Hacettepe Journal of Mathematics and Statistics
Volume 45 (2) (2016), 609 – 628

On the Bayesian analysis of 3-component mixture of exponential distributions under different loss functions

Muhammad Tahir* †, Muhammad Aslam‡ and Zawar Hussain§

Abstract

The memory-less property of the Exponential distribution is a strong reason of its use for testing lifetimes of objects in many lifetime modeling applications. Also, mixture models have extensively been used in survival analysis and reliability studies. This article focuses on the Bayesian analysis of the 3-component mixture of Exponential distributions under type-I right censoring scheme. Taking different non-informative and informative priors, Bayes estimators and posterior risks for the unknown parameters (parameters of component distributions and mixing proportions) are derived under squared error loss function, precautionary loss function and DeGroot loss function. The elicitation of the hyperparameters is also done using prior predictive distribution.The Bayes estimators and posterior risks are looked at as a function of the test termination time. Some important properties and comparisons of the Bayes estimates are presented. Simulated results and real data example are also given to illustrate the study.

Keywords: 3-Component mixture distribution, Non-informative and informative priors, Loss function, Bayes estimators, Posterior risks, Test termination time.

2000 AMS Classification: 62F15, 62N05.

Received : 26.06.2014 *Accepted* : 13.02.2015 *Doi* : 10.15672/HJMS.2015519451

---

*Department of Statistics, Government College University, Faisalabad, Pakistan.
†Department of Statistics, Quaid-i-Azam University, Islamabad, Pakistan.
Email: tahirqaustat@yahoo.com Corresponding Author.
‡Department of Mathematics and Statistics, Riphah International University, Islamabad, Pakistan.
Email: aslamsdqu@yahoo.com
§Department of Statistics, Quaid-i-Azam University, Islamabad, Pakistan.
Email: zhlangah@yahoo.com

1. Introduction

The Exponential distribution, because of its memory-less property, has many real life applications in testing lifetimes of objects where lifetimes do not depend upon their ages. There are many electronic devices whose failure rate does not depend on their ages, therefore, the Exponential distribution is suitable to model the lifetimes. Generally, in lifetime modeling, population is supposed to be composed of more than one subpopulations mixed by unknown mixing proportions. In our study, we take the data from a population which is characterized by three different members of the Exponential family of distributions. McCullagh (1994) derived some conditions under which quadratic and polynomial Exponential models can be generated as mixtures of the Exponential models. Raqab and Ahsanullah (2001) discussed the location and scale parameters of generalized Exponential distribution based on order statistic. Hebert and Scariano (2005) compared the location estimators for the Exponential mixtures under Pitman's measure of closeness. Ali et al. (2005) studied the Bayes estimators of the Exponential distribution and Abu-Taleb et al. (2007) presented the Bayesian estimation of lifetime parameters of Exponential distributions when survival time and censoring time are both exponentially distributed.

The use of mixture models in situations where data are given only for overall mixture distributions is known as direct application of the mixture models. Li (1983) and Li and Sedransk (1982, 1988) discussed different features of mixture models and defined two types of mixture models. The mixture of the probability density functions from the same family is known as type-I mixture model and type-II mixture model is defined as a mixture of density functions from several families. In this study, the direct application of mixture model (with the unknown component and mixing proportion parameters of the 3-component mixture of Exponential distributions) is considered under type-I mixture modeling.

Due to the development of advanced computational facilities, researchers are now able to find the Bayes estimates, infer and predict about complex systems such as mixture models. With the provision of these computational facilities, the Bayesian technique to analyze a 3-component mixture model has developed the interest among many researchers. The posterior distribution, which is obtained when prior information is combined with likelihood, is the workbench of Bayesian inference. Thus, the prior information, a subjective assessment by an expert before the data are actually gathered, is very important and necessary for Bayesian inference.In this study, the Bayesian analysis of a 3-component mixture of Exponential distributions using the non-informative (uniform and Jeffreys') priors and the informative prior (IP) under squared error loss function (SELF), precautionary loss function (PLF) and DeGroot loss function (DLF) is considered.

There are many fields such as engineering, biological sciences, physical sciences and social sciences where mixture models have been used quite effectively. Most of the researchers worked on the Bayesian analysis of 2-component mixture models. For example, Sinha (1998) used the Bayesian counterpart of the maximum likelihood estimates of the 2-component mixture model considered by Mendenhall and Hader (1958). Saleem and Aslam (2008) discussed the use of the informative and the non-informative priors for Bayesian analysis of the 2-component mixture of Rayleigh distributions. Saleem et al. (2010) presented the Bayesian analysis of the 2-component mixture of Power distributions using the complete and censored data. Kazmi et al. (2012) developed the Bayesian analysis for the 2-component mixture of Maxwell distributions.

In real life applications, most of the times, it is not suitable to continue the testing procedure until failure of the last object under testing. In such situations, censored samples are observed. Censoring is an important and valuable aspect of lifetime applications. A valuable account on censoring is given in Romeu (2004), Gijbels (2010) and Kalbfleisch and Prentice (2011). In this paper, an ordinary type-I right censoring is used with fixed life-test termination time for all objects.

The rest of the paper is organized as follows. In Section 2, the 3-component mixture of Exponential distributions is presented. Posterior distributions using the uniform prior (UP), the Jeffreys' prior (JP) and the informative prior (IP) are derived in Section 3. The Bayes estimators and posterior risks using the UP, the JP and the IP under SELF, PLF and DLF are presented in Sections 4, 5 and 6, respectively. The elicitation of hyperparameters is described in Section 7. The limiting expressions are derived in Section 8. A simulation study and real data example are discussed in Sections 9 and 10, respectively. Finally, the conclusion of the study is given in Section 11.

2. 3-component mixture of exponential distributions

If X is exponentially distributed with parameter θ_m, its probability density function is given as:

(2.1) $$f_m\left(x;\ \theta_m\right) = \theta_m\ \exp\left(-\theta_m\ x\right),\ x \geq 0,\ \theta_m > 0,\ m = 1,\ 2,\ 3.$$

According to Barger (2006) and Strelec and Stehlk (2012), a finite 3-component mixture of Exponential distributions with unknown mixing proportions p_1 and p_2 is defined as:

(2.2) $$f\left(x\right) = p_1\ f_1\left(x\right) + p_2\ f_2\left(x\right) + \left(1 - p_1 - p_2\right)\ f_3\left(x\right),p_1,\ p_2 \geq 0,\ p_1 + p_2 \leq 1$$

(2.3) $$\begin{aligned} f\left(x;\ \theta_1,\ \theta_2,\ \theta_3,\ p_1,\ p_2\right) &= p_1\ \theta_1 \exp\left(-\theta_1\ x\right) + p_2\ \theta_2 \exp\left(-\theta_2\ x\right) \\ &+ \left(1 - p_1 - p_2\right)\ \theta_3 \exp\left(-\theta_3\ x\right) \end{aligned}$$

As cumulative distribution function of the random variable X is given by:

(2.4) $$F_m\left(x\right) = 1 - \exp\left(-\theta_m\ x\right),\ m = 1,\ 2,\ 3,$$

the cumulative distribution function of 3-component mixture distribution is defined as:

(2.5) $$F\left(x\right) = p_1\ F_1\left(x\right) + p_2\ F_2\left(x\right) + \left(1 - p_1 - p_2\right)\ F_3\left(x\right)$$

(2.6) $$F\left(x\right) = 1 - p_1\ \exp\left(-\theta_1\ x\right) - p_2\ \exp\left(-\theta_2\ x\right) - \left(1 - p_1 - p_2\right)\ \exp\left(-\theta_3\ x\right)$$

3. The posterior distribution using the UP, the JP and the IP

The posterior distributions of parameters given data $\mathbf{x}$ are derived using the UP, the JP and the IP.

3.1. The likelihood function. Suppose n units are used in a life testing experiment with the 3-component mixture modeling. Let r out of n units fail before fixed test termination time t and the remaining $n - r$ units are still working. According to Mendenhall and Hader (1958), there are many practical situations in which the failing objects can be pointed out easily as subset of subpopulation-1, subpopulation-2or subpopulation-3. Out of r units, suppose r_1, r_2 and r_3 units belong to subpopulation-1, subpopulation-2 and subpopulation-3,respectively,such that $r = r_1 + r_2 + r_3$. Now, define x_{lk}, $0 < x_{lk} \leq t$, as the failure time of k^{th} $(k = 1,\ 2, \cdots,\ r_l)$ unit belong to l^{th} $(l = 1,\ 2,\ 3)$ subpopulation. Thus, the likelihood function of the 3-component mixture model for the random sample vector x is given as(cf. Everitt and Hand, 1981):

$$L(\boldsymbol{\psi}|\mathbf{x}) \propto \left\{\prod_{k=1}^{r_1} p_1\ f_1(x_{1k})\right\}\left\{\prod_{k=1}^{r_2} p_2\ f_2(x_{2k})\right\}\left\{\prod_{k=1}^{r_3}(1-p_1-p_2)\ f_3(x_{3k})\right\}$$
$$\{1-F(t)\}^{n-r} \tag{3.1}$$

$$L(\boldsymbol{\psi}|\mathbf{x}) \propto \theta_1^{r_1}\theta_2^{r_2}\theta_3^{r_3}\left[\sum_{i=0}^{n-r}\sum_{j=0}^{i}\binom{n-r}{i}\binom{i}{j}\right.$$
$$\exp\left\{-\theta_1\left(nt-rt-it+\sum_{k=1}^{r_1}x_{1k}\right)\right\}$$
$$\exp\left\{-\theta_2\left(it-jt+\sum_{k=1}^{r_2}x_{2k}\right)\right\}\exp\left\{-\theta_3\left(jt+\sum_{k=1}^{r_3}x_{3k}\right)\right\}$$
$$\left.p_1^{n-r-i+r_1}p_2^{i-j+r_2}(1-p_1-p_2)^{j+r_3}\right], \tag{3.2}$$

where $\boldsymbol{\psi}=(\theta_1,\ \theta_2,\ \theta_3,\ p_1,\ p_2)$ and $\mathbf{x}=(x_{11},...,\ x_{1r_1},\ x_{21},...,\ x_{2r_2},\ x_{31},...,\ x_{3r_3})$.

3.2. The posterior distribution using the UP. When no or little prior information is given, usually, the non-informative prior is assumed to be the UP. Bayes (1763), de Laplace (1820) and Geisser (1984) proposed that one may take the UP for the unknown parameters $\boldsymbol{\psi}=(\theta_1,\ \theta_2,\ \theta_3,\ p_1,\ p_2)$. Following Bayes (1763), de Laplace (1820) and Geisser (1984), UPs over the intervals $(0,\ \infty)$ and $(0,\ 1)$ are taken for the parameters ($\theta_1,\ \theta_2$ and θ_3) of Exponential distributions and for the mixing proportions (p_1 and p_2), respectively. With these settings, joint prior distribution of the parameters $\theta_1,\ \theta_2,\ \theta_3,\ p_1$ and p_2, as defined by Saleem (2010), is given by:

$$\pi_1(\boldsymbol{\psi}) \propto 1;\ \theta_1,\ \theta_2,\ \theta_3>0,\ p_1,\ p_2\geq 0,\ p_1+p_2\leq 1. \tag{3.3}$$

The joint posterior distribution of parameters $\theta_1,\ \theta_2,\ \theta_3,\ p_1$ and p_2 given data $\mathbf{x}$, using the UP is defined as:

$$g_1(\boldsymbol{\psi}|\mathbf{x}) = \frac{L(\boldsymbol{\psi}|\mathbf{x})\,\pi_1(\boldsymbol{\psi})}{\int_{\boldsymbol{\psi}} L(\boldsymbol{\psi}|\mathbf{x})\,\pi_1(\boldsymbol{\psi})\,d\boldsymbol{\psi}} \tag{3.4}$$

$$g_1(\boldsymbol{\psi}|\mathbf{x}) = \frac{1}{E_1\theta_1^{1-A_{11}}\theta_2^{1-A_{21}}\theta_3^{1-A_{31}}}\sum_{i=0}^{n-r}\sum_{j=0}^{i}\binom{n-r}{i}\binom{i}{j}\exp(-B_{11}\theta_1)\times$$
$$\exp(-B_{21}\theta_2)\exp(-B_{31}\theta_3)\,p_1^{A_{01}-1}p_2^{B_{01}-1}(1-p_1-p_2)^{C_{01}-1}, \tag{3.5}$$

where $A_{11}=r_1+1,\ A_{21}=r_2+1,\ A_{31}=r_3+1,\ B_{11}=nt-rt-it+\sum_{k=1}^{r_1}x_{1k}$, $B_{21}=it-jt+\sum_{k=1}^{r_2}x_{2k},\ B_{31}=jt+\sum_{k=1}^{r_3}x_{3k}, A_{01}=n-r-i+r_1+1, B_{01}=i-j+r_2+1, C_{01}=j+r_3+1,\ E_1=\Gamma(A_{11})\Gamma(A_{21})\Gamma(A_{31})\ \sum_{i=0}^{n-r}\sum_{j=0}^{i}\binom{n-r}{i}\binom{i}{\mathrm{j}}\times$ $B(A_{01},\ B_{01},\ C_{01})\ \mathrm{B}_{11}^{-A_{11}}\ \mathrm{B}_{21}^{-A_{21}}\ \mathrm{B}_{31}^{-A_{31}}$.

3.3. The posterior distribution using the JP. According to Jeffreys' (1946, 1961), Bernardo (1979) and Berger (1985), the JP is defined as $p(\theta_m)\propto\sqrt{|I(\theta_m)|}$, $m=1,\ 2,\ 3$, where $I(\theta_m)=-E\left[\frac{\partial^2 f(x|\theta_m)}{\partial\theta_m^2}\right]$ is the Fisher's information matrix. The prior distributions of the mixing proportions p_1 and p_2 are again taken to be the uniform on over the interval $(0,\ 1)$. The joint prior distribution of parameters $\theta_1,\ \theta_2,\ \theta_3,\ p_1$ and p_2 is (cf. Sinha, 1998) given by:

$$\pi_2(\psi) \propto \frac{1}{\theta_1\ \theta_2\ \theta_3},\ \theta_1,\ \theta_2,\ \theta_3>0,\ p_1,\ p_2\geq 0,\ p_1+p_2\leq 1 \tag{3.6}$$

The joint posterior distribution of parameters $\theta_1,\ \theta_2,\ \theta_3,\ p_1$ and p_2 given data $\mathbf{x}$, using the JP is:

$$g_2(\boldsymbol{\psi}\,|\mathbf{x}) = \frac{L(\boldsymbol{\psi}\,|\mathbf{x})\,\pi_2(\boldsymbol{\psi})}{\int_{\boldsymbol{\psi}} L(\boldsymbol{\psi}\,|\mathbf{x})\,\pi_2(\boldsymbol{\psi})\,d\boldsymbol{\psi}} \tag{3.7}$$

$$g_2(\boldsymbol{\psi}\,|\mathbf{x}) = \frac{\sum_{i=0}^{n-r}\sum_{j=0}^{i}\binom{n-r}{i}\binom{i}{j}\exp(-B_{12}\theta_1)}{E_2\theta_1^{1-A_{12}}\theta_2^{1-A_{22}}\theta_3^{1-A_{32}}} \times \exp(-B_{22}\theta_2)\exp(-B_{32}\theta_3)\,p_1^{A_{02}-1}p_2^{B_{02}-1}(1-p_1-p_2)^{C_{02}-1}, \tag{3.8}$$

where $A_{12}=r_1,\ A_{22}=r_2,\ A_{32}=r_3,\ B_{12}=nt-rt-it+\sum_{k=1}^{r_1}x_{1k},\ B_{22}=it-jt+\sum_{k=1}^{r_2}x_{2k},\ B_{32}=jt+\sum_{k=1}^{r_3}x_{3k}, A_{02}=n-r-i+r_1+1, B_{02}=i-j+r_2+1, C_{02}=j+r_3+1,$
$E_2=\Gamma(A_{12})\Gamma(A_{22})\Gamma(A_{32})\ \sum_{i=0}^{n-r}\sum_{j=0}^{i}\binom{n-r}{i}\binom{i}{j}\times\ B(A_{02},\ B_{02},\ C_{02})\ \mathrm{B}_{12}^{-A_{12}}\ \mathrm{B}_{22}^{-A_{22}}\ \mathrm{B}_{32}^{-A_{32}}.$

3.4. The posterior distribution using the IP. As an informative prior distribution, we take Gamma distribution for component parameters $\theta_1,\ \theta_2,\ \theta_3$ and bivariate beta distribution for proportion parameters $p_1,\ p_2$, i.e.

$$\pi_4(\theta_1;\ a_1,\ b_1) = \frac{b_1^{a_1}}{\Gamma(a_1)}\ \theta_1^{a_1-1}\exp(-b_1\ \theta_1),\ \theta_1>0,\ a_1,\ b_1>0 \tag{3.9}$$

$$\pi_5(\theta_2;\ a_2,\ b_2) = \frac{b_2^{a_2}}{\Gamma(a_2)}\ \theta_2^{a_2-1}\exp(-b_2\ \theta_2),\ \theta_2>0,\ a_2,\ b_2>0 \tag{3.10}$$

$$\pi_6(\theta_3;\ a_3,\ b_3) = \frac{b_3^{a_3}}{\Gamma(a_3)}\ \theta_3^{a_3-1}\exp(-b_3\ \theta_3),\ \theta_3>0,\ a_3,\ b_3>0 \tag{3.11}$$

$$\pi_7(p_1,\ p_2;\ a,\ b,\ c) = \frac{1}{B(a,\ b,\ c)}\ p_1^{a-1}\ p_2^{b-1}\ (1-p_1-p_2)^{c-1}, \tag{3.12}$$

$$p_1,\ \ p_2\geq 0,\ p_1+p_2\leq 1,\ a,\ b,\ c>0.$$

So, the joint prior distribution of parameters $\theta_1,\ \theta_2,\ \theta_3,\ p_1$ and p_2 using the IP is

$$\pi_3(\boldsymbol{\psi}) \propto \theta_1^{a_1-1}\exp(-b_1\theta_1)\,\theta_2^{a_2-1}\exp(-b_2\theta_2)\,\theta_3^{a_3-1}\times \exp(-b_3\theta_3)\,p_1^{a-1}p_2^{b-1}(1-p_1-p_2)^{c-1} \tag{3.13}$$

The joint posterior distribution of parameters $\theta_1,\ \theta_2,\ \theta_3,\ p_1$ and p_2 given data $\mathbf{x}$, using the IP is:

$$g_3(\boldsymbol{\psi}\,|\mathbf{x}) = \frac{L(\boldsymbol{\psi}\,|\mathbf{x})\,\pi_3(\boldsymbol{\psi})}{\int_{\boldsymbol{\psi}} L(\boldsymbol{\psi}\,|\mathbf{x})\,\pi_3(\boldsymbol{\psi})\,d\boldsymbol{\psi}} \tag{3.14}$$

$$g_3(\boldsymbol{\psi}\,|\mathbf{x}) = \frac{\sum_{i=0}^{n-r}\sum_{j=0}^{i}\binom{n-r}{i}\binom{i}{j}\exp(-B_{13}\theta_1)\exp(-B_{23}\theta_2)}{E_3\theta_1^{1-A_{13}}\theta_2^{1-A_{23}}\theta_3^{1-A_{33}}} \times \exp(-B_{33}\theta_3)\,p_1^{A_{03}-1}p_2^{B_{03}-1}(1-p_1-p_2)^{C_{03}-1}, \tag{3.15}$$

where $A_{13}=r_1+a_1,\ A_{23}=r_2+a_2,\ A_{33}=r_3+a_3,\ B_{13}=nt-rt-it+\sum_{k=1}^{r_1}x_{1k}+b_1,$ $B_{23}=it-jt+\sum_{k=1}^{r_2}x_{2k}+b_2,\ B_{33}=jt+\sum_{k=1}^{r_3}x_{3k}+b_3, A_{03}=n-r-i+r_1+a, B_{03}=$ $i-j+r_2+b, C_{03}=j+r_3+c,\ E_3=\Gamma(A_{13})\Gamma(A_{23})\Gamma(A_{33})\sum_{i=0}^{n-r}\sum_{j=0}^{i}\binom{n-r}{i}\binom{i}{j}\times$ $B(A_{03},B_{03},C_{03})\,\mathrm{B}_{13}^{-A_{13}}\mathrm{B}_{23}^{-A_{23}}\mathrm{B}_{33}^{-A_{33}}.$

4. The Bayes estimators and posterior risks using the UP, the JP and IP under SELF

If $L(\theta,\ d)$ is a loss function then the expected value of the loss function for a given decision with respect to the posterior distribution is posterior risk function and if $\hat{d}$ is a Bayes estimator then $\rho\left(\hat{d}\right)$ is called posterior risk and is given by $\rho\left(\hat{d}\right) = E_{\theta|\mathbf{x}}\left\{L\left(\theta,\ \hat{d}\right)\right\}$. The SELF is suggested by Legendre (1806) and is defined as: $L(\theta, d) = (\theta - d)^2$. The Bayes estimator and posterior risk under SELF are: $\hat{d} = E_{\theta|\mathbf{x}}(\theta)$ and $\rho\left(\hat{d}\right) = E_{\theta|\mathbf{x}}\left(\theta^2\right) - \left\{E_{\theta|\mathbf{x}}(\theta)\right\}^2$, respectively. So, the Bayes estimators and posterior risks using the UP, the JP and IP for parameters θ_1, θ_2, θ_3, p_1 and p_2 under SELF are obtained with their respective marginal posterior distributions as given below:

$$\hat{\theta}_{1v} = \frac{\Gamma(A_{1v}+1)\Gamma(A_{2v})\Gamma(A_{3v})}{E_v} \sum_{i=0}^{n-r}\sum_{j=0}^{i} \binom{n-r}{i}\binom{i}{j} \times$$

$$\mathrm{B}_{1v}^{-(A_{1v}+1)}\ \mathrm{B}_{2v}^{-A_{2v}}\ \mathrm{B}_{3v}^{-A_{3v}}\ B(A_{0v},\ C_{0v})\ B(B_{0v},\ A_{0v}+C_{0v}) \tag{4.1}$$

$$\hat{\theta}_{2v} = \frac{\Gamma(A_{1v})\Gamma(A_{2v}+1)\Gamma(A_{3v})}{E_v} \sum_{i=0}^{n-r}\sum_{j=0}^{i} \binom{n-r}{i}\binom{i}{j} \times$$

$$\mathrm{B}_{1v}^{-A_{1v}}\ \mathrm{B}_{2v}^{-(A_{2v}+1)}\ \mathrm{B}_{3v}^{-A_{3v}}\ B(A_{0v},\ C_{0v})\ B(B_{0v},\ A_{0v}+C_{0v}) \tag{4.2}$$

$$\hat{\theta}_{3v} = \frac{\Gamma(A_{1v})\Gamma(A_{2v})\Gamma(A_{3v}+1)}{E_v} \sum_{i=0}^{n-r}\sum_{j=0}^{i} \binom{n-r}{i}\binom{i}{j} \times$$

$$\mathrm{B}_{1v}^{-A_{1v}}\ \mathrm{B}_{2v}^{-A_{2v}}\ \mathrm{B}_{3v}^{-(A_{3v}+1)}\ B(A_{0v},\ C_{0v})\ B(B_{0v},\ A_{0v}+C_{0v}) \tag{4.3}$$

$$\hat{p}_{1v} = \frac{\Gamma(A_{1v})\Gamma(A_{2v})\Gamma(A_{3v})}{E_v} \sum_{i=0}^{n-r}\sum_{j=0}^{i} \binom{n-r}{i}\binom{i}{j} \times$$

$$\mathrm{B}_{1v}^{-A_{1v}}\ \mathrm{B}_{2v}^{-A_{2v}}\ \mathrm{B}_{3v}^{-A_{3v}}\ B(B_{0v},\ C_{0v})\ B(A_{0v}+1,\ B_{0v}+C_{0v}) \tag{4.4}$$

$$\hat{p}_{2v} = \frac{\Gamma(A_{1v})\Gamma(A_{2v})\Gamma(A_{3v})}{E_v} \sum_{i=0}^{n-r}\sum_{j=0}^{i} \binom{n-r}{i}\binom{i}{j} \times$$

$$\mathrm{B}_{1v}^{-A_{1v}}\ \mathrm{B}_{2v}^{-A_{2v}}\ \mathrm{B}_{3v}^{-A_{3v}}\ B(A_{0v},\ C_{0v})\ B(B_{0v}+1,\ A_{0v}+C_{0v}) \tag{4.5}$$

$$\rho\left(\hat{\theta}_{1v}\right) = \frac{\Gamma(A_{1v}+2)\Gamma(A_{2v})\Gamma(A_{3v})}{E_v} \sum_{i=0}^{n-r}\sum_{j=0}^{i} \binom{n-r}{i}\binom{i}{j} \times$$

$$\mathrm{B}_{1v}^{-(A_{1v}+2)}\mathrm{B}_{2v}^{-A_{2v}}\mathrm{B}_{3v}^{-A_{3v}} B(A_{0v}, C_{0v})\, B(B_{0v}, A_{0v}+C_{0v}) -$$

$$\left\{\begin{array}{l} \frac{\Gamma(A_{1v}+1)\Gamma(A_{2v})\Gamma(A_{3v})}{E_v} \sum_{i=0}^{n-r}\sum_{j=0}^{i} \binom{n-r}{i}\binom{i}{j} \times \\ \mathrm{B}_{1v}^{-(A_{1v}+1)}\mathrm{B}_{2v}^{-A_{2v}}\mathrm{B}_{3v}^{-A_{3v}} B(A_{0v}, C_{0v})\, B(B_{0v}, A_{0v}+C_{0v}) \end{array}\right\}^2 \tag{4.6}$$

$$\rho\left(\hat{\theta}_{2v}\right) = \frac{\Gamma(A_{1v})\Gamma(A_{2v}+2)\Gamma(A_{3v})}{E_v} \sum_{i=0}^{n-r}\sum_{j=0}^{i} \binom{n-r}{i}\binom{i}{j} \times$$

$$\mathrm{B}_{1v}^{-A_{1v}}\mathrm{B}_{2v}^{-(A_{2v}+2)}\mathrm{B}_{3v}^{-A_{3v}} B(A_{0v}, C_{0v})\, B(B_{0v}, A_{0v}+C_{0v}) -$$

$$\left\{\begin{array}{l} \frac{\Gamma(A_{1v})\Gamma(A_{2v}+1)\Gamma(A_{3v})}{E_v} \sum_{i=0}^{n-r}\sum_{j=0}^{i} \binom{n-r}{i}\binom{i}{j} \times \\ \mathrm{B}_{1v}^{-A_{1v}}\mathrm{B}_{2v}^{-(A_{2v}+1)}\mathrm{B}_{3v}^{-A_{3v}} B(A_{0v}, C_{0v})\, B(B_{0v}, A_{0v}+C_{0v}) \end{array}\right\}^2 \tag{4.7}$$

$$\rho\left(\hat{\theta}_{3v}\right) = \frac{\Gamma(A_{1v})\Gamma(A_{2v})\Gamma(A_{3v}+2)}{E_v}\sum_{i=0}^{n-r}\sum_{j=0}^{i}\binom{n-r}{i}\binom{i}{j}\times$$

$$\mathrm{B}_{1v}^{-A_{1v}}\mathrm{B}_{2v}^{-A_{2v}}\mathrm{B}_{3v}^{-(A_{3v}+2)}B(A_{0v},C_{0v})B(B_{0v},A_{0v}+C_{0v}) -$$

$$\left\{\begin{array}{l}\frac{\Gamma(A_{1v})\Gamma(A_{2v})\Gamma(A_{3v}+1)}{E_v}\sum_{i=0}^{n-r}\sum_{j=0}^{i}\binom{n-r}{i}\binom{i}{j}\times \\ \mathrm{B}_{1v}^{-A_{1v}}\mathrm{B}_{2v}^{-A_{2v}}\mathrm{B}_{3v}^{-(A_{3v}+1)}B(A_{0v},C_{0v})B(B_{0v},A_{0v}+C_{0v})\end{array}\right\}^2 \quad (4.8)$$

$$\rho\left(\hat{p}_{1v}\right) = \frac{\Gamma(A_{1v})\Gamma(A_{2v})\Gamma(A_{3v})}{E_v}\sum_{i=0}^{n-r}\sum_{j=0}^{i}\binom{n-r}{i}\binom{i}{j}\times$$

$$\mathrm{B}_{1v}^{-A_{1v}}\mathrm{B}_{2v}^{-A_{2v}}\mathrm{B}_{3v}^{-A_{3v}}B(B_{0v},C_{0v})B(A_{0v}+2,B_{0v}+C_{0v}) -$$

$$\left\{\begin{array}{l}\frac{\Gamma(A_{1v})\Gamma(A_{2v})\Gamma(A_{3v})}{E_v}\sum_{i=0}^{n-r}\sum_{j=0}^{i}\binom{n-r}{i}\binom{i}{j}\times \\ \mathrm{B}_{1v}^{-A_{1v}}\mathrm{B}_{2v}^{-A_{2v}}\mathrm{B}_{3v}^{-A_{3v}}B(B_{0v},C_{0v})B(A_{0v}+1,B_{0v}+C_{0v})\end{array}\right\}^2 \quad (4.9)$$

$$\rho\left(\hat{p}_{2v}\right) = \frac{\Gamma(A_{1v})\Gamma(A_{2v})\Gamma(A_{3v})}{E_v}\sum_{i=0}^{n-r}\sum_{j=0}^{i}\binom{n-r}{i}\binom{i}{j}\times$$

$$\mathrm{B}_{1v}^{-A_{1v}}\mathrm{B}_{2v}^{-A_{2v}}\mathrm{B}_{3v}^{-A_{3v}}B(A_{0v},C_{0v})B(B_{0v}+2,A_{0v}+C_{0v}) -$$

$$\left\{\begin{array}{l}\frac{\Gamma(A_{1v})\Gamma(A_{2v})\Gamma(A_{3v})}{E_v}\sum_{i=0}^{n-r}\sum_{j=0}^{i}\binom{n-r}{i}\binom{i}{j}\times \\ \mathrm{B}_{1v}^{-A_{1v}}\mathrm{B}_{2v}^{-A_{2v}}\mathrm{B}_{3v}^{-A_{3v}}B(A_{0v},C_{0v})B(B_{0v}+1,A_{0v}+C_{0v})\end{array}\right\}^2, \quad (4.10)$$

where $v = 1$ for the UP, $v = 2$ for the JP and $v = 3$ for the IP.

5. The Bayes estimators and posterior risks using the UP, the JP and IP under PLF

Norstrom (1996) discussed an asymmetric PLF and a special case of general class of PLFs is $L(\theta,\ d) = \frac{(\theta-d)^2}{d}$. The Bayes estimator and posterior risk are: $\hat{d} = \left\{E_{\theta|\mathbf{x}}\left(\theta^2\right)\right\}^{\frac{1}{2}}$ and $\rho\left(\hat{d}\right) = 2\left\{E_{\theta|\mathbf{x}}\left(\theta^2\right)\right\}^{\frac{1}{2}} - 2E_{\theta|\mathbf{x}}(\theta)$, respectively. The respective marginal posterior distributions yield the Bayes estimators and posterior risks using the UP, the JP and the IP for parameters $\theta_1,\ \theta_2,\ \theta_3,\ p_1$ and p_2 under PLF as:

$$\hat{\theta}_{1v} = \left\{\begin{array}{l}\frac{\Gamma(A_{1v}+2)\Gamma(A_{2v})\Gamma(A_{3v})}{E_v}\sum_{i=0}^{n-r}\sum_{j=0}^{i}\binom{n-r}{i}\binom{i}{j}\times \\ \mathrm{B}_{1v}^{-(A_{1v}+2)}\mathrm{B}_{2v}^{-A_{2v}}\mathrm{B}_{3v}^{-A_{3v}}B(A_{0v},C_{0v})B(B_{0v},A_{0v}+C_{0v})\end{array}\right\}^{\frac{1}{2}} \quad (5.1)$$

$$\hat{\theta}_{2v} = \left\{\begin{array}{l}\frac{\Gamma(A_{1v})\Gamma(A_{2v}+2)\Gamma(A_{3v})}{E_v}\sum_{i=0}^{n-r}\sum_{j=0}^{i}\binom{n-r}{i}\binom{i}{j}\times \\ \mathrm{B}_{1v}^{-A_{1v}}\mathrm{B}_{2v}^{-(A_{2v}+2)}\mathrm{B}_{3v}^{-A_{3v}}B(A_{0v},C_{0v})B(B_{0v},A_{0v}+C_{0v})\end{array}\right\}^{\frac{1}{2}} \quad (5.2)$$

$$\hat{\theta}_{3v} = \left\{\begin{array}{l}\frac{\Gamma(A_{1v})\Gamma(A_{2v})\Gamma(A_{3v}+2)}{E_v}\sum_{i=0}^{n-r}\sum_{j=0}^{i}\binom{n-r}{i}\binom{i}{j}\times \\ \mathrm{B}_{1v}^{-A_{1v}}\mathrm{B}_{2v}^{-A_{2v}}\mathrm{B}_{3v}^{-(A_{3v}+2)}B(A_{0v},C_{0v})B(B_{0v},A_{0v}+C_{0v})\end{array}\right\}^{\frac{1}{2}} \quad (5.3)$$

$$\hat{p}_{1v} = \left\{\begin{array}{l}\frac{\Gamma(A_{1v})\Gamma(A_{2v})\Gamma(A_{3v})}{E_v}\sum_{i=0}^{n-r}\sum_{j=0}^{i}\binom{n-r}{i}\binom{i}{j}\times \\ \mathrm{B}_{1v}^{-A_{1v}}\mathrm{B}_{2v}^{-A_{2v}}\mathrm{B}_{3v}^{-A_{3v}}B(B_{0v},C_{0v})B(A_{0v}+2,B_{0v}+C_{0v})\end{array}\right\}^{\frac{1}{2}} \quad (5.4)$$

$$\hat{p}_{2v} = \left\{ \begin{array}{l} \frac{\Gamma(A_{1v})\Gamma(A_{2v})\Gamma(A_{3v})}{E_v} \sum_{i=0}^{n-r} \sum_{j=0}^{i} \binom{n-r}{i} \binom{i}{j} \times \\ \mathrm{B}_{1v}^{-A_{1v}} \mathrm{B}_{2v}^{-A_{2v}} \mathrm{B}_{3v}^{-A_{3v}} B\left(A_{0v}, C_{0v}\right) B\left(B_{0v}+2, A_{0v}+C_{0v}\right) \end{array} \right\}^{\frac{1}{2}} \tag{5.5}$$

$$\begin{aligned} \rho\left(\hat{\theta}_{1v}\right) = 2 & \left\{ \begin{array}{l} \frac{\Gamma(A_{1v}+2)\Gamma(A_{2v})\Gamma(A_{3v})}{E_v} \sum_{i=0}^{n-r} \sum_{j=0}^{i} \binom{n-r}{i} \binom{i}{j} \mathrm{B}_{1v}^{-(A_{1v}+2)} \times \\ \mathrm{B}_{2v}^{-A_{2v}} \mathrm{B}_{3v}^{-A_{3v}} B\left(A_{0v}, C_{0v}\right) B\left(B_{0v}, A_{0v}+C_{0v}\right) \end{array} \right\}^{\frac{1}{2}} \\ & - 2 \left\{ \begin{array}{l} \frac{\Gamma(A_{1v}+1)\Gamma(A_{2v})\Gamma(A_{3v})}{E_v} \sum_{i=0}^{n-r} \sum_{j=0}^{i} \binom{n-r}{i} \binom{i}{j} \mathrm{B}_{1v}^{-(A_{1v}+1)} \times \\ \mathrm{B}_{2v}^{-A_{2v}} \mathrm{B}_{3v}^{-A_{3v}} B\left(A_{0v}, C_{0v}\right) B\left(B_{0v}, A_{0v}+C_{0v}\right) \end{array} \right\} \end{aligned} \tag{5.6}$$

$$\begin{aligned} \rho\left(\hat{\theta}_{2v}\right) = 2 & \left\{ \begin{array}{l} \frac{\Gamma(A_{1v})\Gamma(A_{2v}+2)\Gamma(A_{3v})}{E_v} \sum_{i=0}^{n-r} \sum_{j=0}^{i} \binom{n-r}{i} \binom{i}{j} \mathrm{B}_{1v}^{-A_{1v}} \times \\ \mathrm{B}_{2v}^{-(A_{2v}+2)} \mathrm{B}_{3v}^{-A_{3v}} B\left(A_{0v}, C_{0v}\right) B\left(B_{0v}, A_{0v}+C_{0v}\right) \end{array} \right\}^{\frac{1}{2}} \\ & - 2 \left\{ \begin{array}{l} \frac{\Gamma(A_{1v})\Gamma(A_{2v}+1)\Gamma(A_{3v})}{E_v} \sum_{i=0}^{n-r} \sum_{j=0}^{i} \binom{n-r}{i} \binom{i}{j} \mathrm{B}_{1v}^{-A_{1v}} \times \\ \mathrm{B}_{2v}^{-(A_{2v}+1)} \mathrm{B}_{3v}^{-A_{3v}} B\left(A_{0v}, C_{0v}\right) B\left(B_{0v}, A_{0v}+C_{0v}\right) \end{array} \right\} \end{aligned} \tag{5.7}$$

$$\begin{aligned} \rho\left(\hat{\theta}_{3v}\right) = 2 & \left\{ \begin{array}{l} \frac{\Gamma(A_{1v})\Gamma(A_{2v})\Gamma(A_{3v}+2)}{E_v} \sum_{i=0}^{n-r} \sum_{j=0}^{i} \binom{n-r}{i} \binom{i}{j} \mathrm{B}_{1v}^{-A_{1v}} \times \\ \mathrm{B}_{2v}^{-A_{2v}} \mathrm{B}_{3v}^{-(A_{3v}+2)} B\left(A_{0v}, C_{0v}\right) B\left(B_{0v}, A_{0v}+C_{0v}\right) \end{array} \right\}^{\frac{1}{2}} \\ & - 2 \left\{ \begin{array}{l} \frac{\Gamma(A_{1v})\Gamma(A_{2v})\Gamma(A_{3v}+1)}{E_v} \sum_{i=0}^{n-r} \sum_{j=0}^{i} \binom{n-r}{i} \binom{i}{j} \mathrm{B}_{1v}^{-A_{1v}} \times \\ \mathrm{B}_{2v}^{-A_{2v}} \mathrm{B}_{3v}^{-(A_{3v}+1)} B\left(A_{0v}, C_{0v}\right) B\left(B_{0v}, A_{0v}+C_{0v}\right) \end{array} \right\} \end{aligned} \tag{5.8}$$

$$\begin{aligned} \rho\left(\hat{p}_{1v}\right) = 2 & \left\{ \begin{array}{l} \frac{\Gamma(A_{1v})\Gamma(A_{2v})\Gamma(A_{3v})}{E_v} \sum_{i=0}^{n-r} \sum_{j=0}^{i} \binom{n-r}{i} \binom{i}{j} \mathrm{B}_{1v}^{-A_{1v}} \times \\ \mathrm{B}_{2v}^{-A_{2v}} \mathrm{B}_{3v}^{-A_{3v}} B\left(B_{0v}, C_{0v}\right) B\left(A_{0v}+2, B_{0v}+C_{0v}\right) \end{array} \right\}^{\frac{1}{2}} \\ & - 2 \left\{ \begin{array}{l} \frac{\Gamma(A_{1v})\Gamma(A_{2v})\Gamma(A_{3v})}{E_v} \sum_{i=0}^{n-r} \sum_{j=0}^{i} \binom{n-r}{i} \binom{i}{j} \mathrm{B}_{1v}^{-A_{1v}} \times \\ \mathrm{B}_{2v}^{-A_{2v}} \mathrm{B}_{3v}^{-A_{3v}} B\left(B_{0v}, C_{0v}\right) B\left(A_{0v}+1, B_{0v}+C_{0v}\right) \end{array} \right\} \end{aligned} \tag{5.9}$$

$$\begin{aligned} \rho\left(\hat{p}_{2v}\right) = 2 & \left\{ \begin{array}{l} \frac{\Gamma(A_{1v})\Gamma(A_{2v})\Gamma(A_{3v})}{E_v} \sum_{i=0}^{n-r} \sum_{j=0}^{i} \binom{n-r}{i} \binom{i}{j} \mathrm{B}_{1v}^{-A_{1v}} \times \\ \mathrm{B}_{2v}^{-A_{2v}} \mathrm{B}_{3v}^{-A_{3v}} B\left(A_{0v}, C_{0v}\right) B\left(B_{0v}+2, A_{0v}+C_{0v}\right) \end{array} \right\}^{\frac{1}{2}} \\ & - 2 \left\{ \begin{array}{l} \frac{\Gamma(A_{1v})\Gamma(A_{2v})\Gamma(A_{3v})}{E_v} \sum_{i=0}^{n-r} \sum_{j=0}^{i} \binom{n-r}{i} \binom{i}{j} \mathrm{B}_{1v}^{-A_{1v}} \times \\ \mathrm{B}_{2v}^{-A_{2v}} \mathrm{B}_{3v}^{-A_{3v}} B\left(A_{0v}, C_{0v}\right) B\left(B_{0v}+1, A_{0v}+C_{0v}\right) \end{array} \right\} \end{aligned} \tag{5.10}$$

6. The Bayes estimators and posterior risks using the UP, the JP and the IP under DLF

DeGroot (2005) introduced the asymmetric loss function, $L(\theta,\ d) = \left(\frac{\theta-d}{d}\right)^2$, known as DLF.The Bayes estimator and its posterior risk under DLF are: $\hat{d} = \frac{E_{\theta|\mathbf{x}}\left(\theta^2\right)}{E_{\theta|\mathbf{x}}(\theta)}$ and

$\rho\left(\hat{d}\right) = 1 - \frac{\left\{E_{\theta|\mathbf{x}}(\theta)\right\}^2}{E_{\theta|\mathbf{x}}\left(\theta^2\right)}$, respectively. The Bayes estimators and posterior risks using the UP, the JP and the IP for parameters $\theta_1,\ \theta_2,\ \theta_3,\ p_1$ and p_2 under DLF are:

$$\hat{\theta}_{1v} = \frac{\Gamma(A_{1v}+2)\Gamma(A_{2v})\Gamma(A_{3v})}{\Gamma(A_{1v}+1)\Gamma(A_{2v})\Gamma(A_{3v})}\times \frac{\sum_{i=0}^{n-r}\sum_{j=0}^{i}\binom{n-r}{i}\binom{i}{j}\mathrm{B}_{1v}^{-(A_{1v}+2)}\mathrm{B}_{2v}^{-A_{2v}}\mathrm{B}_{3v}^{-A_{3v}}B(A_{0v},C_{0v})B(B_{0v},A_{0v}+C_{0v})}{\sum_{i=0}^{n-r}\sum_{j=0}^{i}\binom{n-r}{i}\binom{i}{j}\mathrm{B}_{1v}^{-(A_{1v}+1)}\mathrm{B}_{2v}^{-A_{2v}}\mathrm{B}_{3v}^{-A_{3v}}B(A_{0v},C_{0v})B(B_{0v},A_{0v}+C_{0v})} \tag{6.1}$$

$$\hat{\theta}_{2v} = \frac{\Gamma(A_{1v})\Gamma(A_{2v}+2)\Gamma(A_{3v})}{\Gamma(A_{1v})\Gamma(A_{2v}+1)\Gamma(A_{3v})}\times \frac{\sum_{i=0}^{n-r}\sum_{j=0}^{i}\binom{n-r}{i}\binom{i}{j}\mathrm{B}_{1v}^{-A_{1v}}\mathrm{B}_{2v}^{-(A_{2v}+2)}\mathrm{B}_{3v}^{-A_{3v}}B(A_{0v},C_{0v})B(B_{0v},A_{0v}+C_{0v})}{\sum_{i=0}^{n-r}\sum_{j=0}^{i}\binom{n-r}{i}\binom{i}{j}\mathrm{B}_{1v}^{-A_{1v}}\mathrm{B}_{2v}^{-(A_{2v}+1)}\mathrm{B}_{3v}^{-A_{3v}}B(A_{0v},C_{0v})B(B_{0v},A_{0v}+C_{0v})} \tag{6.2}$$

$$\hat{\theta}_{3v} = \frac{\Gamma(A_{1v})\Gamma(A_{2v})\Gamma(A_{3v}+2)}{\Gamma(A_{1v})\Gamma(A_{2v})\Gamma(A_{3v}+1)}\times \frac{\sum_{i=0}^{n-r}\sum_{j=0}^{i}\binom{n-r}{i}\binom{i}{j}\mathrm{B}_{1v}^{-A_{1v}}\mathrm{B}_{2v}^{-A_{2v}}\mathrm{B}_{3v}^{-(A_{3v}+2)}B(A_{0v},C_{0v})B(B_{0v},A_{0v}+C_{0v})}{\sum_{i=0}^{n-r}\sum_{j=0}^{i}\binom{n-r}{i}\binom{i}{j}\mathrm{B}_{1v}^{-A_{1v}}\mathrm{B}_{2v}^{-A_{2v}}\mathrm{B}_{3v}^{-(A_{3v}+1)}B(A_{0v},C_{0v})B(B_{0v},A_{0v}+C_{0v})} \tag{6.3}$$

$$\hat{p}_{1v} = \frac{\Gamma(A_{1v})\Gamma(A_{2v})\Gamma(A_{3v})}{\Gamma(A_{1v})\Gamma(A_{2v})\Gamma(A_{3v})}\times \frac{\sum_{i=0}^{n-r}\sum_{j=0}^{i}\binom{n-r}{i}\binom{i}{j}\mathrm{B}_{1v}^{-A_{1v}}\mathrm{B}_{2v}^{-A_{2v}}\mathrm{B}_{3v}^{-A_{3v}}B(B_{0v},C_{0v})B(A_{0v}+2,B_{0v}+C_{0v})}{\sum_{i=0}^{n-r}\sum_{j=0}^{i}\binom{n-r}{i}\binom{i}{j}\mathrm{B}_{1v}^{-A_{1v}}\mathrm{B}_{2v}^{-A_{2v}}\mathrm{B}_{3v}^{-A_{3v}}B(B_{0v},C_{0v})B(A_{0v}+1,B_{0v}+C_{0v})} \tag{6.4}$$

$$\hat{p}_{2v} = \frac{\Gamma(A_{1v})\Gamma(A_{2v})\Gamma(A_{3v})}{\Gamma(A_{1v})\Gamma(A_{2v})\Gamma(A_{3v})}\times \frac{\sum_{i=0}^{n-r}\sum_{j=0}^{i}\binom{n-r}{i}\binom{i}{j}\mathrm{B}_{1v}^{-A_{1v}}\mathrm{B}_{2v}^{-A_{2v}}\mathrm{B}_{3v}^{-A_{3v}}B(A_{0v},C_{0v})B(B_{0v}+2,A_{0v}+C_{0v})}{\sum_{i=0}^{n-r}\sum_{j=0}^{i}\binom{n-r}{i}\binom{i}{j}\mathrm{B}_{1v}^{-A_{1v}}\mathrm{B}_{2v}^{-A_{2v}}\mathrm{B}_{3v}^{-A_{3v}}B(A_{0v},C_{0v})B(B_{0v}+1,A_{0v}+C_{0v})} \tag{6.5}$$

$$\rho\left(\hat{\theta}_{1v}\right) = 1 - \frac{\{\Gamma(A_{1v}+1)\Gamma(A_{2v})\Gamma(A_{3v})\}^2}{E_v\Gamma(A_{1v}+2)\Gamma(A_{2v})\Gamma(A_{3v})}\times \frac{\left\{\sum_{i=0}^{n-r}\sum_{j=0}^{i}\binom{n-r}{i}\binom{i}{j}\mathrm{B}_{1v}^{-(A_{1v}+1)}\mathrm{B}_{2v}^{-A_{2v}}\mathrm{B}_{3v}^{-A_{3v}}B(A_{0v},C_{0v})B(B_{0v},A_{0v}+C_{0v})\right\}^2}{\sum_{i=0}^{n-r}\sum_{j=0}^{i}\binom{n-r}{i}\binom{i}{j}\mathrm{B}_{1v}^{-(A_{1v}+2)}\mathrm{B}_{2v}^{-A_{2v}}\mathrm{B}_{3v}^{-A_{3v}}B(A_{0v},C_{0v})B(B_{0v},A_{0v}+C_{0v})} \tag{6.6}$$

$$\rho\left(\hat{\theta}_{2v}\right) = 1 - \frac{\{\Gamma(A_{1v})\Gamma(A_{2v}+1)\Gamma(A_{3v})\}^2}{E_v\Gamma(A_{1v})\Gamma(A_{2v}+2)\Gamma(A_{3v})}\times \frac{\left\{\sum_{i=0}^{n-r}\sum_{j=0}^{i}\binom{n-r}{i}\binom{i}{j}\mathrm{B}_{1v}^{-A_{1v}}\mathrm{B}_{2v}^{-(A_{2v}+1)}\mathrm{B}_{3v}^{-A_{3v}}B(A_{0v},C_{0v})B(B_{0v},A_{0v}+C_{0v})\right\}^2}{\sum_{i=0}^{n-r}\sum_{j=0}^{i}\binom{n-r}{i}\binom{i}{j}\mathrm{B}_{1v}^{-A_{1v}}\mathrm{B}_{2v}^{-(A_{2v}+2)}\mathrm{B}_{3v}^{-A_{3v}}B(A_{0v},C_{0v})B(B_{0v},A_{0v}+C_{0v})} \tag{6.7}$$

$$\rho\left(\hat{\theta}_{3v}\right) = 1 - \tfrac{\{\Gamma(A_{1v})\Gamma(A_{2v})\Gamma(A_{3v}+1)\}^2}{E_v\Gamma(A_{1v})\Gamma(A_{2v})\Gamma(A_{3v}+2)} \times \frac{\left\{\sum_{i=0}^{n-r}\sum_{j=0}^{i}\binom{n-r}{i}\binom{i}{j} \mathrm{B}_{1v}^{-A_{1v}}\mathrm{B}_{2v}^{-A_{2v}}\mathrm{B}_{3v}^{-(A_{3v}+1)} B(A_{0v},C_{0v})B(B_{0v},A_{0v}+C_{0v})\right\}^2}{\sum_{i=0}^{n-r}\sum_{j=0}^{i}\binom{n-r}{i}\binom{i}{j} \mathrm{B}_{1v}^{-A_{1v}}\mathrm{B}_{2v}^{-A_{2v}}\mathrm{B}_{3v}^{-(A_{3v}+2)} B(A_{0v},C_{0v})B(B_{0v},A_{0v}+C_{0v})} \tag{6.8}$$

$$\rho\left(\hat{p}_{1v}\right) = 1 - \tfrac{\{\Gamma(A_{1v})\Gamma(A_{2v})\Gamma(A_{3v})\}^2}{E_v\Gamma(A_{1v})\Gamma(A_{2v})\Gamma(A_{3v})} \times \frac{\left\{\sum_{i=0}^{n-r}\sum_{j=0}^{i}\binom{n-r}{i}\binom{i}{j} \mathrm{B}_{1v}^{-A_{1v}}\mathrm{B}_{2v}^{-A_{2v}}\mathrm{B}_{3v}^{-A_{3v}} B(B_{0v},C_{0v})B(A_{0v}+1,B_{0v}+C_{02})\right\}^2}{\sum_{i=0}^{n-r}\sum_{j=0}^{i}\binom{n-r}{i}\binom{i}{j} \mathrm{B}_{1v}^{-A_{1v}}\mathrm{B}_{2v}^{-A_{2v}}\mathrm{B}_{3v}^{-A_{3v}} B(B_{0v},C_{0v})B(A_{0v}+2,B_{0v}+C_{02})} \tag{6.9}$$

$$\rho\left(\hat{p}_{2v}\right) = 1 - \tfrac{\{\Gamma(A_{1v})\Gamma(A_{2v})\Gamma(A_{3v})\}^2}{E_v\Gamma(A_{1v})\Gamma(A_{2v})\Gamma(A_{3v})} \times \frac{\left\{\sum_{i=0}^{n-r}\sum_{j=0}^{i}\binom{n-r}{i}\binom{i}{j} \mathrm{B}_{1v}^{-A_{1v}}\mathrm{B}_{2v}^{-A_{2v}}\mathrm{B}_{3v}^{-A_{3v}} B(A_{0v},C_{0v})B(B_{0v}+1,A_{0v}+C_{0v})\right\}^2}{\sum_{i=0}^{n-r}\sum_{j=0}^{i}\binom{n-r}{i}\binom{i}{j} \mathrm{B}_{1v}^{-A_{1v}}\mathrm{B}_{2v}^{-A_{2v}}\mathrm{B}_{3v}^{-A_{3v}} B(A_{0v},C_{0v})B(B_{0v}+2,A_{0v}+C_{0v})}. \tag{6.10}$$

7. Elicitation of hyperparameters

Elicitation is a tool used to quantify a person's belief and knowledge about the parameter(s) of interest. In Bayesian perspective, elicitation, most often, arises as a method for specifying the prior distribution of the random parameter(s). Elicitation is simply the quantification of prior knowledge about the random parameter(s) so that this can then be combined with the likelihood to obtain posterior distribution for further statistical analysis. Elicitation has remained a challenging problem for the statistician.Authors who have discussed this problem include Kadane et al. (1980), Birch and Bartollucci (1983), Chaloner and Duncan (1983), Gavasakar (1988), Al-Awadhi and Gartwaite (1998), Aslam (2003), Hahn (2006), Saleem and Aslam (2008) and references cited therein. In this study, we adopted a method based on predictive probabilities, given by Aslam (2003).

For eliciting the hyperparameters, prior predictive distribution (PPD) is used. The PPD for a random variable X is:

$$p(x) = \int_{\boldsymbol{\psi}} p(x\,|\boldsymbol{\psi})\,\pi_3(\boldsymbol{\psi})\,d\boldsymbol{\psi} \tag{7.1}$$

$$p(x) = \frac{1}{(a+b+c)}\left[\frac{a\,a_1\,b_1^{a_1}}{(b_1+x)^{a_1+1}} + \frac{b\,a_2\,b_2^{a_2}}{(b_2+x)^{a_2+1}} + \frac{c\,a_3\,b_3^{a_3}}{(b_3+x)^{a_3+1}}\right]. \tag{7.2}$$

We choose the prior predictive probabilities, satisfying the laws of probability, to elicit the hyperparameters of the prior density. By following these laws of probability, some minor inconsistencies may arise which are expected to be ignorable. Using the prior predictive distribution given in (7.2) we consider nine intervals (0, 1), (1, 2), (2, 3), (3, 4), (4, 5), (5, 6), (6, 7), (7, 8) and (8, 9) with probabilities 0.57, 0.20, 0.10, 0.05, 0.02, 0.015, 0.01, 0.005 and 0.003, respectively, given as expert opinion. The following nine equations are derived from the given information using the (7.2) as:

$$\frac{1}{(a+b+c)}\int_0^1\left[\frac{a\,a_1\,b_1^{a_1}}{(b_1+x)^{a_1+1}} + \frac{b\,a_2\,b_2^{a_2}}{(b_2+x)^{a_2+1}} + \frac{c\,a_3\,b_3^{a_3}}{(b_3+x)^{a_3+1}}\right]dx = 0.57 \tag{7.3}$$

$$\frac{1}{(a+b+c)}\int_1^2\left[\frac{a\,a_1\,b_1^{a_1}}{(b_1+x)^{a_1+1}}+\frac{b\,a_2\,b_2^{a_2}}{(b_2+x)^{a_2+1}}+\frac{c\,a_3\,b_3^{a_3}}{(b_3+x)^{a_3+1}}\right]dx=0.20 \tag{7.4}$$

$$\frac{1}{(a+b+c)}\int_2^3\left[\frac{a\,a_1\,b_1^{a_1}}{(b_1+x)^{a_1+1}}+\frac{b\,a_2\,b_2^{a_2}}{(b_2+x)^{a_2+1}}+\frac{c\,a_3\,b_3^{a_3}}{(b_3+x)^{a_3+1}}\right]dx=0.10 \tag{7.5}$$

$$\frac{1}{(a+b+c)}\int_3^4\left[\frac{a\,a_1\,b_1^{a_1}}{(b_1+x)^{a_1+1}}+\frac{b\,a_2\,b_2^{a_2}}{(b_2+x)^{a_2+1}}+\frac{c\,a_3\,b_3^{a_3}}{(b_3+x)^{a_3+1}}\right]dx=0.05 \tag{7.6}$$

$$\frac{1}{(a+b+c)}\int_4^5\left[\frac{a\,a_1\,b_1^{a_1}}{(b_1+x)^{a_1+1}}+\frac{b\,a_2\,b_2^{a_2}}{(b_2+x)^{a_2+1}}+\frac{c\,a_3\,b_3^{a_3}}{(b_3+x)^{a_3+1}}\right]dx=0.02 \tag{7.7}$$

$$\frac{1}{(a+b+c)}\int_5^6\left[\frac{a\,a_1\,b_1^{a_1}}{(b_1+x)^{a_1+1}}+\frac{b\,a_2\,b_2^{a_2}}{(b_2+x)^{a_2+1}}+\frac{c\,a_3\,b_3^{a_3}}{(b_3+x)^{a_3+1}}\right]dx=0.015 \tag{7.8}$$

$$\frac{1}{(a+b+c)}\int_6^7\left[\frac{a\,a_1\,b_1^{a_1}}{(b_1+x)^{a_1+1}}+\frac{b\,a_2\,b_2^{a_2}}{(b_2+x)^{a_2+1}}+\frac{c\,a_3\,b_3^{a_3}}{(b_3+x)^{a_3+1}}\right]dx=0.01 \tag{7.9}$$

$$\frac{1}{(a+b+c)}\int_7^8\left[\frac{a\,a_1\,b_1^{a_1}}{(b_1+x)^{a_1+1}}+\frac{b\,a_2\,b_2^{a_2}}{(b_2+x)^{a_2+1}}+\frac{c\,a_3\,b_3^{a_3}}{(b_3+x)^{a_3+1}}\right]dx=0.005 \tag{7.10}$$

$$\frac{1}{(a+b+c)}\int_8^9\left[\frac{a\,a_1\,b_1^{a_1}}{(b_1+x)^{a_1+1}}+\frac{b\,a_2\,b_2^{a_2}}{(b_2+x)^{a_2+1}}+\frac{c\,a_3\,b_3^{a_3}}{(b_3+x)^{a_3+1}}\right]dx=0.003 \tag{7.11}$$

The above nine equations (7.3-7.11) are solved simultaneously by using Mathematica software for eliciting the hyperparameters (a_1, b_1, a_2, b_2, a_3, b_3, a, b, c). Through this criteria, the values of the hyperparameters are obtained as (3.8330, 3.7310, 3.3570, 3.1360, 2.9030, 2.7330, 3.0280, 0.6995, 2.7350).

8. The limiting expressions

When t tends to ∞, r tends to n and r_l tends to n_l, $l = 1, 2, 3$, then all the values which are censored become uncensored in our analysis. So, the information contained in the sample is increased. Consequently, the posterior risks of the Bayes estimates diminish. The efficiency of the Bayes estimates is increased because all the values are incorporated in our sample. The limiting (complete sample) expressions for Bayes estimators and posterior risks using the UP, the JP and the IP under SELF, PLF and DLF are given in the Tables 1-6.

Table 1. Limiting Expressions for the Bayes Estimators as $t \to \infty$ using the UP, the JP and the IP under SELF

	Bayes Estimators		
Parameters	UP	JP	IP
θ_1	$\frac{n_1+1}{\sum_{k=1}^{n_1} x_{1k}}$	$\frac{n_1}{\sum_{k=1}^{n_1} x_{1k}}$	$\frac{n_1+a_1}{\sum_{k=1}^{n_1} x_{1k}+b_1}$
θ_2	$\frac{n_2+1}{\sum_{k=1}^{n_2} x_{2k}}$	$\frac{n_2}{\sum_{k=1}^{n_2} x_{2k}}$	$\frac{n_2+a_2}{\sum_{k=1}^{n_2} x_{2k}+b_2}$
θ_3	$\frac{n_3+1}{\sum_{k=1}^{n_3} x_{3k}}$	$\frac{n_3}{\sum_{k=1}^{n_3} x_{3k}}$	$\frac{n_3+a_3}{\sum_{k=1}^{n_3} x_{3k}+b_3}$
p_1	$\frac{n_1+1}{n+3}$	$\frac{n_1+1}{n+3}$	$\frac{n_1+a}{n+a+b+c}$
p_2	$\frac{n_2+1}{n+3}$	$\frac{n_2+1}{n+3}$	$\frac{n_2+b}{n+a+b+c}$

Table 2. Limiting Expressions for the Posterior Risks as $t \to \infty$ using the UP, the JP and the IP under SELF

	Posterior Risks		
Parameters	UP	JP	IP
θ_1	$\frac{n_1+1}{\left(\sum_{k=1}^{n_1} x_{1k}\right)^2}$	$\frac{n_1}{\left(\sum_{k=1}^{n_1} x_{1k}\right)^2}$	$\frac{n_1+a_1}{\left(\sum_{k=1}^{n_1} x_{1k}+b_1\right)^2}$
θ_2	$\frac{n_2+1}{\left(\sum_{k=1}^{n_2} x_{2k}\right)^2}$	$\frac{n_2}{\left(\sum_{k=1}^{n_2} x_{2k}\right)^2}$	$\frac{n_2+a_2}{\left(\sum_{k=1}^{n_2} x_{2k}+b_2\right)^2}$
θ_3	$\frac{n_3+1}{\left(\sum_{k=1}^{n_3} x_{3k}\right)^2}$	$\frac{n_3}{\left(\sum_{k=1}^{n_3} x_{3k}\right)^2}$	$\frac{n_3+a_3}{\left(\sum_{k=1}^{n_3} x_{3k}+b_3\right)^2}$
p_1	$\frac{(n_1+1)(n_2+n_3+2)}{(n+3)^2(n+4)}$	$\frac{(n_1+1)(n_2+n_3+2)}{(n+3)^2(n+4)}$	$\frac{(n_1+a)(n_2+n_3+b+c)}{(n+a+b+c)^2(n+a+b+c+1)}$
p_2	$\frac{(n_2+1)(n_1+n_3+2)}{(n+3)^2(n+4)}$	$\frac{(n_2+1)(n_1+n_3+2)}{(n+3)^2(n+4)}$	$\frac{(n_2+b)(n_1+n_3+a+c)}{(n+a+b+c)^2(n+a+b+c+1)}$

Table 3. Limiting expressions for the Bayes estimators as $t \to \infty$ using the UP, the JP and the IP under PLF

	Bayes Estimators		
Parameters	UP	JP	IP
θ_1	$\frac{(n_1+1)^{1/2}(n_1+2)^{1/2}}{\left(\sum_{k=1}^{n_1} x_{1k}\right)^{1/2}}$	$\frac{(n_1)^{1/2}(n_1+1)^{1/2}}{\left(\sum_{k=1}^{n_1} x_{1k}\right)^{1/2}}$	$\frac{(n_1+a_1)^{1/2}(n_1+a_1+1)^{1/2}}{\left(\sum_{k=1}^{n_1} x_{1k}+b_1\right)^{1/2}}$
θ_2	$\frac{(n_2+1)^{1/2}(n_2+2)^{1/2}}{\left(\sum_{k=1}^{n_2} x_{2k}\right)^{1/2}}$	$\frac{(n_2)^{1/2}(n_2+1)^{1/2}}{\left(\sum_{k=1}^{n_2} x_{2k}\right)^{1/2}}$	$\frac{(n_2+a_2)^{1/2}(n_2+a_2+1)^{1/2}}{\left(\sum_{k=1}^{n_2} x_{2k}+b_2\right)^{1/2}}$
θ_3	$\frac{(n_3+1)^{1/2}(n_3+2)^{1/2}}{\left(\sum_{k=1}^{n_3} x_{3k}\right)^{1/2}}$	$\frac{(n_3)^{1/2}(n_3+1)^{1/2}}{\left(\sum_{k=1}^{n_3} x_{3k}\right)^{1/2}}$	$\frac{(n_3+a_3)^{1/2}(n_3+a_3+1)^{1/2}}{\left(\sum_{k=1}^{n_3} x_{3k}+b_3\right)^{1/2}}$
p_1	$\frac{(n_1+1)^{1/2}(n_1+2)^{1/2}}{(n+3)^{1/2}(n+4)^{1/2}}$	$\frac{(n_1+1)^{1/2}(n_1+2)^{1/2}}{(n+3)^{1/2}(n+4)^{1/2}}$	$\frac{(n_1+a)^{1/2}(n_1+a+1)^{1/2}}{(n+a+b+c)^{1/2}(n+a+b+c+1)^{1/2}}$
p_2	$\frac{(n_2+1)^{1/2}(n_2+2)^{1/2}}{(n+3)^{1/2}(n+4)^{1/2}}$	$\frac{(n_2+1)^{1/2}(n_2+2)^{1/2}}{(n+3)^{1/2}(n+4)^{1/2}}$	$\frac{(n_2+b)^{1/2}(n_2+b+1)^{1/2}}{(n+a+b+c)^{1/2}(n+a+b+c+1)^{1/2}}$

Table 4. Limiting expressions for the posterior risks as $t \to \infty$ using the UP, the JP and the IP under PLF

	Posterior Risks		
Parameters	UP	JP	IP
θ_1	$\frac{2(n_1+1)}{\sum_{k=1}^{n_1} x_{1k}}\left\{\frac{(n_1+2)^{1/2}}{(n_1+1)^{1/2}}-1\right\}$	$\frac{2n_1}{\sum_{k=1}^{n_1} x_{1k}}\left\{\frac{(n_1+1)^{1/2}}{(n_1)^{1/2}}-1\right\}$	$\frac{2(n_1+a_1)}{\left(\sum_{k=1}^{n_1} x_{1k}+b_1\right)}\left\{\frac{(n_1+a_1+1)^{1/2}}{(n_1+a_1)^{1/2}}-1\right\}$
θ_2	$\frac{2(n_2+1)}{\sum_{k=1}^{n_2} x_{2k}}\left\{\frac{(n_2+2)^{1/2}}{(n_2+1)^{1/2}}-1\right\}$	$\frac{2n_2}{\sum_{k=1}^{n_2} x_{2k}}\left\{\frac{(n_2+1)^{1/2}}{(n_2)^{1/2}}-1\right\}$	$\frac{2(n_2+a_2)}{\left(\sum_{k=1}^{n_2} x_{2k}+b_2\right)}\left\{\frac{(n_2+a_2+1)^{1/2}}{(n_2+a_2)^{1/2}}-1\right\}$
θ_3	$\frac{2(n_3+1)}{\sum_{k=1}^{n_3} x_{3k}}\left\{\frac{(n_3+2)^{1/2}}{(n_3+1)^{1/2}}-1\right\}$	$\frac{2n_3}{\sum_{k=1}^{n_3} x_{3k}}\left\{\frac{(n_3+1)^{1/2}}{(n_3)^{1/2}}-1\right\}$	$\frac{2(n_3+a_3)}{\left(\sum_{k=1}^{n_3} x_{3k}+b_3\right)}\left\{\frac{(n_3+a_3+1)^{1/2}}{(n_3+a_3)^{1/2}}-1\right\}$
p_1	$\frac{2(n_1+1)}{(n+3)}\left\{\frac{\frac{(n_1+2)^{1/2}}{(n_1+1)^{1/2}}}{\frac{(n+4)^{1/2}}{(n+3)^{1/2}}}-1\right\}$	$\frac{2(n_1+1)}{(n+3)}\left\{\frac{\frac{(n_1+2)^{1/2}}{(n_1+1)^{1/2}}}{\frac{(n+4)^{1/2}}{(n+3)^{1/2}}}-1\right\}$	$\frac{2(n_1+a)}{(n+a+b+c)}\left\{\frac{\frac{(n_1+a+1)^{1/2}}{(n_1+a)^{1/2}}}{\frac{(n+a+b+c+1)^{1/2}}{(n+a+b+c)^{1/2}}}-1\right\}$
p_2	$\frac{2(n_2+1)}{(n+3)}\left\{\frac{\frac{(n_2+2)^{1/2}}{(n_2+1)^{1/2}}}{\frac{(n+4)^{1/2}}{(n+3)^{1/2}}}-1\right\}$	$\frac{2(n_2+1)}{(n+3)}\left\{\frac{\frac{(n_2+2)^{1/2}}{(n_2+1)^{1/2}}}{\frac{(n+4)^{1/2}}{(n+3)^{1/2}}}-1\right\}$	$\frac{2(n_2+b)}{(n+a+b+c)}\left\{\frac{\frac{(n_2+b+1)^{1/2}}{(n_2+b)^{1/2}}}{\frac{(n+a+b+c+1)^{1/2}}{(n+a+b+c)^{1/2}}}-1\right\}$

9. Simulation study

Simulation study is a flexible methodology to illustrate the properties of the Bayes estimates of the 3-component mixture of Exponential distributions using the UP, the JP and the IP under SELF, PLF and DLF in terms of different sample sizes and test termination times. The samples of different sizes $n = 30,\ 100,\ 200$ are generated from the 3-component mixture of Exponential distributions for each choice of the vector of the parameters $(\theta_1,\ \theta_2,\ \theta_3,\ p_1,\ p_2) = \{(4,\ 3,\ 2,\ 0.5,\ 0.3),\ (3,\ 3,\ 3,\ 0.4,\ 0.4),\ (2,\ 3,\ 4,\ 0.3,\ 0.5)\}$.

Table 5. Limiting expressions for the Bayes estimators as $t \to \infty$ using the UP, the JP and the IP under DLF

Parameters	Bayes Estimators		
	UP	JP	IP
θ_1	$\frac{n_1+2}{\sum_{k=1}^{n_1} x_{1k}}$	$\frac{n_1+1}{\sum_{k=1}^{n_1} x_{1k}}$	$\frac{n_1+a_1+1}{\sum_{k=1}^{n_1} x_{1k}+b_1}$
θ_2	$\frac{n_2+2}{\sum_{k=1}^{n_2} x_{2k}}$	$\frac{n_2+1}{\sum_{k=1}^{n_2} x_{2k}}$	$\frac{n_2+a_2+1}{\sum_{k=1}^{n_2} x_{2k}+b_2}$
θ_3	$\frac{n_3+2}{\sum_{k=1}^{n_3} x_{3k}}$	$\frac{n_3+1}{\sum_{k=1}^{n_3} x_{3k}}$	$\frac{n_3+a_3+1}{\sum_{k=1}^{n_3} x_{3k}+b_3}$
p_1	$\frac{n_1+2}{n+4}$	$\frac{n_1+2}{n+4}$	$\frac{n_1+a+1}{n+a+b+c+1}$
p_2	$\frac{n_2+2}{n+4}$	$\frac{n_2+2}{n+4}$	$\frac{n_2+b+1}{n+a+b+c+1}$

Table 6. Limiting expressions for the posterior risks as $t \to \infty$ using the UP, the JP and the IP under DLF

	Posterior Risks		
Parameters	UP	JP	IP
θ_1	$\frac{1}{n_1+2}$	$\frac{1}{n_1+1}$	$\frac{1}{n_1+a_1+1}$
θ_2	$\frac{1}{n_2+2}$	$\frac{1}{n_2+1}$	$\frac{1}{n_2+a_2+1}$
θ_3	$\frac{1}{n_3+2}$	$\frac{1}{n_3+1}$	$\frac{1}{n_3+a_3+1}$
p_1	$\frac{(n_2+n_3+2)}{(n_1+2)(n+3)}$	$\frac{(n_2+n_3+2)}{(n_1+2)(n+3)}$	$\frac{(n_2+n_3+b+c)}{(n_1+a+1)(n+a+b+c)}$
p_2	$\frac{(n_1+n_3+2)}{(n_2+2)(n+3)}$	$\frac{(n_1+n_3+2)}{(n_2+2)(n+3)}$	$\frac{(n_1+n_3+a+c)}{(n_2+b+1)(n+a+b+c)}$

The simulation is repeated 1000 times and the results are then averaged. Sample of sizes p_1n, p_2n and $(1-p_1-p_2)\,n$ are chosen randomly from first component density $f_1\,(x;\,\theta_1)$, second component density $f_2\,(x;\,\theta_2)$ and third component density $f_3\,(x;\,\theta_3)$, respectively. To check the impact of test termination time on Bayes estimates, we estimate the parameters of the 3-component mixture of Exponential distributions based on a sample censored at fixed test termination times $t = 0.5,\ 0.8$. The observations which are greater than test termination time t are taken as censored. Only failures can be considered as members of subpopulation-1, subpopulation-2 or subpopulation-3 of the 3-component mixture of Exponential distributions. For the sake of brevity, simulated results only for $n = 30, 100, 200$ and $(\theta_1, \theta_2, \theta_3, p_1, p_2) = (4,\ 3,\ 2,\ 0.5,\ 0.3)$ are presented in the Tables 8-10 (see appendix). The simulated results for $(\theta_1,\ \theta_2,\ \theta_3,\ p_1,\ p_2) = \{(3,\ 3,\ 3,\ 0.4,\ 0.4)\,,\ (2,\ 3,\ 4,\ 0.3,\ 0.5)\}$ are available with the first author and can be obtained on demand.

From Tables 8-10 (see appendix), it can be seen that differences of Bayes estimates of component and proportion parameters from assumed parameters reduce with an increase in sample size at different test termination times and same is the case with large test termination time as compared to small test termination time for different sample sizes.Also, if $\theta_1 > \theta_2 > \theta_3$ and $p_1 > p_2$, first and second component parameters and second proportion parameter using the IP under SELF, PLF and DLF are under-estimated but third component and first proportion parameters are over-estimated at different sample sizes and test termination times with a few exceptions.By using the IP under SELF, PLF and DLF, three component parameters and second proportion parameter are under-estimated, however,first proportion parameter is over-estimated with a few exceptions in case of $\theta_1 = \theta_2 = \theta_3$ and $p_1 = p_2$. Also, if $\theta_1 < \theta_2 < \theta_3$ and $p_1 < p_2$, third component and second proportion parameters using the IP under SELF, PLF and DLF are under-estimated but there is a mixed pattern (over-estimation or under-estimation) for first and second component and first proportion parameters using the IP. Similarly, the component parameters using the UP and the JP under SELF, PLF and DLF are over-estimated but there is a mix pattern (under-estimation or over-estimation) for proportion parameters using the UP and the JP under SELF, PLF and DLF at different sample sizes and test termination times.

It is, also, clear from the Tables 8-10 that for a fixed test termination time, the posterior risks of the Bayes estimates, using the UP, the JP and the IP under SELF, PLF and DLF, reduce with an increase in sample size. On the other hand, for all priors, loss functions and sample sizes considered in this study, posterior risks decrease with an increase in test termination time.The posterior risks using the IP are smaller than the posterior risks using the UP and the JP for different sample sizes and test termination times.Also, the posterior risks using the JP are smaller than that using the UP for different sample sizes and test termination times. It is also observed that in estimating the component parameters θ_1, θ_2 and θ_3, posterior risks are smaller under DLF than under SELF and PLF at different sample sizes and test termination times considered in this study. However, for estimating the mixing proportions, SELF yields smaller posterior risks than SELF and DLF, at different sample sizes and test termination times. Thus, DLF is more suitable for estimating component parameters and SELF is a preferable choice for estimating proportion parameters p_1 and p_2.

10. Real data example

Davis (1952) reported a mixture data on lifetimes (in thousand hours) of many components used in aircraft sets. To illustrate the proposed methodology, we take the data on three components, namely, Transmitter Tube, Combination of Transformers and Combination of Relays. It is unknown that which component (Tubes, Transformers and Relays) fails until a failure (of a radar set) occurs at or before the test termination time $t = 0.4$. The total number of tests are conducted 702 times.For test termination time $t = 0.4$, the data are summarized as below. $n = 702$, $r_1 = 310$, $r_2 = 148$, $r_3 = 181$, $r = 639$, $n-r = 63$, $\sum_{k=1}^{r_1} x_{1k} = 36.875$, $\sum_{k=1}^{r_2} x_{2k} = 22.90$, $\sum_{k=1}^{r_3} x_{3k} = 19.125$. Since $n - r = 63$, we have almost 9 percent censored sample. Thus, this is a type-I right censored data. Bayes estimates and their posterior risks using the UP, the JP and the IP under SELF, PLF and DLF are showcased in Table 7 given below.

From the Table 10, it is noticed that results obtained through real data are compatible with simulation results, however, there are some exceptions which can be attributed to using large data set. The Table 10also reveals that the performance of the IP is best. In addition, results are relatively more precise under the JP than the UP.It is also observed that DLF (SELF) performance better than PLF and SELF (PLF and DLF) for estimating component (proportion) parameters.

11. Conclusion

The importance and application of the 3-component mixture models in real life problems is undeniable.An extensive simulation study is performed to compare and highlight some important and interesting properties of the Bayes estimates of a 3-component mixture of Exponential distributions using the UP, the JP and the IP under SELF, PLF and DLF. The simulation results revealed that an increase in sample size and/or test termination time produced improved (in terms of closeness)and reliable (in terms of posterior risk) Bayes estimates. It is concluded that with an increase in sample size and/or test termination time, the posterior risks decrease.To estimate component as well as proportion parameters, priors can be ordered with respect to their performance as: IP $<$ JP $<$ UP. The ordering of loss functions depends upon the parameters being estimated.

Table 7. Bayes estimates (BEs) and posterior risks (PRs) using the UP, the JP and the IP under SELF, PLF and DLF with Davis (1952) mixture data

Prior	Loss Function		$\hat{\theta}_1$	$\hat{\theta}_2$	$\hat{\theta}_3$	$\hat{p}_1$	$\hat{p}_2$
UP	SELF	BE	6.916945	4.026699	8.372263	0.470263	0.262025
		PR	**0.288346**	**0.194104**	**0.639500**	**0.000426**	**0.000360**
	PLF	BE	6.937758	4.050730	8.410368	0.470716	0.262711
		PR	**0.041624**	**0.048061**	**0.076210**	**0.000905**	**0.001371**
	DLF	BE	6.958632	4.074904	8.448647	0.471169	0.263398
		PR	**0.005991**	**0.011830**	**0.009041**	**0.001922**	**0.005212**
JP	SELF	BE	6.900167	3.999295	8.313222	0.470132	0.262032
		PR	**0.286064**	**0.191420**	**0.635543**	**0.000425**	**0.000359**
	PLF	BE	6.920864	4.023155	8.351360	0.470584	0.262716
		PR	**0.041396**	**0.047721**	**0.076275**	**0.000903**	**0.001368**
	DLF	BE	6.941624	4.047158	8.389672	0.471036	0.263402
		PR	**0.005972**	**0.011826**	**0.009112**	**0.001919**	**0.005201**
IP	SELF	BE	6.339530	3.948387	7.209497	0.473607	0.253837
		PR	**0.212304**	**0.168152**	**0.469845**	**0.000416**	**0.000341**
	PLF	BE	6.356253	3.969624	7.242009	0.474045	0.254509
		PR	**0.033445**	**0.042473**	**0.065024**	**0.000878**	**0.001343**
	DLF	BE	6.373019	3.990975	7.274667	0.474485	0.255182
		PR	**0.005255**	**0.010671**	**0.008958**	**0.001850**	**0.005271**

Specifically, for estimating component parameters, ordering of loss functions is: DLF < PLF < SELF, while it changes to SELF < PLF < DLF when proportion parameters are being estimated. The results obtained through real data coincide with the simulated results. Finally,it can be concluded that for a Bayesian analysis of mixture data, the IP paired with SELF and the IP paired with DLF are preferable choices for estimating proportion and component parameters, respectively.

References

[1] Abu-Taleb, A. A., Smadi, M. M. and Alawneh, A. J. *Bayes estimation of the lifetime parameters for the Exponential distribution*, Journal of Mathematics and Statistics **3**(3), 106-108, 2007.

[2] Al-Awadhi, S. A. and Gartwaite, P. H. *An elicitation method for multivariate normal distributions*, Communication in Statistics, Part A-Theory and Methods **27**, 1123-1142, 1998.

[3] Ali, M. M., Woo, J. and Nadarajah, S. *Bayes estimators of the Exponential distribution*, Journal of Statistics and Management Systems **8**(1), 53-58, 2005.

[4] Aslam, M. *An Application of Prior Predictive Distribution to Elicit the Prior Density*, Journal of Statistical Theory and Applications **2**, 70-83, 2003.

[5] Bayes, T. *An essay towards solving a problem in the doctrine of chances*, Philosophical Transactions of the Royal Society of London **53**, 370-418, 1763.

[6] Barger, K. J. *Mixtures of Exponential distributions to describe the distribution of Poisson means in estimating the number of unobserved* classes, M. Sc. Thesis, Cornell University, 2006.

[7] Berger, J. O. *Statistical decision theory and Bayesian analysis*(New York: Springer-Verlag, 1985).

[8] Bernardo, J. M. *Reference posterior distributions for Bayesian inference*, Journal of the Royal Statistical Society, Series B (Methodological), **41**(2), 113-147, 1979.

[9] Birch, R. and Bartolucci, A. A. *Determination of the hyperparameters of a prior probability model in survival analysis*, Computer Programs in Biomedicine **17**, 89-94, 1983.

[10] Chaloner, K. M. and Duncan, G. T. *Assessment of beta prior distribution: PM elicitation*, The Statistician **32**, 174-180, 1983.

[11] Davis, D. J. *An analysis of some failure data*, Journal of the American Statistical Association **47**, 113-150, 1952.

[12] DeGroot, M. H. *Optimal statistical decision*(McGraw-Hill, 2005).

[13] de Laplace, P. S. *Theorie analytique des probabilities (*Paris: Gautheir-Villars, 1820).

[14] Everitt, B. and Hand, D. J. *Finite mixture distribution* (New York: Chapman and Hall, 1981).

[15] Gavasakar, U. *A comparison of two elicitation methods for a prior distribution for a binomial parameter*, Management Science **34**, 784-790, 1988.

[16] Geisser, S. *On prior distributions for binary trials*, The American Statistician **38**(4), 244-247, 1984.

[17] Gijbels, I. *Censored data*, Wiley Interdisciplinary Reviews: Computational Statistics **2**(2), 178-188, 2010.

[18] Hahn, E. D. *Re-examining informative prior elicitation through the lens of Markovchain Monte Carlo methods*, Journal of the Royal Statistical Society **169**, Series A, 37-48, 2006.

[19] Hebert, J. L. and Scariano, S. M. *Comparing location estimators for Exponential mixtures under Pitman's measure of closeness*, Communications in Statistics- Theory and Methods **33**(1), 29-46, 2005.

[20] Jeffreys, H. *An invariant form for the prior probability in estimation problems*, Proceeding of the Royal Society of London, Series A, Mathematical and Physical Sciences **186**(1007), 453-461, 1946.

[21] Jeffreys, H. *Theory of Probability*(Oxford, UK: Claredon Press, 1961).

[22] Kadane, J. B., Dickey, J. M., Winkler, R. L., Smith, W. and Peter, S. C. *Interactive elicitation of opinion for a normal linear model*, Journal of the American Statistical Association **75**, 845-854, 1980.

[23] Kalbfleisch, J. D. and Prentice, R. L. *The Statistical analysis of failure time data*(New York: John Wiley & Sons, 2011).

[24] Kazmi, S. M. A., Aslam, M. and Ali, S. *On the Bayesian estimation for two-component mixture of Maxwell distribution assuming type-I censored data*, International Journal of Applied Science and Technology **2**(1), 197-218, 2012.

[25] Legendre, A. M. Nouvelles methodes pour la determination des orbites des cometes: Appendice sur la Methode des Moindres Carres (Paris: Gautheir-Villars, 1806)

[26] Li, L. A. *Decomposition theorems, conditional probability, and finite mixtures distributions*, Thesis, State University, New York, Albany, 1983.

[27] Li, L. A. and Sedransk, N. *Inference about the presence of a mixture*, Technical Report, State University, New York, Albany, 1982.

[28] Li, L. A. and Sedransk, N. *Mixtures of distributions: A topological approach*, The annals of Statistics **16**(4), 1623-1634, 1988.

[29] McCullagh, P. *Exponential mixtures and quadratic Exponential families*, Biometrika **81**(4), 721-729, 1994.

[30] Mendenhall, W. and Hader, R. J. *Estimation of parameters of mixed exponentially distributed failure time distributions from censored life test data*, Biometrika **45**(3-4), 504-520, 1958.

[31] Norstrom, J. G. *The use of precautionary loss function in risk analysis*, Reliability, IEEE Transactions on **45**(3), 400-403, 1996.

[32] Raqab, M. M. and Ahsanullah, M. *Estimation of the location and scale parameters of generalized Exponential distribution based on order statistic*, Journal of Statistical Computation and Simulation **69**(2), 109-123, 2001.

[33] Romeu, L. J. *Censored data*, Strategic Arms Reduction Treaty **11**(3), 1-8, 2004.

[34] Saleem. M. *Bayesian analysis of mixture distributions*. Ph. D. Dissertation, Dept. of Statistics, Quaid-i-Azam University, Islamabad, Pakistan, 2010.

[35] Saleem, M. and Aslam, M. *On prior selection for the mixture of Raleigh distribution using predictive intervals*, Pakistan Journal of Statistics **24**(1), 21-35, 2008.

[36] Saleem, M. and Aslam, M. *Bayesian analysis of the two-component mixture of the Rayleigh distribution assuming the uniform and the Jeffreys' priors*, Journal of Applied Statistical Science **16**(4), 493-502, 2009.

[37] Saleem, M., Aslam, M. and Economus, P. *On the Bayesian analysis of the mixture of Power distribution using the complete and censored sample*, Journal of Applied Statistics **37**(1), 25-40, 2010.

[38] Sinha, S. K. *Bayesian estimation* (New Delhi: New Age International (p) limited, Publisher, 1998).

[39] Strelec, L. and Stehlk, M. *On simulation of exact tests in Rayleigh and Normal families*, AIP (American Institute of Physics) Conference Proceedings **1479**, 2012.

Appendix

Table 8. Bayes Estimates (BEs) and Posterior Risks (PRs) of 3-Component Mixture of an Exponential Distribution using the UP under SELF, PLF and DLF with $\theta_1 = 4,\ \theta_2 = 3,\ \theta_3 = 2,\ p_1 = 0.5,\ p_2 = 0.3$ and $t = 0.5,\ 0.8$

t	n	Loss Functions		$\hat{\theta}_1$	$\hat{\theta}_2$	$\hat{\theta}_3$	$\hat{p}_1$	$\hat{p}_2$
0.5	30	SELF	BE	5.03127	4.81717	14.4058	0.493360	0.305990
			PR	**3.78733**	**7.04383**	**393.721**	**0.010409**	**0.009351**
		PLF	BE	5.18096	5.34651	7.10057	0.506304	0.321826
			PR	**0.65794**	**1.14487**	**2.26392**	**0.021169**	**0.030100**
		DLF	BE	5.69559	6.08516	32.9960	0.517202	0.336103
			PR	**0.12627**	**0.21448**	**0.31318**	**0.042015**	**0.094862**
	100	SELF	BE	4.32408	3.48119	3.18176	0.501051	0.306898
			PR	**1.06177**	**1.47960**	**2.49490**	**0.004021**	**0.003927**
		PLF	BE	4.43269	3.75565	3.47246	0.505030	0.311397
			PR	**0.23514**	**0.39934**	**0.66400**	**0.008038**	**0.012548**
		DLF	BE	4.51580	3.99458	3.89480	0.510777	0.317893
			PR	**0.05348**	**0.10395**	**0.18747**	**0.015990**	**0.039946**
	200	SELF	BE	4.15969	3.30336	2.65355	0.500567	0.304289
			PR	**0.54772**	**0.77122**	**1.06800**	**0.002233**	**0.002223**
		PLF	BE	4.18300	3.38534	2.92080	0.504513	0.307870
			PR	**0.12781**	**0.22162**	**0.37151**	**0.004457**	**0.007263**
		DLF	BE	4.29189	3.53659	3.03098	0.506274	0.309894
			PR	**0.03027**	**0.06450**	**0.12266**	**0.008705**	**0.023440**
0.8	30	SELF	BE	4.60708	4.06730	3.95910	0.491784	0.305857
			PR	**2.06164**	**3.05704**	**6.71197**	**0.008108**	**0.007047**
		PLF	BE	4.78976	4.41100	4.44132	0.499841	0.317388
			PR	**0.41090**	**0.64347**	**1.02895**	**0.016344**	**0.022641**
		DLF	BE	5.04824	4.71186	4.92032	0.50771	0.328947
			PR	**0.08470**	**0.14425**	**0.21952**	**0.032500**	**0.070789**
	100	SELF	BE	4.19187	3.34385	2.65896	0.498415	0.303129
			PR	**0.57004**	**0.73496**	**0.89049**	**0.002707**	**0.002426**
		PLF	BE	4.27433	3.41190	2.72112	0.499986	0.307765
			PR	**0.13347**	**0.20684**	**0.29326**	**0.005432**	**0.007970**
		DLF	BE	4.34097	3.53208	2.93405	0.503415	0.311406
			PR	**0.03143**	**0.06083**	**0.10652**	**0.010834**	**0.025890**
	200	SELF	BE	4.08408	3.13722	2.34034	0.499107	0.302543
			PR	**0.28609**	**0.35794**	**0.38474**	**0.001398**	**0.001282**
		PLF	BE	4.14270	3.17768	2.43561	0.500479	0.305603
			PR	**0.06892**	**0.10915**	**0.15321**	**0.002780**	**0.004201**
		DLF	BE	4.13182	3.26269	2.52608	0.502583	0.306298
			PR	**0.01687**	**0.03470**	**0.06245**	**0.005568**	**0.013850**

Table 9. Bayes Estimates (BEs) and Posterior Risks (PRs) of 3-Component Mixture of an Exponential Distribution using the JP under SELF, PLF and DLF with $\theta_1 = 4,\ \theta_2 = 3,\ \theta_3 = 2,\ p_1 = 0.5,\ p_2 = 0.3$ and $t = 0.5,\ 0.8$

t	n	Loss Functions		$\hat{\theta}_1$	$\hat{\theta}_2$	$\hat{\theta}_3$	$\hat{p}_1$	$\hat{p}_2$
0.5	30	SELF	BE	4.68438	4.14788	4.14810	0.486255	0.307003
			PR	**3.28607**	**5.84447**	**112.213**	**0.010047**	**0.009113**
		PLF	BE	5.06787	4.50278	4.97240	0.492724	0.320652
			PR	**0.63799**	**1.05847**	**2.13408**	**0.020695**	**0.029650**
		DLF	BE	5.43172	5.13024	11.0016	0.505493	0.336744
			PR	**0.12292**	**0.22330**	**0.37228**	**0.041414**	**0.090771**
	100	SELF	BE	4.28903	3.44976	2.65577	0.496809	0.303804
			PR	**1.00298**	**1.45326**	**1.90329**	**0.003884**	**0.003721**
		PLF	BE	4.38009	3.58409	3.01029	0.503270	0.309635
			PR	**0.22362**	**0.37814**	**0.60703**	**0.007759**	**0.012199**
		DLF	BE	4.48458	3.75851	3.37003	0.504503	0.316679
			PR	**0.05133**	**0.10363**	**0.19409**	**0.015604**	**0.039164**
	200	SELF	BE	4.17641	3.21003	2.41831	0.497047	0.303295
			PR	**0.53203**	**0.72122**	**0.90182**	**0.002145**	**0.002173**
		PLF	BE	4.19861	3.37741	2.64199	0.502472	0.305365
			PR	**0.12219**	**0.21241**	**0.33145**	**0.004263**	**0.006915**
		DLF	BE	4.25922	3.47091	2.83717	0.503951	0.310409
			PR	**0.02966**	**0.06356**	**0.12632**	**0.008578**	**0.022924**
0.8	30	SELF	BE	4.39979	3.52249	3.04594	0.486904	0.306358
			PR	**1.94160**	**2.57549**	**4.29295**	**0.008017**	**0.007075**
		PLF	BE	4.60389	3.91436	3.79902	0.495912	0.316711
			PR	**0.40382**	**0.62240**	**0.98201**	**0.016328**	**0.022631**
		DLF	BE	4.82216	4.34275	3.93999	0.504094	0.327098
			PR	**0.08604**	**0.15228**	**0.25101**	**0.032569**	**0.070579**
	100	SELF	BE	4.13305	3.21866	2.38451	0.498135	0.302131
			PR	**0.55222**	**0.69769**	**0.76144**	**0.002695**	**0.002421**
		PLF	BE	4.20130	3.26838	2.54904	0.499784	0.307532
			PR	**0.13079**	**0.20230**	**0.28690**	**0.005401**	**0.007958**
		DLF	BE	4.22792	3.43384	2.74990	0.502684	0.310692
			PR	**0.03143**	**0.06125**	**0.10953**	**0.010812**	**0.025775**
	200	SELF	BE	4.06307	3.10621	2.24022	0.498726	0.302046
			PR	**0.28071**	**0.35039**	**0.35378**	**0.001388**	**0.001271**
		PLF	BE	4.09626	3.13286	2.34996	0.499902	0.305688
			PR	**0.06796**	**0.10693**	**0.14942**	**0.002775**	**0.004169**
		DLF	BE	4.14565	3.22203	2.39826	0.501271	0.306324
			PR	**0.01668**	**0.03474**	**0.06291**	**0.005545**	**0.013783**

Table 10. Bayes Estimates (BEs) and Posterior Risks (PRs) of 3-Component Mixture of an Exponential Distribution using the IP under SELF, PLF and DLF with $\theta_1 = 4,\ \theta_2 = 3,\ \theta_3 = 2,\ p_1 = 0.5,\ p_2 = 0.3$ and $t = 0.5,\ 0.8$

t	n	Loss Functions		$\hat{\theta}_1$	$\hat{\theta}_2$	$\hat{\theta}_3$	$\hat{p}_1$	$\hat{p}_2$
0.5	30	SELF	BE	2.33666	1.99205	1.56456	0.516529	0.253366
			PR	**0.38446**	**0.44742**	**0.42467**	**0.008301**	**0.006275**
		PLF	BE	2.40593	2.12002	1.67158	0.52332	0.269693
			PR	**0.16074**	**0.21795**	**0.26020**	**0.016032**	**0.024229**
		DLF	BE	2.50499	2.22293	1.82387	0.530591	0.282341
			PR	**0.06619**	**0.10097**	**0.15199**	**0.030634**	**0.089499**
	100	SELF	BE	3.12388	2.56172	1.92370	0.513419	0.284786
			PR	**0.32993**	**0.38752**	**0.37916**	**0.003298**	**0.002739**
		PLF	BE	3.17347	2.64326	2.03329	0.518166	0.288056
			PR	**0.10266**	**0.14824**	**0.18994**	**0.006375**	**0.009540**
		DLF	BE	3.23605	2.73451	2.14476	0.520769	0.295013
			PR	**0.03215**	**0.05510**	**0.09357**	**0.012292**	**0.032628**
	200	SELF	BE	3.47498	2.77357	2.06363	0.511406	0.292558
			PR	**0.25846**	**0.30721**	**0.31254**	**0.001865**	**0.001619**
		PLF	BE	3.49013	2.80368	2.16324	0.512843	0.295125
			PR	**0.07334**	**0.10910**	**0.14811**	**0.003661**	**0.005535**
		DLF	BE	3.54183	2.88947	2.22229	0.514936	0.297619
			PR	**0.02086**	**0.03829**	**0.06791**	**0.007115**	**0.018739**
0.8	30	SELF	BE	2.41149	2.03909	1.63540	0.505104	0.263843
			PR	**0.37034**	**0.41121**	**0.40135**	**0.007235**	**0.005650**
		PLF	BE	2.50768	2.14343	1.76339	0.51284	0.27436
			PR	**0.15051**	**0.19540**	**0.23654**	**0.014208**	**0.021037**
		DLF	BE	2.59032	2.26105	1.88480	0.519925	0.284365
			PR	**0.05938**	**0.08976**	**0.13082**	**0.027575**	**0.070114**
	100	SELF	BE	3.21605	2.57804	1.93565	0.504645	0.288391
			PR	**0.27556**	**0.30674**	**0.29392**	**0.002599**	**0.002186**
		PLF	BE	3.26907	2.61318	2.02791	0.508064	0.293030
			PR	**0.08505**	**0.11580**	**0.14777**	**0.005135**	**0.007558**
		DLF	BE	3.33053	2.77935	2.14013	0.510514	0.296837
			PR	**0.02567**	**0.04355**	**0.07129**	**0.010059**	**0.025568**
	200	SELF	BE	3.53859	2.78925	2.06133	0.504253	0.295431
			PR	**0.18994**	**0.21303**	**0.20664**	**0.001364**	**0.001188**
		PLF	BE	3.57238	2.82378	2.10400	0.505779	0.296186
			PR	**0.05289**	**0.07625**	**0.09668**	**0.002695**	**0.004006**
		DLF	BE	3.58655	2.89477	2.13775	0.507708	0.298415
			PR	**0.01471**	**0.02687**	**0.04567**	**0.005307**	**0.013426**