathcal{S}$. Note that the quantity $a_3 - a_2^2$ represents $S_f(0)/6$, where S_f denotes the Schwarzian derivative $(f''/f')' - (f''/f')^2/2$ of locally univalent functions f in $\mathbb{D}$. It is interesting to consider the behavior of $\mathcal{L}_\lambda$ for subclasses of the class $\mathcal{S}$. The Fekete–Szegö problem is to determine sharp upper bound for Fekete–Szegö functional $\mathcal{L}_\lambda$ over a family $\mathcal{F} \subset \mathcal{S}$. In the literature, there exists a large number of results about inequalities for $a_3 - a_2^2$ corresponding to various subclasses of $\mathcal{S}$. In the present paper we obtain the Fekete–Szegö inequalities for the class $\mathcal{SL}^k$. Before we consider how the Taylor series coefficients of functions in the class $\mathcal{SL}^k$ might be bounded, let us first recall this problem for the Caratheodory functions. Let $\mathcal{P}$ denote the class of analytic functions p in $\mathbb{D}$ with $p(0) = 1$ and $\mathfrak{Re}\{p(z)\} > 0$.

2.1. Lemma. [2] *Let $p \in \mathcal{P}$ with $p(z) = 1 + c_1 z + c_2 z + \cdots$, then*

$$|c_n| \leq 2, \quad for \quad n \geq 1. \tag{2.1}$$

If $|c_1| = 2$, then $p(z) \equiv p_1(z) = (1 + xz)/(1 - xz)$ with $x = c_1/2$. Conversely, if $p(z) \equiv p_1(z)$ for some $|x| = 1$, then $c_1 = 2x$. Furthermore, we have

$$|c_2 - c_1/2| \leq 2 - |c_1|^2/2. \tag{2.2}$$

If $|c_1 < 2|$ and $|c_2 - c_1/2| = 2 - |c_1|^2/2$, then $p(z) \equiv p_2(z)$, where

$$p_2(z) = \frac{1 + \overline{x}wz + z(wz + x)}{1 + \overline{x}wz - z(wz + x)}$$

and $x = c_1/2$, $w = (2c_2 - c_1^2)/(4 - |c_1|^2)$. *Conversely, if if* $p(z) \equiv p_2(z)$ *for some* $|x| < 1$ *and* $w = 1$, *then* $c_1 = 2x$, $w = (2c_2 - c_1^2)/(4 - |c_1|^2)$ *and* $|c_2 - c_1/2| = 2 - |c_1|^2/2$.

2.2. Theorem. *If* $p(z) = 1 + p_1 z + p_2 z^2 + \cdots$ *and*

$$p(z) \prec \widetilde{p}_k(z) = \frac{1 + \tau_k^2 z^2}{1 - k\tau_k z - \tau_k^2 z^2},\ \tau_k = \frac{k - \sqrt{k^2+4}}{2},\ z \in \mathbb{D},$$

then we have

$$|p_1| \leq \frac{\left(\sqrt{k^2+4} - k\right) k}{2} \tag{2.3}$$

and

$$|p_2| \leq (k^2 + 2)\left\{\frac{(k - \sqrt{k^2+4})k}{2} + 1\right\}. \tag{2.4}$$

The above estimations are sharp.

Proof. If $p \prec \widetilde{p}_k$, then there exists an analytic function w such that $|w(z)| \leq |z|$ in $\mathbb{D}$ and $p(z) = \widetilde{p}_k(w(z))$. Therefore, the function

$$h(z) = \frac{1 + w(z)}{1 - w(z)} = 1 + c_1 z + c_2 z + \cdots \quad (z \in \mathbb{D})$$

is in the class $\mathcal{P}(0)$. It follows that

$$w(z) = \frac{c_1 z}{2} + \left(c_2 - \frac{c_1^2}{2}\right)\frac{z^2}{2} + \cdots \tag{2.5}$$

and

$$\begin{aligned} \widetilde{p}_k(w(z)) &= 1 + \widetilde{p}_{k,1}\left\{\frac{c_1 z}{2} + \left(c_2 - \frac{c_1^2}{2}\right)\frac{z^2}{2} + \cdots\right\} + \widetilde{p}_{k,2}\left\{\frac{c_1 z}{2} + \left(c_2 - \frac{c_1^2}{2}\right)\frac{z^2}{2} + \cdots\right\}^2 + \cdots \\ &= 1 + \frac{\widetilde{p}_{k,1} c_1}{2} z + \left\{\frac{1}{2}\left(c_2 - \frac{c_1^2}{2}\right)\widetilde{p}_{k,1} + \frac{1}{4} c_1^2 \widetilde{p}_{k,2}\right\} z^2 + \cdots \\ &= p(z). \end{aligned} \tag{2.6}$$

From (1.5), we find the coefficients $\widetilde{p}_{k,n}$ of the function $\widetilde{p}_k$ given by

$$\widetilde{p}_{k,n} = (F_{k,n-1} + F_{k,n+1})\tau^n.$$

This shows the relevant connection $\widetilde{p}_k$ with the sequence of k-Fibonacci numbers

$$\begin{aligned} \widetilde{p}_k(z) &= 1 + \sum_{n=1}^{\infty} \widetilde{p}_{k,n} z^n \\ &= 1 + (F_{k,0} + F_{k,2})\tau_k z + (F_{k,1} + F_{k,3})\tau_k^2 z^2 + \cdots \\ &= 1 + k\tau_k z + (k^2 + 2)\tau_k^2 z^2 + (k^3 + 3k)\tau_k^3 z^3 + \cdots. \end{aligned} \tag{2.7}$$

If $p(z) = 1 + p_1 z + p_2 z^2 + \cdots$, then by (2.6) and (2.7), we have

$$p_1 = \frac{k\tau_k c_1}{2} \tag{2.8}$$

and

$$p_2 = \frac{k\tau_k}{2}\left(c_2 - \frac{c_1^2}{2}\right) + \frac{(k^2+2)}{4} c_1^2 \tau_k^2. \tag{2.9}$$

From (2.8) and (2.1) we directly obtain (2.3). From (2.9) and (2.2), we obtain

$$\begin{aligned}
|p_2| &= \left|\frac{k\tau_k}{2}\left(c_2 - \frac{c_1^2}{2}\right) + \frac{(k^2+2)}{4}c_1^2\tau_k^2\right| \\
&\leq \left|\frac{k\tau_k}{2}\left(c_2 - \frac{c_1^2}{2}\right)\right| + \left|\frac{(k^2+2)}{4}c_1^2\tau_k^2\right| \\
&\leq \frac{k|\tau_k|}{2}\left(2 - \frac{1}{2}|c_1|^2\right) + \frac{(k^2+2)}{4}|c_1|^2\tau_k^2 \\
&= k|\tau_k| + \frac{|c_1|^2}{4}\left((k^2+2)\tau_k^2 - k|\tau_k|\right). \qquad (2.10)
\end{aligned}$$

Since $\tau_k = (k - \sqrt{k^2+4})/2$, so it is easily verified that

$$(k^2+2)\tau_k^2 - k|\tau_k| = \frac{(k(k-\sqrt{k^2+4}))(k^2+3)}{2} + k^2 + 2. \qquad (2.11)$$

We want to show that (2.11) is positive for $k > 0$. Notice that

$$\frac{(k-\sqrt{k^2+4})(k^3+3k)}{2} + k^2 + 2 = \frac{(k^2+2)\sqrt{k^2+4} - k^3 - 4k}{k + \sqrt{k^2+4}}. \qquad (2.12)$$

Thus, (2.11) is positive when

$$(k^2+2)\sqrt{k^2+4} > k^3 + 4k, \quad k > 0, \qquad (2.13)$$

or equivalently, when

$$\left\{(k^2+2)\sqrt{k^2+4}\right\}^2 > \left\{k^3+4k\right\}^2, \quad k > 0. \qquad (2.14)$$

The inequality (2.14) yields the inequality

$$4k^2 + 16 > 0, \quad k > 0, \qquad (2.15)$$

which is evidently true, and hence (2.11) is positive. Therefore, $(k^2+2)\tau_k^2 - |\tau_k| > 0$ and from (2.10), we obtain

$$\begin{aligned}
|p_2| &\leq k|\tau_k| + \frac{|c_1|^2}{4}\left((k^2+2)\tau_k^2 - k|\tau_k|\right) \\
&\leq k|\tau_k| + (k^2+2)\tau_k^2 - k|\tau_k| \\
&= (k^2+2)\tau_k^2 \\
&= (k^2+2)\left\{\frac{(k-\sqrt{k^2+4})k}{2} + 1\right\}.
\end{aligned}$$

Thus, the equality in estimations (2.3), (2.4) are attained by the coefficients of the function given by(2.7). □

2.3. Theorem. *Let λ be real. If $f(z) = z + a_2z^2 + a_3z^3 + \cdots$ belongs to $\mathcal{SL}^k$, then*

$$|a_3 - \lambda a_2^2| \leq (k(k - \sqrt{k^2+4})/2 + 1)(k^2 + 1 + k^2|\lambda|). \qquad (2.16)$$

The above estimation is sharp. If $\lambda \leq 0$, then the equality in (2.16) is attained by the function f_k given in (1.6), and by the function $-f_k(-z)$ when $\lambda \geq 0$.

Proof. For given $f \in \mathcal{SL}^k$, define $p(z) = 1 + p_1z + p_2z^2 + \cdots$ by

$$\frac{zf'(z)}{f(z)} = p(z) \quad (z \in \mathbb{D}),$$

where $p \prec \widetilde{p}_k$ in $\mathbb{D}$. Hence

$$z + 2a_2z^2 + 3a_3z^3 + \cdots = \left\{z + a_2z^2 + a_3z^3 + \cdots\right\}\left\{1 + p_1z + p_2z^2 + \cdots\right\}$$

and

$$a_2 = p_1, \quad 2a_3 = p_1 a_2 + p_2.$$

Therefore, $|a_3 - \lambda a_2| = |(p_1 a_2 + p_2)/2 + \lambda p_1^2|$. Using this and the bounds (2.3), (2.4) and (1.6), we obtain

$$\begin{aligned}
|a_3 - \lambda a_2^2| &= |(p_1 a_2 + p_2)/2 - \lambda p_1^2| \\
&\leq \frac{|p_1||a_2| + |p_2|}{2} + |\lambda||p_1^2| \\
&\leq \frac{k(k-\sqrt{k^2+4})/2 \cdot k(k-\sqrt{k^2+4})/2 + (k^2+2)(k(k-\sqrt{k^2+4})/2+1)}{2} \\
&\quad + |\lambda| \left\{ \frac{\left(\sqrt{k^2+4}-k\right)k}{2} \right\}^2 \\
&= \frac{k^2(k(k-\sqrt{k^2+4})/2+1) + (k^2+2)(k(k-\sqrt{k^2+4})/2+1)}{2} \\
&\quad + |\lambda| \left\{ \frac{\left(\sqrt{k^2+4}-k\right)k}{2} \right\}^2 \\
&= (k^2+1)(k(k-\sqrt{k^2+4})/2+1) + |\lambda| \left\{ \frac{\left(\sqrt{k^2+4}-k\right)k}{2} \right\}^2 \\
&= (k(k-\sqrt{k^2+4})/2+1)(k^2+1+k^2|\lambda|).
\end{aligned}$$

□

2.4. Corollary. *If $g(z) = z + \sum_{n=2}^{\infty} b_n z^n$, $|z| < r_0(g)$, $r_0(g) \geq 1/4$, is an inverse to $f \in \mathcal{SL}^k$, then we have*

$$(2.17) \quad |b_2| \leq \frac{(k-\sqrt{k^2+4})k}{2},$$

$$(2.18) \quad |b_3| \leq (k(k-\sqrt{k^2+4})/2+1)(3k^2+1).$$

The above estimation is sharp. The equalities are attained by the function $-if_k^{-1}(iz)$, where f_k is given in (1.6).

Proof. For each $f \in \mathcal{S}$, the Koebe one-quarter theorem ensures that the image of $\mathbb{D}$ under f contains the disc of radius $1/4$. If $f(z) = z + a_2 z^2 + a_3 z^3 + \cdots$ is univalent in $\mathbb{D}$ then, f has the inverse f^{-1} with the expansion

$$(2.19) \quad f^{-1}(z) = z - a_2 z^2 + (2a_2^2 - a_3)z^3 + \cdots, \ |z| < r_0(f), \ r_0(f) \geq 1/4.$$

It was proved in [5] that functions in the class $\mathcal{SL}^k$ are univalent in $\mathbb{D}$. From Lemma 1.2 and (2.19), we obtain the inequality (2.17). Also, from Theorem 2.3 (with $\lambda = 2$) and (2.19), we obtain the inequality (2.18). If $f \in \mathcal{SL}^k$, then the function $-if_k(iz)$ satisfies

(1.1), so it belongs to the class $\mathcal{SL}^k$ too. Moreover, from (1.6), we have

$$\begin{aligned}
&-if_k^{-1}(iz)\\
&= z + i\frac{(k-\sqrt{k^2+4})k}{2}z^2\\
&-\left\{2\left(\frac{(k-\sqrt{k^2+4})k}{2}\right)^2 + (k^2+1)\left(\frac{(k-\sqrt{k^2+4})k}{2}+1\right)\right\}z^3+\cdots\\
&= z + i\frac{(k-\sqrt{k^2+4})k}{2}z^2 - (k(k-\sqrt{k^2+4})/2+1)(3k^2+1)z^3+\cdots.
\end{aligned}$$

This shows that the equalities in (2.17) and (2.18) are attained by the second and third coefficients of the function $-if_k^{-1}(iz)$. □

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