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Exact travelling wave solutions of nonlinear pseudoparabolic equations by using the G'/G Expansion Method

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Abstract: In this paper, the $\left(\frac{G'}{G}\right)$ expansion method with the aid of computer algebraic system Maple, is proposed for seeking the travelling wave solutions for the a class of nonlinear pseudoparabolic equations. The method is straightforward and concise, and it be also applied to other nonlinear pseudoparabolic equations. We studied mostly important four nonlinear pseudoparabolic physical models : the Benjamin-Bona-Mahony-Peregrine-Burger(BBMPB) equation, the Oskolkov-Benjamin-Bona-Mahony-Burgers(OBBMB) equation, the one-dimensional Oskolkov equation and the generalized hyperbolic-elastic-rod wave equation.

Keywords: The $\left(\frac{G'}{G}\right)$ expansion method, Travelling wave solution; Nonlinear pseudoparabolic equation Benjamin-Bona-Mahony-Peregrine-Burger(BBMPB) equation, Oskolkov-Benjamin-Bona-Mahony-Burgers(OBBMB) equation, one-dimensional Oskolkov equation, Generalized hyperbolic-elastic-rod wave equation.

1 Introduction

Nonlinear partial differential equations arise in a large number of physics, mathematics and engineering problems. In the soliton theory, the study of exact solutions to these nonlinear equations plays a very germane role, as they provide much information about the physical models they describe. Various powerful methods have been employed to construct exact travelling wave solutions to nonlinear partial differential equations. These methods include the inverse scattering transform [1], the Backlund transform [2,3], the Darboux transform [4], the Hirota bilinear method [5], the tanh-function method [6,7], the sine-cosine method [8], the exp-function method [9], the generalized Riccati equation [10], the homogeneous balance method [11], the first integral method [12,13], the $\left(\frac{G'}{G}\right)$ expansion method[14,15], and the modified simple equation method [16,18].

The objective of this paper is to use a powerful method called the $\left(\frac{G'}{G}\right)$ expansion method to obtain travelling wave solution for the a class of nonlinear pseudoparabolic equations. The method , first introduced by Wang and Zhang [19], has been widely used to obtain exact solutions of nonlinear equations [20 – 29].

Equations with one-time derivative appearing in the highest order term are called pseudoparabolic and arise in many areas of mathematics and physics. They have been used, for instance, for fluid flow in fissured rock, consolidation of clay, shear in second-order fluids, thermodynamics and propagation of long waves of small amplitude. For more details,

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we refer reader to $[30-34]$ and references therein.

An important special case of pseudoparabolic-type equations is the generalized Benjamin-Bona-Mahony-Burgers (BBMB) equation,

$$u_t - u_{xxt} - \alpha u_{xx} + \gamma u_x + f(u)_x = 0, \tag{1}$$

where $u(x,t)$ represents the fluid velocity in the horizantal direction x, α is a positive constant, γ is any given real constant and $f(u)$ is a C^2-smooth nonlinear function. If we take $f(u)_x = \theta u u_x + \beta u_{xxx}$ in Eq.(1.1), the we obtain a general form of the Benjamin-Bona-Mahony-Peregrine-Burgers (BBMPB) equation

$$u_t - u_{xxt} - \alpha u_{xx} + \gamma u_x + \theta u u_x + \beta u_{xxx} = 0. \tag{2}$$

For $\beta = 0$ in Eq.(1.2), we obtain a general form of the Oskolkov-Benjamin-Bona-Mahony-Burgers (OBBMB) equation

$$u_t - u_{xxt} - \alpha u_{xx} + \gamma u_x + \theta u u_x = 0. \tag{3}$$

This nonlinear, one-dimensional and pseudoparabolic equation describes nonlinear surface waves that spread along the axis Ox and αu_{xx} is the viscosity term.

The one-dimensional Oskolkov equation,

$$u_t - \lambda u_{xxt} - \alpha u_{xx} + u u_x = 0 \tag{4}$$

describes the dynamics of an incompressible viscoelastic Kelvin-Voigt fluid.

The generalized hyperelastic-rod wave equation,

$$u_t - u_{xxt} + \alpha u_x + 2\beta u u_x + 3\theta u^2 u_x - \gamma u_x u_{xx} - u u_{xxx} = 0 \tag{5}$$

where α, β, γ and θ are constant parameters. This eqaution includes many important physical models in mathematical physics. For $\beta = \frac{3}{2}, \theta = 0$ and $\gamma = 2$, we obtain the Camassa-Holm(CH) equation

$$u_t - u_{xxt} + \alpha u_x + 3 u u_x - 2 u_x u_{xx} - u u_{xxx} = 0, \tag{6}$$

where u is the fluid velocity in the direction x, α is a constant related to the critical shallow water wave speed. Taking $\alpha = 1, \beta = 2, \theta = 0$ and $\gamma = 3$, the Eq.(1.5) leads to the Fornberg-Whitham(FW) equation used to study the qualitative behaviour of wave-breaking

$$u_t - u_{xxt} + u_x + u u_x - 3 u_x u_{xx} - u u_{xxx} = 0. \tag{7}$$

As stated before, pseudoparabolic-type equation arise in many areas of mathematics and physics to describe many physical phenomena. In recent years considerable attention has been paid to the study of pseudoparabolic -type equations. In this paper, $\left(\frac{G'}{G}\right)$ expansion method is used to find the solutions for the pseudoparabolic-type equations stated above.

The main ideas are that the travelling wave solutions of nonlinear equation can be expressed by a polynomial in $\left(\frac{G'}{G}\right)$, where $G = G(\xi)$ satisfies the second order linear ordinary differential equation: $G''(\xi) + \lambda G'(\xi) + \mu G(\xi) = 0$, where $\xi = x - ct$ and λ, μ, c are constants. The degree of this polynomial can be determined by considering the homogenous balance between the highest order derivative and nonlinear terms appearing in the given nonlinear equations. The coefficients of the polynomial λ, μ and c can be obtained by solving a set of algebraic equations resulting from the process of using the proposed method. Moreover, the travelling wave solutions obtained via this method are expressed by the hyperbolic functions, the trigonometric functions and the rational functions.

2 Description of the $\left(\frac{G'}{G}\right)$ expansion method

In this section, we describe the $\left(\frac{G'}{G}\right)$ expansion method for finding travelling wave solutions of nonlinear partial differential equations. Suppose that a nonlinear partial differential equation(PDE), say in two independent variables x and t, is given by

$$P(u, u_t, u_x, u_{xt}, u_{tt}, u_{xx}, \ldots) = 0 \tag{8}$$

where $u(x,t)$ is an unknown function, P is a polynomial in $u = u(x,t)$ and its various partial derivatives, in which highest order derivatives and nonlinear terms are involved.

The summary of the $\left(\frac{G'}{G}\right)$ expansion method, can be presented in the following six steps:

Step 1: To find the travelling wave solutions of Eq.(8) we introduce the wave variable

$$u(x,t) = u(\xi), \xi = x - ct, \tag{9}$$

where the constant c is generally termed the wave velocity. Substituting Eq.(9) into Eq.(8), we obtain the following ordinary differential equations(ODE) in ξ (which illustrates a principal advantage of a travelling wave solution, i.e., a PDE is reduced to an ODE).

$$P(U, cU', U', cU'', c^2U'', U'', \ldots) = 0 \tag{10}$$

Step 2: If necessary we integrate Eq.(10) as many times as possible and set the constants of integration to be zero for simplicity.

Step 3: We suppose the solution of nonlinear partial differential equation can be expressed by a polynomial in $\left(\frac{G'}{G}\right)$ as

$$u(\xi) = \sum_{i=0}^{m} a_i \left(\frac{G'}{G}\right)^i, \tag{11}$$

where $G = G(\xi)$ satisfies the second-order linear ordinary differential equation

$$G''(\xi) + \lambda G'(\xi) + \mu G(\xi) = 0, \tag{12}$$

where $G' = \frac{dG}{d\xi}, G'' = \frac{d^2G}{d\xi^2}$, and a_i, λ and μ are real constants with $a_m \neq 0$. Here the prime denotes the derivative with respect to ξ. Using the general solutions of Eq.(12), we have

$$\left(\frac{G'}{G}\right) = \begin{cases} -\frac{\lambda}{2} + \frac{\sqrt{\lambda^2-4\mu}}{2}\left(\frac{c_1 \sinh\left(\frac{\sqrt{\lambda^2-4\mu}}{2}\xi\right) + c_2 \cosh\left(\frac{\sqrt{\lambda^2-4\mu}}{2}\xi\right)}{c_1 \cosh\left(\frac{\sqrt{\lambda^2-4\mu}}{2}\xi\right) + c_2 \sinh\left(\frac{\sqrt{\lambda^2-4\mu}}{2}\xi\right)}\right), \lambda^2 - 4\mu > 0 \\ -\frac{\lambda}{2} + \frac{\sqrt{4\mu-\lambda^2}}{2}\left(\frac{-c_1 \sin\left(\frac{\sqrt{4\mu-\lambda^2}}{2}\xi\right) + c_2 \cos\left(\frac{\sqrt{4\mu-\lambda^2}}{2}\xi\right)}{c_1 \cos\left(\frac{\sqrt{4\mu-\lambda^2}}{2}\xi\right) + c_2 \sin\left(\frac{\sqrt{4\mu-\lambda^2}}{2}\xi\right)}\right), \lambda^2 - 4\mu < 0 \\ \left(\frac{c_2}{c_1 + c_2\xi}\right) - \frac{\lambda}{2}, \lambda^2 - 4\mu = 0 \end{cases} \tag{13}$$

The above results can be written in simplified forms as

$$\left(\frac{G'}{G}\right)=\begin{cases}-\frac{\lambda}{2}+\frac{\sqrt{\lambda^2-4\mu}}{2}\tanh\left(\frac{\sqrt{\lambda^2-4\mu}}{2}\xi+\xi_0\right),\lambda^2-4\mu>0,\tanh\xi_0=\frac{c_1}{c_2},\left|\frac{c_1}{c_2}\right|>1,\\ -\frac{\lambda}{2}+\frac{\sqrt{4\mu-\lambda^2}}{2}\coth\left(\frac{\sqrt{4\mu-\lambda^2}}{2}\xi+\xi_0\right),\lambda^2-4\mu<0,\coth\xi_0=\frac{c_1}{c_2},\left|\frac{c_1}{c_2}\right|<1,\\ -\frac{\lambda}{2}+\frac{\sqrt{4\mu-\lambda^2}}{2}\cot\left(\frac{\sqrt{4\mu-\lambda^2}}{2}\xi+\xi_0\right),\lambda^2-4\mu<0,\cot\xi_0=\frac{c_1}{c_2},\\ \left(\frac{c_2}{c_1+c_2\xi}\right)-\frac{\lambda}{2},\lambda^2-4\mu=0\end{cases} \tag{14}$$

Moreover, it follows from Eq.(11) and (12) that

$$\begin{cases}U'=-\sum_{i=1}^{m} ia_i\left(\left(\frac{G'}{G}\right)^{i+1}+\lambda\left(\frac{G'}{G}\right)^{i}+\mu\left(\frac{G'}{G}\right)^{i-1}\right),\\ U''=\sum_{i=1}^{m} ia_i((i+1)\left(\frac{G'}{G}\right)^{i+2}+(2i+1)\lambda\left(\frac{G'}{G}\right)^{i+1}+i\left(\lambda^2+2\mu\right)\left(\frac{G'}{G}\right)^{i}\\ \qquad +(2i-1)\lambda\mu\left(\frac{G'}{G}\right)^{i-1}+(i-1)\mu^2\left(\frac{G'}{G}\right)^{i-2}),\end{cases} \tag{15}$$

and so on, here the prime denotes the derivative with respect to ξ.

Step 4: The positive integer m can be accomplished by considering the homogeneous balance between the highest order derivatives and nonlinear terms appearing in Eq.(10) as follows: if we define the degree of $u(\xi)$ as $D[u(\xi)]=m$, then the degree of other expressions is defined by

$$D\left[\frac{d^q u}{d\xi^q}\right]=m+q,\ \ D\left[u^r\left(\frac{d^q u}{d\xi^q}\right)^s\right]=mr+s(q+m).$$

Therefore, we can get the value of m in Eq.(2.4).

Step 5: Substituting Eq.(11) into Eq.(10) using general solutions of Eq.(12) and collecting all terms with the same order of $\left(\frac{G'}{G}\right)$ together, then setting each coefficient of this polynomial to zero yield a set of algebraic equations for a_i, c, λ and μ.

Step 6: Substitute a_i, c, λ and μ obtained in step 5 and the general solutions of Eq.(12) into Eq.(11). Next, depending on the sign of discriminant $(\lambda^2-4\mu)$, we can obtain the explicit solution of Eq.(8) immediately.

3 Benjamin-Bona-Mahony-Peregrine-Burgers(BBMPB) equation

The Benjamin-Bona-Mahony-Peregrine-Burgers (BBMPB) equation is given by

$$u_t-u_{xxt}-\alpha u_{xx}+\gamma u_x+\theta uu_x+\beta u_{xxx}=0 \tag{16}$$

where α is a positive constant, θ and β are nonzero real numbers. Using the wave variable $\xi=x-ct$ in Eq.(16), then integrating this equation and considering the integration constant to be zero, we obtain

$$(\gamma-c)U-\alpha U'+\frac{\theta}{2}U^2+(\beta+c)U''=0 \tag{17}$$

According to step 4, balancing U^2 and U'' gives N=2. Therefore, the solutions of Eq.(17) can be written in the form

$$U(\xi)=a_0+a_1\left(\frac{G'}{G}\right)+a_2\left(\frac{G'}{G}\right)^2, \tag{18}$$

where a_0, a_1 and a_2 are constants which are unknowns to be determined later. By Eq.(12) or Eq.(**??**) we derive

$$\begin{cases} U' = -2a_2\left(\frac{G'}{G}\right)^3 - (2a_2\lambda + a_1)\left(\frac{G'}{G}\right)^2 - (2a_2\mu + a_1\lambda)\left(\frac{G'}{G}\right) - a_1\mu. \\ U'' = 6a_2\left(\frac{G'}{G}\right)^4 + (10a_2\lambda + 2a_1)\left(\frac{G'}{G}\right)^3 + \left(8a_2\mu + 4a_2\lambda^2 + 3a_1\lambda\right)\left(\frac{G'}{G}\right)^2 \\ \quad + \left(6a_2\lambda\mu + 2a_1\mu + a_1\lambda^2\right)\left(\frac{G'}{G}\right) + 2a_2\mu^2 + a_1\lambda\mu, \end{cases} \tag{19}$$

Substituting Eq.(18) and its derivatives Eq.s (18) into Eq.s (17) and equating each coefficient of $\left(\frac{G'}{G}\right)$ to zero, we obtain a set of nonlinear algebraic equations for a_0, a_1, a_2, λ and c.Solving this system using Maple, we obtain,

Set 1. $c = -\beta, \lambda = \sqrt{4\mu + \frac{(\gamma+\beta)^2}{\alpha^2}}, a_2 = 0, a_1 = -\frac{2\alpha}{\theta}, a_0 = -\frac{\gamma+\alpha\lambda+\beta}{\theta};$

Set 2. $c = -\frac{\beta}{2} + \frac{\gamma}{2} + \frac{\sqrt{25(\beta+\gamma)^2 - 24\alpha^2}}{10}, \lambda = \sqrt{4\mu + \frac{\gamma-c}{6(\beta+c)}}, a_2 = -\frac{12(\beta+c)}{\theta}, a_1 = -\frac{12\alpha}{5\theta} - \frac{12\lambda(\beta+c)}{\theta},$

$a_0 = \frac{-15\lambda^2(\beta+\gamma)-6\lambda\alpha}{5\theta} + \frac{3\alpha^2(6\lambda^2-1)}{25\theta(\beta+c)};$

Set 3. $c = -\frac{\beta}{2} + \frac{\gamma}{2} - \frac{\sqrt{25(\beta+\gamma)^2 - 24\alpha^2}}{10}, \lambda = \sqrt{4\mu + \frac{\gamma-c}{6(\beta+c)}}, a_2 = -\frac{12(\beta+c)}{\theta}, a_1 = -\frac{12\alpha}{5\theta} - \frac{12\lambda(\beta+c)}{\theta},$

$a_0 = \frac{-15\lambda^2(\beta+\gamma)-6\lambda\alpha}{5\theta} + \frac{3\alpha^2(6\lambda^2-1)}{25\theta(\beta+c)};$

Set 4. $c = -\frac{\beta}{2} + \frac{\gamma}{2} + \frac{\sqrt{25(\beta+\gamma)^2 + 24\alpha^2}}{10}, \lambda = \sqrt{4\mu + \frac{c-\gamma}{6(\beta+c)}}, a_2 = -\frac{12(\beta+c)}{\theta}, a_1 = -\frac{12\alpha}{5\theta} - \frac{12\lambda(\beta+c)}{\theta},$

$a_0 = \frac{-15\lambda^2(\beta+\gamma)-6\lambda\alpha}{5\theta} - \frac{9\alpha^2(2\lambda^2-1)}{25\theta(\beta+c)};$

Set 5. $c = -\frac{\beta}{2} + \frac{\gamma}{2} - \frac{\sqrt{25(\beta+\gamma)^2 + 24\alpha^2}}{10}, \lambda = \sqrt{4\mu + \frac{c-\gamma}{6(\beta+c)}}, a_2 = -\frac{12(\beta+c)}{\theta}, a_1 = -\frac{12\alpha}{5\theta} - \frac{12\lambda(\beta+c)}{\theta},$

$a_0 = \frac{-15\lambda^2(\beta+\gamma)-6\lambda\alpha}{5\theta} - \frac{9\alpha^2(2\lambda^2-1)}{25\theta(\beta+c)}.$

Using these values in Eq.(18) when $\lambda^2 - 4\mu > 0$, we obtain the hyperbolic solutions respectively:

$$u_1(x,t) = -\frac{\gamma+\beta}{\theta} - \frac{\gamma+\beta}{\theta}\left(\frac{c_1\sinh\left(\frac{\gamma+\beta}{2\alpha}\xi\right) + c_2\cosh\left(\frac{\gamma+\beta}{2\alpha}\xi\right)}{c_1\cosh\left(\frac{\gamma+\beta}{2\alpha}\xi\right) + c_2\sinh\left(\frac{\gamma+\beta}{2\alpha}\xi\right)}\right) \tag{20}$$

where $\xi = x + \beta t$,

$$\begin{aligned} u_2(x,t) = {} & \frac{3\lambda^2(c-\gamma)}{\theta} + \frac{3\alpha^2\left(6\lambda^2-1\right)}{25\theta(\beta+c)} - \frac{6\alpha}{5\theta}\sqrt{\frac{\gamma-c}{6(\beta+c)}}\left(\frac{c_1\sinh(w\xi) + c_2\cosh(w\xi)}{c_1\cosh(w\xi) + c_2\sinh(w\xi)}\right) \\ & + \frac{c-\gamma}{2\theta}\left(\frac{c_1\sinh(w\xi) + c_2\cosh(w\xi)}{c_1\cosh(w\xi) + c_2\sinh(w\xi)}\right)^2, \end{aligned} \tag{21}$$

where $w = \frac{1}{2}\sqrt{\frac{\gamma-c}{6(\beta+c)}}$ and $\xi = x + \left(\frac{\beta}{2} - \frac{\gamma}{2} - \frac{\sqrt{25(\beta+\gamma)^2 - 24\alpha^2}}{10}\right)t,$

$$\begin{aligned} u_3(x,t) = {} & \frac{3\lambda^2(c-\gamma)}{\theta} + \frac{3\alpha^2\left(6\lambda^2-1\right)}{25\theta(\beta+c)} - \frac{6\alpha}{5\theta}\sqrt{\frac{\gamma-c}{6(\beta+c)}}\left(\frac{c_1\sinh(w\xi) + c_2\cosh(w\xi)}{c_1\cosh(w\xi) + c_2\sinh(w\xi)}\right) \\ & + \frac{c-\gamma}{2\theta}\left(\frac{c_1\sinh(w\xi) + c_2\cosh(w\xi)}{c_1\cosh(w\xi) + c_2\sinh(w\xi)}\right)^2, \end{aligned} \tag{22}$$

where $w=\frac{1}{2}\sqrt{\frac{\gamma-c}{6(\beta+c)}}$ and $\xi=x+\left(\frac{\beta}{2}-\frac{\gamma}{2}+\frac{\sqrt{25(\beta+\gamma)^2-24\alpha^2}}{10}\right)t$,

$$u_4(x,t)=\frac{3\lambda^2(c-\gamma)}{\theta}-\frac{9\alpha^2(2\lambda^2-1)}{25\theta(\beta+c)}-\frac{6\alpha}{5\theta}\sqrt{\frac{c-\gamma}{6(\beta+c)}}\left(\frac{c_1\sinh(w\xi)+c_2\cosh(w\xi)}{c_1\cosh(w\xi)+c_2\sinh(w\xi)}\right)+\frac{\gamma-c}{2\theta}\left(\frac{c_1\sinh(w\xi)+c_2\cosh(w\xi)}{c_1\cosh(w\xi)+c_2\sinh(w\xi)}\right)^2, \tag{23}$$

where $w=\frac{1}{2}\sqrt{\frac{c-\gamma}{6(\beta+c)}}$ and $\xi=x+\left(\frac{\beta}{2}-\frac{\gamma}{2}-\frac{\sqrt{25(\beta+\gamma)^2+24\alpha^2}}{10}\right)t$,

$$u_5(x,t)=\frac{3\lambda^2(c-\gamma)}{\theta}-\frac{9\alpha^2(2\lambda^2-1)}{25\theta(\beta+c)}-\frac{6\alpha}{5\theta}\sqrt{\frac{c-\gamma}{6(\beta+c)}}\left(\frac{c_1\sinh(w\xi)+c_2\cosh(w\xi)}{c_1\cosh(w\xi)+c_2\sinh(w\xi)}\right)+\frac{\gamma-c}{2\theta}\left(\frac{c_1\sinh(w\xi)+c_2\cosh(w\xi)}{c_1\cosh(w\xi)+c_2\sinh(w\xi)}\right)^2, \tag{24}$$

where $w=\frac{1}{2}\sqrt{\frac{c-\gamma}{6(\beta+c)}}$ and $\xi=x+\left(\frac{\beta}{2}-\frac{\gamma}{2}+\frac{\sqrt{25(\beta+\gamma)^2+24\alpha^2}}{10}\right)t$.

When $\lambda^2-4\mu<0$, we obtain the trigonometric solutions respectively:

$$u_6(x,t)=\frac{3\lambda^2(c-\gamma)}{\theta}+\frac{3\alpha^2(6\lambda^2-1)}{25\theta(\beta+c)}-\frac{6\alpha}{5\theta}\sqrt{\frac{c-\gamma}{6(\beta+c)}}\left(\frac{-c_1\sin(w\xi)+c_2\cos(w\xi)}{c_1\cos(w\xi)+c_2\sin(w\xi)}\right)+\frac{\gamma-c}{2\theta}\left(\frac{-c_1\sin(w\xi)+c_2\cos(w\xi)}{c_1\cos(w\xi)+c_2\sin(w\xi)}\right)^2, \tag{25}$$

where $w=\frac{1}{2}\sqrt{\frac{c-\gamma}{6(\beta+c)}}$ and $\xi=x+\left(\frac{\beta}{2}-\frac{\gamma}{2}-\frac{\sqrt{25(\beta+\gamma)^2-24\alpha^2}}{10}\right)t$,

$$u_7(x,t)=\frac{3\lambda^2(c-\gamma)}{\theta}+\frac{3\alpha^2(6\lambda^2-1)}{25\theta(\beta+c)}-\frac{6\alpha}{5\theta}\sqrt{\frac{c-\gamma}{6(\beta+c)}}\left(\frac{-c_1\sin(w\xi)+c_2\cos(w\xi)}{c_1\cos(w\xi)+c_2\sin(w\xi)}\right)+\frac{\gamma-c}{2\theta}\left(\frac{-c_1\sin(w\xi)+c_2\cos(w\xi)}{c_1\cos(w\xi)+c_2\sin(w\xi)}\right)^2, \tag{26}$$

where $w=\frac{1}{2}\sqrt{\frac{c-\gamma}{6(\beta+c)}}$ and $\xi=x+\left(\frac{\beta}{2}-\frac{\gamma}{2}+\frac{\sqrt{25(\beta+\gamma)^2-24\alpha^2}}{10}\right)t$,

$$u_8(x,t)=\frac{3\lambda^2(c-\gamma)}{\theta}-\frac{9\alpha^2(2\lambda^2-1)}{25\theta(\beta+c)}-\frac{6\alpha}{5\theta}\sqrt{\frac{\gamma-c}{6(\beta+c)}}\left(\frac{-c_1\sin(w\xi)+c_2\cos(w\xi)}{c_1\cos(w\xi)+c_2\sin(w\xi)}\right)+\frac{c-\gamma}{2\theta}\left(\frac{-c_1\sin(w\xi)+c_2\cos(w\xi)}{c_1\cos(w\xi)+c_2\sin(w\xi)}\right)^2, \tag{27}$$

where $w=\frac{1}{2}\sqrt{\frac{\gamma-c}{6(\beta+c)}}$ and $\xi=x+\left(\frac{\beta}{2}-\frac{\gamma}{2}-\frac{\sqrt{25(\beta+\gamma)^2+24\alpha^2}}{10}\right)t$,

$$u_9(x,t)=\frac{3\lambda^2(c-\gamma)}{\theta}-\frac{9\alpha^2(2\lambda^2-1)}{25\theta(\beta+c)}-\frac{6\alpha}{5\theta}\sqrt{\frac{\gamma-c}{6(\beta+c)}}\left(\frac{-c_1\sin(w\xi)+c_2\cos(w\xi)}{c_1\cos(w\xi)+c_2\sin(w\xi)}\right)+\frac{c-\gamma}{2\theta}\left(\frac{-c_1\sin(w\xi)+c_2\cos(w\xi)}{c_1\cos(w\xi)+c_2\sin(w\xi)}\right)^2, \tag{28}$$

where $w = \frac{1}{2}\sqrt{\frac{\gamma-c}{6(\beta+c)}}$ and $\xi = x + \left(\frac{\beta}{2} - \frac{\gamma}{2} + \frac{\sqrt{25(\beta+\gamma)^2+24\alpha^2}}{10}\right)t$.

When $\lambda^2 - 4\mu = 0$, we obtain the rational solutions respectively:

$$u_{10}(x,t) = -\frac{2\alpha}{\theta}\left(\frac{c_2}{c_1+c_2\xi}\right), \tag{29}$$

where $\xi = x + \beta t$,

$$u_{11}(x,t) = \frac{3\alpha^2\left(6\lambda^2-1\right)}{25\theta\left(\beta+c\right)} - \frac{12\alpha}{5\theta}\left(\frac{c_2}{c_1+c_2\xi}\right) - \frac{12\left(\beta+c\right)}{\theta}\left(\frac{c_2}{c_1+c_2\xi}\right)^2,$$

where $\xi = x + \left(\frac{\beta}{2} - \frac{\gamma}{2} - \frac{\sqrt{25(\beta+\gamma)^2-24\alpha^2}}{10}\right)t$,

$$u_{12}(x,t) = \frac{3\alpha^2\left(6\lambda^2-1\right)}{25\theta\left(\beta+c\right)} - \frac{12\alpha}{5\theta}\left(\frac{c_2}{c_1+c_2\xi}\right) - \frac{12\left(\beta+c\right)}{\theta}\left(\frac{c_2}{c_1+c_2\xi}\right)^2,$$

where $\xi = x + \left(\frac{\beta}{2} - \frac{\gamma}{2} + \frac{\sqrt{25(\beta+\gamma)^2-24\alpha^2}}{10}\right)t$,

$$u_{13}(x,t) = -\frac{9\alpha^2\left(2\lambda^2-1\right)}{25\theta\left(\beta+c\right)} - \frac{12\alpha}{5\theta}\left(\frac{c_2}{c_1+c_2\xi}\right) - \frac{12\left(\beta+c\right)}{\theta}\left(\frac{c_2}{c_1+c_2\xi}\right)^2,$$

where $\xi = x + \left(\frac{\beta}{2} - \frac{\gamma}{2} - \frac{\sqrt{25(\beta+\gamma)^2+24\alpha^2}}{10}\right)t$,

$$u_{14}(x,t) = -\frac{9\alpha^2\left(2\lambda^2-1\right)}{25\theta\left(\beta+c\right)} - \frac{12\alpha}{5\theta}\left(\frac{c_2}{c_1+c_2\xi}\right) - \frac{12\left(\beta+c\right)}{\theta}\left(\frac{c_2}{c_1+c_2\xi}\right)^2,$$

where $\xi = x + \left(\frac{\beta}{2} - \frac{\gamma}{2} + \frac{\sqrt{25(\beta+\gamma)^2+24\alpha^2}}{10}\right)t$.

4 The Oskolkov-Benjamin-Bona-Mahony-Burgers(OBBMB) equation

Consider the Oskolkov-Benjamin-Bona-Mahony-Burgers(OBBMB) equation

$$u_t - u_{xxt} - \alpha u_{xx} + \gamma u_x + \theta u u_x = 0 \tag{30}$$

where α is positive and θ is a nonzero constant. Using the wave variable $\xi = x - ct$ in Eq.(16), then integrating this equation and considering the integration constant to be zero, we obtain

$$(\gamma - c)U - \alpha U' + \frac{\theta}{2}U^2 + cU'' = 0 \tag{31}$$

According to step 4, balancing U^2 and U'' gives N=2. Therefore, the solutions of Eq.(31) can be written in the form

$$U(\xi) = a_0 + a_1\left(\frac{G'}{G}\right) + a_2\left(\frac{G'}{G}\right)^2, \tag{32}$$

where a_0, a_1 and a_2 are constants which are unknowns to be determined later. Substituting Eq.(32) and its derivatives Eq.s (19) into Eq.s (31) and equating each coefficient of $\left(\frac{G'}{G}\right)$ to zero, we obtain a set of nonlinear algebraic equations for a_0, a_1, a_2, λ and c. Solving this system using Maple, we obtain,

Set 1. $c = 0, \lambda = \sqrt{4\mu + \frac{\gamma^2}{\alpha^2}}, a_2 = 0, a_1 = -\frac{2\alpha}{\theta}, a_0 = -\frac{\gamma+\alpha\lambda}{\theta}$;

Set 2. $c = \frac{\gamma}{2} + \frac{\sqrt{25\gamma^2+24\alpha^2}}{10}, \lambda = \sqrt{4\mu + \frac{c-\gamma}{6c}}, a_2 = -\frac{12c}{\theta}, a_1 = -\frac{12\alpha}{5\theta} - \frac{12\lambda c}{\theta}$,

$a_0 = \frac{-12\mu\left(18\alpha^3+30\gamma\lambda\alpha^2+125\lambda c\gamma^2+75c\gamma\alpha+30c\lambda\alpha^2\right)}{5\theta\left(6\lambda\alpha^2+25c\lambda\gamma+5\alpha c\right)}$;

Set 3. $c = \frac{\gamma}{2} - \frac{\sqrt{25\gamma^2+24\alpha^2}}{10}, \lambda = \sqrt{4\mu + \frac{c-\gamma}{6c}}, a_2 = -\frac{12c}{\theta}, a_1 = -\frac{12\alpha}{5\theta} - \frac{12\lambda c}{\theta}$,

$a_0 = \frac{-12\mu\left(18\alpha^3+30\gamma\lambda\alpha^2+125\lambda c\gamma^2+75c\gamma\alpha+30c\lambda\alpha^2\right)}{5\theta\left(6\lambda\alpha^2+25c\lambda\gamma+5\alpha c\right)}$;

Set 4. $c = \frac{\gamma}{2} - \frac{\sqrt{25\gamma^2-24\alpha^2}}{10}, \lambda = \sqrt{4\mu + \frac{\gamma-c}{6c}}, a_2 = -\frac{12c}{\theta}, a_1 = -\frac{12\alpha}{5\theta} - \frac{12\lambda c}{\theta}$,

$a_0 = \frac{\frac{12}{5}\alpha^3 - \frac{216}{5}\mu\alpha^3 - 72\lambda\mu\gamma\alpha^2 + 300\lambda\mu c\gamma^2 + 180\mu\alpha c\gamma + 12c\lambda\alpha^2 - 72c\lambda\mu\alpha^2}{\theta\left(6\lambda\alpha^2-25c\lambda\gamma-5\alpha c\right)}$;

Set 5. $c = \frac{\gamma}{2} + \frac{\sqrt{25\gamma^2-24\alpha^2}}{10}, \lambda = \sqrt{4\mu + \frac{\gamma-c}{6c}}, a_2 = -\frac{12c}{\theta}, a_1 = -\frac{12\alpha}{5\theta} - \frac{12\lambda c}{\theta}$,

$a_0 = \frac{\frac{12}{5}\alpha^3 - \frac{216}{5}\mu\alpha^3 - 72\lambda\mu\gamma\alpha^2 + 300\lambda\mu c\gamma^2 + 180\mu\alpha c\gamma + 12c\lambda\alpha^2 - 72c\lambda\mu\alpha^2}{\theta\left(6\lambda\alpha^2-25c\lambda\gamma-5\alpha c\right)}$.

Using these values in Eq.(32) when $\lambda^2 - 4\mu > 0$, we obtain the hyperbolic solutions respectively:

$$u_1(x,t) = -\frac{\gamma}{\theta} - \frac{\gamma}{\theta}\left(\frac{c_1 \sinh\left(\frac{\gamma}{2\alpha}x\right) + c_2 \cosh\left(\frac{\gamma}{2\alpha}x\right)}{c_1 \cosh\left(\frac{\gamma}{2\alpha}x\right) + c_2 \sinh\left(\frac{\gamma}{2\alpha}x\right)}\right), \tag{33}$$

$$\begin{aligned} u_2(x,t) = a_0 + \frac{6\alpha\lambda}{5\theta} + \frac{3\lambda^2 c}{\theta} - \frac{6\alpha}{5\theta}\sqrt{\frac{c-\gamma}{6c}}\left(\frac{c_1 \sinh(w\xi) + c_2 \cosh(w\xi)}{c_1 \cosh(w\xi) + c_2 \sinh(w\xi)}\right) \\ + \frac{\gamma - c}{2\theta}\left(\frac{c_1 \sinh(w\xi) + c_2 \cosh(w\xi)}{c_1 \cosh(w\xi) + c_2 \sinh(w\xi)}\right)^2, \end{aligned} \tag{34}$$

where $a_0 = \frac{-12\mu\left(18\alpha^3+30\gamma\lambda\alpha^2+125\lambda c\gamma^2+75c\gamma\alpha+30c\lambda\alpha^2\right)}{5\theta\left(6\lambda\alpha^2+25c\lambda\gamma+5\alpha c\right)}, w = \frac{1}{2}\sqrt{\frac{c-\gamma}{6c}}$ and $\xi = x - \left(\frac{\gamma}{2} + \frac{\sqrt{25\gamma^2+24\alpha^2}}{10}\right)t$,

$$\begin{aligned} u_3(x,t) = a_0 + \frac{6\alpha\lambda}{5\theta} + \frac{3\lambda^2 c}{\theta} - \frac{6\alpha}{5\theta}\sqrt{\frac{c-\gamma}{6c}}\left(\frac{c_1 \sinh(w\xi) + c_2 \cosh(w\xi)}{c_1 \cosh(w\xi) + c_2 \sinh(w\xi)}\right) \\ + \frac{\gamma - c}{2\theta}\left(\frac{c_1 \sinh(w\xi) + c_2 \cosh(w\xi)}{c_1 \cosh(w\xi) + c_2 \sinh(w\xi)}\right)^2, \end{aligned} \tag{35}$$

where $a_0 = \frac{-12\mu\left(18\alpha^3+30\gamma\lambda\alpha^2+125\lambda c\gamma^2+75c\gamma\alpha+30c\lambda\alpha^2\right)}{5\theta\left(6\lambda\alpha^2+25c\lambda\gamma+5\alpha c\right)}, w = \frac{1}{2}\sqrt{\frac{c-\gamma}{6c}}$ and $\xi = x - \left(\frac{\gamma}{2} - \frac{\sqrt{25\gamma^2+24\alpha^2}}{10}\right)t$,

$$\begin{aligned} u_4(x,t) = a_0 + \frac{6\alpha\lambda}{5\theta} + \frac{3\lambda^2 c}{\theta} - \frac{6\alpha}{5\theta}\sqrt{\frac{\gamma-c}{6c}}\left(\frac{c_1 \sinh(w\xi) + c_2 \cosh(w\xi)}{c_1 \cosh(w\xi) + c_2 \sinh(w\xi)}\right) \\ + \frac{c - \gamma}{2\theta}\left(\frac{c_1 \sinh(w\xi) + c_2 \cosh(w\xi)}{c_1 \cosh(w\xi) + c_2 \sinh(w\xi)}\right)^2, \end{aligned} \tag{36}$$

where $a_0 = \frac{\frac{12}{5}\alpha^3 - \frac{216}{5}\mu\alpha^3 - 72\lambda\mu\gamma\alpha^2 + 300\lambda\mu c\gamma^2 + 180\mu\alpha c\gamma + 12c\lambda\alpha^2 - 72c\lambda\mu\alpha^2}{\theta\left(6\lambda\alpha^2 - 25c\lambda\gamma - 5\alpha c\right)}$, $w = \frac{1}{2}\sqrt{\frac{\gamma - c}{6c}}$ and $\xi = x - \left(\frac{\gamma}{2} - \frac{\sqrt{25\gamma^2 - 24\alpha^2}}{10}\right)t$,

$$u_5(x,t) = a_0 + \frac{6\alpha\lambda}{5\theta} + \frac{3\lambda^2 c}{\theta} - \frac{6\alpha}{5\theta}\sqrt{\frac{\gamma - c}{6c}}\left(\frac{c_1\sinh(w\xi) + c_2\cosh(w\xi)}{c_1\cosh(w\xi) + c_2\sinh(w\xi)}\right) + \frac{c - \gamma}{2\theta}\left(\frac{c_1\sinh(w\xi) + c_2\cosh(w\xi)}{c_1\cosh(w\xi) + c_2\sinh(w\xi)}\right)^2, \tag{37}$$

where $a_0 = \frac{\frac{12}{5}\alpha^3 - \frac{216}{5}\mu\alpha^3 - 72\lambda\mu\gamma\alpha^2 + 300\lambda\mu c\gamma^2 + 180\mu\alpha c\gamma + 12c\lambda\alpha^2 - 72c\lambda\mu\alpha^2}{\theta\left(6\lambda\alpha^2 - 25c\lambda\gamma - 5\alpha c\right)}$, $w = \frac{1}{2}\sqrt{\frac{\gamma - c}{6c}}$ and $\xi = x - \left(\frac{\gamma}{2} + \frac{\sqrt{25\gamma^2 - 24\alpha^2}}{10}\right)t$.

When $\lambda^2 - 4\mu < 0$, we obtain the trigonometric solutions respectively:

$$u_6(x,t) = a_0 + \frac{6\alpha\lambda}{5\theta} + \frac{3\lambda^2 c}{\theta} - \frac{6\alpha}{5\theta}\sqrt{\frac{\gamma - c}{6c}}\left(\frac{-c_1\sin(w\xi) + c_2\cos(w\xi)}{c_1\cos(w\xi) + c_2\sin(w\xi)}\right) + \frac{c - \gamma}{2\theta}\left(\frac{-c_1\sin(w\xi) + c_2\cos(w\xi)}{c_1\cos(w\xi) + c_2\sin(w\xi)}\right)^2, \tag{38}$$

where $a_0 = \frac{-12\mu\left(18\alpha^3 + 30\gamma\lambda\alpha^2 + 125\lambda c\gamma^2 + 75c\gamma\alpha + 30c\lambda\alpha^2\right)}{5\theta\left(6\lambda\alpha^2 + 25c\lambda\gamma + 5\alpha c\right)}$, $w = \frac{1}{2}\sqrt{\frac{\gamma - c}{6c}}$ and $\xi = x - \left(\frac{\gamma}{2} + \frac{\sqrt{25\gamma^2 + 24\alpha^2}}{10}\right)t$,

$$u_7(x,t) = a_0 + \frac{6\alpha\lambda}{5\theta} + \frac{3\lambda^2 c}{\theta} - \frac{6\alpha}{5\theta}\sqrt{\frac{\gamma - c}{6c}}\left(\frac{-c_1\sin(w\xi) + c_2\cos(w\xi)}{c_1\cos(w\xi) + c_2\sin(w\xi)}\right) + \frac{c - \gamma}{2\theta}\left(\frac{-c_1\sin(w\xi) + c_2\cos(w\xi)}{c_1\cos(w\xi) + c_2\sin(w\xi)}\right)^2, \tag{39}$$

where $a_0 = \frac{-12\mu\left(18\alpha^3 + 30\gamma\lambda\alpha^2 + 125\lambda c\gamma^2 + 75c\gamma\alpha + 30c\lambda\alpha^2\right)}{5\theta\left(6\lambda\alpha^2 + 25c\lambda\gamma + 5\alpha c\right)}$, $w = \frac{1}{2}\sqrt{\frac{\gamma - c}{6c}}$ and $\xi = x - \left(\frac{\gamma}{2} - \frac{\sqrt{25\gamma^2 + 24\alpha^2}}{10}\right)t$,

$$u_8(x,t) = a_0 + \frac{6\alpha\lambda}{5\theta} + \frac{3\lambda^2 c}{\theta} - \frac{6\alpha}{5\theta}\sqrt{\frac{c - \gamma}{6c}}\left(\frac{c_1\sinh(w\xi) + c_2\cosh(w\xi)}{c_1\cosh(w\xi) + c_2\sinh(w\xi)}\right) + \frac{\gamma - c}{2\theta}\left(\frac{c_1\sinh(w\xi) + c_2\cosh(w\xi)}{c_1\cosh(w\xi) + c_2\sinh(w\xi)}\right)^2, \tag{40}$$

where $a_0 = \frac{\frac{12}{5}\alpha^3 - \frac{216}{5}\mu\alpha^3 - 72\lambda\mu\gamma\alpha^2 + 300\lambda\mu c\gamma^2 + 180\mu\alpha c\gamma + 12c\lambda\alpha^2 - 72c\lambda\mu\alpha^2}{\theta\left(6\lambda\alpha^2 - 25c\lambda\gamma - 5\alpha c\right)}$, $w = \frac{1}{2}\sqrt{\frac{c - \gamma}{6c}}$ and $\xi = x - \left(\frac{\gamma}{2} - \frac{\sqrt{25\gamma^2 - 24\alpha^2}}{10}\right)t$,

$$u_9(x,t) = a_0 + \frac{6\alpha\lambda}{5\theta} + \frac{3\lambda^2 c}{\theta} - \frac{6\alpha}{5\theta}\sqrt{\frac{c - \gamma}{6c}}\left(\frac{c_1\sinh(w\xi) + c_2\cosh(w\xi)}{c_1\cosh(w\xi) + c_2\sinh(w\xi)}\right) + \frac{\gamma - c}{2\theta}\left(\frac{c_1\sinh(w\xi) + c_2\cosh(w\xi)}{c_1\cosh(w\xi) + c_2\sinh(w\xi)}\right)^2, \tag{41}$$

where $a_0 = \frac{\frac{12}{5}\alpha^3 - \frac{216}{5}\mu\alpha^3 - 72\lambda\mu\gamma\alpha^2 + 300\lambda\mu c\gamma^2 + 180\mu\alpha c\gamma + 12c\lambda\alpha^2 - 72c\lambda\mu\alpha^2}{\theta\left(6\lambda\alpha^2 - 25c\lambda\gamma - 5\alpha c\right)}$, $w = \frac{1}{2}\sqrt{\frac{c - \gamma}{6c}}$ and $\xi = x - \left(\frac{\gamma}{2} + \frac{\sqrt{25\gamma^2 - 24\alpha^2}}{10}\right)t$.

When $\lambda^2 - 4\mu = 0$, we obtain the rational solutions respectively:

$$u_9(x,t) = -\frac{2\alpha}{\theta}\left(\frac{c_2}{c_1 + c_2 x}\right) \tag{42}$$

$$u_{10}(x,t) = a_0 + \frac{6\lambda\alpha}{5\theta} + \frac{3\lambda^2 c}{\theta} - \frac{12\alpha}{5\theta}\left(\frac{c_2}{c_1+c_2\xi}\right) - \frac{12c}{\theta}\left(\frac{c_2}{c_1+c_2\xi}\right)^2 \tag{43}$$

where $a_0 = \frac{-12\mu\left(18\alpha^3+30\gamma\lambda\alpha^2+125\lambda c\gamma^2+75c\gamma\alpha+30c\lambda\alpha^2\right)}{5\theta\left(6\lambda\alpha^2+25c\lambda\gamma+5\alpha c\right)}$ and $\xi = x - \left(\frac{\gamma}{2} + \frac{\sqrt{25\gamma^2+24\alpha^2}}{10}\right)t,$

$$u_{11}(x,t) = a_0 + \frac{6\lambda\alpha}{5\theta} + \frac{3\lambda^2 c}{\theta} - \frac{12\alpha}{5\theta}\left(\frac{c_2}{c_1+c_2\xi}\right) - \frac{12c}{\theta}\left(\frac{c_2}{c_1+c_2\xi}\right)^2 \tag{44}$$

where $a_0 = \frac{-12\mu\left(18\alpha^3+30\gamma\lambda\alpha^2+125\lambda c\gamma^2+75c\gamma\alpha+30c\lambda\alpha^2\right)}{5\theta\left(6\lambda\alpha^2+25c\lambda\gamma+5\alpha c\right)}$ and $\xi = x - \left(\frac{\gamma}{2} - \frac{\sqrt{25\gamma^2+24\alpha^2}}{10}\right)t,$

$$u_{12}(x,t) = a_0 + \frac{6\lambda\alpha}{5\theta} + \frac{3\lambda^2 c}{\theta} - \frac{12\alpha}{5\theta}\left(\frac{c_2}{c_1+c_2\xi}\right) - \frac{12c}{\theta}\left(\frac{c_2}{c_1+c_2\xi}\right)^2 \tag{45}$$

where $a_0 = \frac{\frac{12}{5}\alpha^3 - \frac{216}{5}\mu\alpha^3 - 72\lambda\mu\gamma\alpha^2+300\lambda\mu c\gamma^2+180\mu\alpha c\gamma+12c\lambda\alpha^2-72c\lambda\mu\alpha^2}{\theta\left(6\lambda\alpha^2-25c\lambda\gamma-5\alpha c\right)}$ and $\xi = x - \left(\frac{\gamma}{2} - \frac{\sqrt{25\gamma^2-24\alpha^2}}{10}\right)t,$

$$u_{13}(x,t) = a_0 + \frac{6\lambda\alpha}{5\theta} + \frac{3\lambda^2 c}{\theta} - \frac{12\alpha}{5\theta}\left(\frac{c_2}{c_1+c_2\xi}\right) - \frac{12c}{\theta}\left(\frac{c_2}{c_1+c_2\xi}\right)^2 \tag{46}$$

where $a_0 = \frac{\frac{12}{5}\alpha^3 - \frac{216}{5}\mu\alpha^3 - 72\lambda\mu\gamma\alpha^2+300\lambda\mu c\gamma^2+180\mu\alpha c\gamma+12c\lambda\alpha^2-72c\lambda\mu\alpha^2}{\theta\left(6\lambda\alpha^2-25c\lambda\gamma-5\alpha c\right)}$ and $\xi = x - \left(\frac{\gamma}{2} + \frac{\sqrt{25\gamma^2-24\alpha^2}}{10}\right)t.$

5 The one-dimensional Oskolkov equation

The one-dimensional Oskolkov equation is given by

$$u_t - \beta u_{xxt} - \alpha u_{xx} + u u_x = 0. \tag{47}$$

We will investigate the equation for $\beta \neq 0$ and $\alpha \in R$.Using the wave variable $\xi = x - ct$ in Eq.(47), then integrating this equation and considering the integration constant to be zero, we obtain

$$-cU - \alpha U' + \frac{1}{2}U^2 + \beta U'' = 0 \tag{48}$$

According to step 4, balancing U^2 and U'' gives N=2. Therefore, the solutions of Eq.(48) can be written in the form

$$U(\xi) = a_0 + a_1\left(\frac{G'}{G}\right) + a_2\left(\frac{G'}{G}\right)^2, \tag{49}$$

where a_0, a_1 and a_2 are constants which are unknowns to be determined later. Substituting Eq.(49) and its derivatives Eq.s (19) into Eq.s (48) and equating each coefficient of $\left(\frac{G'}{G}\right)$ to zero, we obtain a set of nonlinear algebraic equations for a_0, a_1, a_2, λ and c. Solving this system using Maple, we obtain,

Set 1. $c = \frac{6\alpha^2}{25\beta}, \lambda = \sqrt{4\mu + \frac{\alpha^2}{25\beta^2}}, a_2 = -12\beta, a_1 = -12\beta\lambda - \frac{12\alpha}{5}, a_0 = -\frac{12\beta\mu(5\beta\lambda+3\alpha)}{5\beta\lambda+\alpha};$

Set 2. $c = \frac{6\alpha^2}{25\beta}, \lambda = -\sqrt{4\mu + \frac{\alpha^2}{25\beta^2}}, a_2 = -12\beta, a_1 = -12\beta\lambda - \frac{12\alpha}{5}, a_0 = -\frac{12\beta\mu(5\beta\lambda+3\alpha)}{5\beta\lambda+\alpha};$

Set 3. $c=-\frac{6\alpha^2}{25\beta}, \lambda=\sqrt{4\mu+\frac{\alpha^2}{25\beta^2}}, a_2=-12\beta, a_1=-12\beta\lambda-\frac{12\alpha}{5}$,

$a_0=\frac{-12\alpha^2}{25}-\frac{12\mu\beta(3\alpha+5\beta\lambda)}{5\beta\lambda+\alpha}$;

Set 4. $c=-\frac{6\alpha^2}{25\beta}, \lambda=-\sqrt{4\mu+\frac{\alpha^2}{25\beta^2}}, a_2=-12\beta, a_1=-12\beta\lambda-\frac{12\alpha}{5}$,

$a_0=\frac{-12\alpha^2}{25}-\frac{12\mu\beta(3\alpha+5\beta\lambda)}{5\beta\lambda+\alpha}$.

Using these values in Eq.(49) when $\lambda^2-4\mu>0$, we obtain the hyperbolic solutions respectively:

$$u_1(x,t)=-\frac{12\beta\mu(5\beta\lambda+3\alpha)}{5\beta\lambda+\alpha}+3\beta\lambda^2+\frac{6\alpha\lambda}{5}-\frac{6\alpha^2}{25\beta}\left(\frac{c_1\sinh\left(\frac{\alpha}{10\beta}\xi\right)+c_2\cosh\left(\frac{\alpha}{10\beta}\xi\right)}{c_1\cosh\left(\frac{\alpha}{10\beta}\xi\right)+c_2\sinh\left(\frac{\alpha}{10\beta}\xi\right)}\right)$$
$$-\frac{3\alpha^2}{25\beta}\left(\frac{c_1\sinh\left(\frac{\alpha}{10\beta}\xi\right)+c_2\cosh\left(\frac{\alpha}{10\beta}\xi\right)}{c_1\cosh\left(\frac{\alpha}{10\beta}\xi\right)+c_2\sinh\left(\frac{\alpha}{10\beta}\xi\right)}\right)^2 \tag{50}$$

where $\xi=x-\frac{6\alpha^2}{25\beta}t$ and $\lambda=\sqrt{4\mu+\frac{\alpha^2}{25\beta^2}}$,

$$u_2(x,t)=-\frac{12\beta\mu(5\beta\lambda+3\alpha)}{5\beta\lambda+\alpha}+3\beta\lambda^2+\frac{6\alpha\lambda}{5}-\frac{6\alpha^2}{25\beta}\left(\frac{c_1\sinh\left(\frac{\alpha}{10\beta}\xi\right)+c_2\cosh\left(\frac{\alpha}{10\beta}\xi\right)}{c_1\cosh\left(\frac{\alpha}{10\beta}\xi\right)+c_2\sinh\left(\frac{\alpha}{10\beta}\xi\right)}\right)$$
$$-\frac{3\alpha^2}{25\beta}\left(\frac{c_1\sinh\left(\frac{\alpha}{10\beta}\xi\right)+c_2\cosh\left(\frac{\alpha}{10\beta}\xi\right)}{c_1\cosh\left(\frac{\alpha}{10\beta}\xi\right)+c_2\sinh\left(\frac{\alpha}{10\beta}\xi\right)}\right)^2 \tag{51}$$

where $\xi=x-\frac{6\alpha^2}{25\beta}t$ and $\lambda=-\sqrt{4\mu+\frac{\alpha^2}{25\beta^2}}$,

$$u_3(x,t)=-\frac{12\beta\mu(5\beta\lambda+3\alpha)}{5\beta\lambda+\alpha}+3\beta\lambda^2+\frac{30\alpha\lambda-12\alpha^2}{25}-\frac{6\alpha^2}{25\beta}\left(\frac{c_1\sinh\left(\frac{\alpha}{10\beta}\xi\right)+c_2\cosh\left(\frac{\alpha}{10\beta}\xi\right)}{c_1\cosh\left(\frac{\alpha}{10\beta}\xi\right)+c_2\sinh\left(\frac{\alpha}{10\beta}\xi\right)}\right)$$
$$-\frac{3\alpha^2}{25\beta}\left(\frac{c_1\sinh\left(\frac{\alpha}{10\beta}\xi\right)+c_2\cosh\left(\frac{\alpha}{10\beta}\xi\right)}{c_1\cosh\left(\frac{\alpha}{10\beta}\xi\right)+c_2\sinh\left(\frac{\alpha}{10\beta}\xi\right)}\right)^2 \tag{52}$$

where $\xi=x+\frac{6\alpha^2}{25\beta}t$ and $\lambda=\sqrt{4\mu+\frac{\alpha^2}{25\beta^2}}$,

$$u_4(x,t)=-\frac{12\beta\mu(5\beta\lambda+3\alpha)}{5\beta\lambda+\alpha}+3\beta\lambda^2+\frac{30\alpha\lambda-12\alpha^2}{25}-\frac{6\alpha^2}{25\beta}\left(\frac{c_1\sinh\left(\frac{\alpha}{10\beta}\xi\right)+c_2\cosh\left(\frac{\alpha}{10\beta}\xi\right)}{c_1\cosh\left(\frac{\alpha}{10\beta}\xi\right)+c_2\sinh\left(\frac{\alpha}{10\beta}\xi\right)}\right)$$
$$-\frac{3\alpha^2}{25\beta}\left(\frac{c_1\sinh\left(\frac{\alpha}{10\beta}\xi\right)+c_2\cosh\left(\frac{\alpha}{10\beta}\xi\right)}{c_1\cosh\left(\frac{\alpha}{10\beta}\xi\right)+c_2\sinh\left(\frac{\alpha}{10\beta}\xi\right)}\right)^2 \tag{53}$$

where $\xi = x + \frac{6\alpha^2}{25\beta}t$ and $\lambda = -\sqrt{4\mu + \frac{\alpha^2}{25\beta^2}}$.

When $\lambda^2 - 4\mu = 0$, we obtain the rational solution:

$$u_5(x,t) = -12\beta\mu + 3\beta\lambda^2 - 12\beta\left(\frac{c_2}{c_1 + c_2 x}\right)^2. \tag{54}$$

6 The generalized Hyperelastic-rod wave equation

The generalized hyperelastic-rod wave equation reads as follows:

$$u_t - u_{xxt} + \alpha u_x + 2\beta u u_x + 3\theta u^2 u_x - \gamma u_x u_{xx} - u u_{xxx} = 0 \tag{55}$$

where α, β, θ and γ are constants parameters, and we assume that θ is nonzero. Using the wave variable $\xi = x - ct$ in Eq.(55), then integrating this equation and considering the integration constant to not be zero, we obtain

$$(\alpha - c)U + cU'' + \beta U^2 + \theta U^3 - \left(\frac{\gamma - 1}{2}\right)U'^2 - UU'' = 0 \tag{56}$$

According to step 4, balancing U^3 and UU'' gives N=2. Therefore, the solutions of Eq.(56) can be written in the form

$$U(\xi) = a_0 + a_1\left(\frac{G'}{G}\right) + a_2\left(\frac{G'}{G}\right)^2, \tag{57}$$

where a_0, a_1 and a_2 are constants which are unknowns to be determined later. Substituting Eq.(57) and its derivatives Eq.s (19) into Eq.s (56) and equating each coefficient of $\left(\frac{G'}{G}\right)$ to zero, we obtain a set of nonlinear algebraic equations for a_0, a_1, a_2, λ and c.Solving this system using Maple, we obtain,

Set 1. $c = \dfrac{\gamma^2+2\gamma+1-2\beta-\sqrt{\begin{array}{c}6\gamma^2+4\gamma+4\gamma^3-8\beta\gamma+1-4\beta+\gamma^4-4\gamma^2\beta+4\beta^2-24\alpha\gamma\theta+8\gamma\beta^2\\-16\gamma^4\mu^2-32\gamma\mu^2-12\alpha\theta\gamma^2-64\gamma^3\mu^2-80\gamma^2\mu^2-12\alpha\theta+4\beta^2\gamma^2\end{array}}}{6\theta}$,

$\lambda = 0,\ a_2 = \frac{2(\gamma+2)}{\theta},\ a_1 = 0, a_0 = \frac{4\gamma^2\mu+8\mu+12\gamma\mu-2\beta-\beta\gamma-3\theta c}{3\theta(\gamma+1)}$;

Set 2. $c = \dfrac{\gamma^2+2\gamma+1-2\beta+\sqrt{\begin{array}{c}6\gamma^2+4\gamma+4\gamma^3-8\beta\gamma+1-4\beta+\gamma^4-4\gamma^2\beta+4\beta^2-24\alpha\gamma\theta+8\gamma\beta^2\\-16\gamma^4\mu^2-32\gamma\mu^2-12\alpha\theta\gamma^2-64\gamma^3\mu^2-80\gamma^2\mu^2-12\alpha\theta+4\beta^2\gamma^2\end{array}}}{6\theta}$,
$\lambda = 0,\ a_2 = \frac{2(\gamma+2)}{\theta},\ a_1 = 0, a_0 = \frac{4\gamma^2\mu+8\mu+12\gamma\mu-2\beta-\beta\gamma-3\theta c}{3\theta(\gamma+1)}$;

Set 3. $c = \dfrac{\gamma^2+2\gamma+1-2\beta-\sqrt{\begin{array}{c}1+4\gamma-4\beta-4\gamma^2\beta+8\mu\lambda^2\gamma^4+16\gamma\mu\lambda^2+40\mu\gamma^2\lambda^2-24\alpha\theta\gamma+32\mu\lambda^2\gamma^3\\-12\alpha\theta\gamma^2-8\beta\gamma-12\alpha\theta-80\gamma^2\mu^2-64\gamma^3\mu^2+8\gamma\beta^2-32\gamma\mu^2\\+4\beta^2\gamma^2-16\gamma^4\mu^2-\lambda^4\gamma^4-5\lambda^4\gamma^2-4\lambda^4\gamma^3-2\lambda^4\gamma+\gamma^4+4\gamma^3+6\gamma^2+4\beta^2\end{array}}}{6\theta}$,
$a_2 = \frac{2(\gamma+2)}{\theta},\ a_1 = \frac{2\lambda(\gamma+2)}{\theta},\ a_0 = \frac{\lambda^2\gamma^2+8\gamma^2\mu-2\beta\gamma+3\lambda^2\gamma+24\gamma\mu-4\beta+16\mu+2\lambda^2-6\theta c}{6\theta(\gamma+1)}$;

Set 4. $c = \dfrac{\gamma^2+2\gamma+1-2\beta+\sqrt{\begin{array}{c}1+4\gamma-4\beta-4\gamma^2\beta+8\mu\lambda^2\gamma^4+16\gamma\mu\lambda^2+40\mu\gamma^2\lambda^2-24\alpha\theta\gamma+32\mu\lambda^2\gamma^3\\-12\alpha\theta\gamma^2-8\beta\gamma-12\alpha\theta-80\gamma^2\mu^2-64\gamma^3\mu^2+8\gamma\beta^2-32\gamma\mu^2\\+4\beta^2\gamma^2-16\gamma^4\mu^2-\lambda^4\gamma^4-5\lambda^4\gamma^2-4\lambda^4\gamma^3-2\lambda^4\gamma+\gamma^4+4\gamma^3+6\gamma^2+4\beta^2\end{array}}}{6\theta}$,
$a_2 = \frac{2(\gamma+2)}{\theta}, a_1 = \frac{2\lambda(\gamma+2)}{\theta},\ a_0 = \frac{\lambda^2\gamma^2+8\gamma^2\mu-2\beta\gamma+3\lambda^2\gamma+24\gamma\mu-4\beta+16\mu+2\lambda^2-6\theta c}{6\theta(\gamma+1)}$.

Using these values in Eq.(6.3) when $\lambda^2 - 4\mu > 0$, we obtain the hyperbolic solutions respectively:

$$u_1(x,t) = \frac{4\gamma^2\mu + 8\mu + 12\gamma\mu - 2\beta - \beta\gamma - 3\theta c}{3\theta(\gamma+1)} + \frac{2(\gamma+2)\sqrt{-\mu}}{\theta}\left(\frac{c_1\sinh(\sqrt{-\mu}\xi) + c_2\cosh(\sqrt{-\mu}\xi)}{c_1\cosh(\sqrt{-\mu}\xi) + c_2\sinh(\sqrt{-\mu}\xi)}\right)^2, \tag{58}$$

where $\xi = x - \frac{\gamma^2+2\gamma+1-2\beta-\sqrt{\begin{array}{c}6\gamma^2+4\gamma+4\gamma^3-8\beta\gamma+1-4\beta+\gamma^4-4\gamma^2\beta+4\beta^2-24\alpha\gamma\theta+8\gamma\beta^2\\ -16\gamma^4\mu^2-32\gamma\mu^2-12\alpha\theta\gamma^2-64\gamma^3\mu^2-80\gamma^2\mu^2-12\alpha\theta+4\beta^2\gamma^2\end{array}}}{6\theta}t$,

$$u_2(x,t) = \frac{4\gamma^2\mu + 8\mu + 12\gamma\mu - 2\beta - \beta\gamma - 3\theta c}{3\theta(\gamma+1)} + \frac{2(\gamma+2)\sqrt{-\mu}}{\theta}\left(\frac{c_1\sinh(\sqrt{-\mu}\xi) + c_2\cosh(\sqrt{-\mu}\xi)}{c_1\cosh(\sqrt{-\mu}\xi) + c_2\sinh(\sqrt{-\mu}\xi)}\right)^2, \tag{59}$$

where $\xi = x - \frac{\gamma^2+2\gamma+1-2\beta+\sqrt{\begin{array}{c}6\gamma^2+4\gamma+4\gamma^3-8\beta\gamma+1-4\beta+\gamma^4-4\gamma^2\beta+4\beta^2-24\alpha\gamma\theta+8\gamma\beta^2\\ -16\gamma^4\mu^2-32\gamma\mu^2-12\alpha\theta\gamma^2-64\gamma^3\mu^2-80\gamma^2\mu^2-12\alpha\theta+4\beta^2\gamma^2\end{array}}}{6\theta}t$,

$$\begin{aligned} u_3(x,t) = {} & \frac{-2\lambda^2\gamma^2 + 8\gamma^2\mu - 2\beta\gamma - 6\lambda^2\gamma + 24\gamma\mu - 4\beta + 16\mu - 4\lambda^2 - 6\theta c}{6\theta(\gamma+1)} \\ & + \frac{(\gamma+2)(\lambda^2-4\mu)}{2\theta}\left(\frac{c_1\sinh\left(\frac{\sqrt{\lambda^2-4\mu}}{2}\xi\right) + c_2\cosh\left(\frac{\sqrt{\lambda^2-4\mu}}{2}\xi\right)}{c_1\cosh\left(\frac{\sqrt{\lambda^2-4\mu}}{2}\xi\right) + c_2\sinh\left(\frac{\sqrt{\lambda^2-4\mu}}{2}\xi\right)}\right)^2 \end{aligned} \tag{60}$$

where $\xi = x - \frac{\gamma^2+2\gamma+1-2\beta-\sqrt{\begin{array}{c}1+4\gamma-4\beta-4\gamma^2\beta+8\mu\lambda^2\gamma^4+16\gamma\mu\lambda^2+40\mu\gamma^2\lambda^2-24\alpha\theta\gamma+32\mu\lambda^2\gamma^3\\ -12\alpha\theta\gamma^2-8\beta\gamma-12\alpha\theta-80\gamma^2\mu^2-64\gamma^3\mu^2+8\gamma\beta^2-32\gamma\mu^2\\ +4\beta^2\gamma^2-16\gamma^4\mu^2-\lambda^4\gamma^4-5\lambda^4\gamma^2-4\lambda^4\gamma^3-2\lambda^4\gamma+\gamma^4+4\gamma^3+6\gamma^2+4\beta^2\end{array}}}{6\theta}t$,

$$\begin{aligned} u_4(x,t) = {} & \frac{-2\lambda^2\gamma^2 + 8\gamma^2\mu - 2\beta\gamma - 6\lambda^2\gamma + 24\gamma\mu - 4\beta + 16\mu - 4\lambda^2 - 6\theta c}{6\theta(\gamma+1)} \\ & + \frac{(\gamma+2)(\lambda^2-4\mu)}{2\theta}\left(\frac{c_1\sinh\left(\frac{\sqrt{\lambda^2-4\mu}}{2}\xi\right) + c_2\cosh\left(\frac{\sqrt{\lambda^2-4\mu}}{2}\xi\right)}{c_1\cosh\left(\frac{\sqrt{\lambda^2-4\mu}}{2}\xi\right) + c_2\sinh\left(\frac{\sqrt{\lambda^2-4\mu}}{2}\xi\right)}\right)^2 \end{aligned} \tag{61}$$

where $\xi = x - \frac{\gamma^2+2\gamma+1-2\beta+\sqrt{\begin{array}{c}1+4\gamma-4\beta-4\gamma^2\beta+8\mu\lambda^2\gamma^4+16\gamma\mu\lambda^2+40\mu\gamma^2\lambda^2-24\alpha\theta\gamma+32\mu\lambda^2\gamma^3\\ -12\alpha\theta\gamma^2-8\beta\gamma-12\alpha\theta-80\gamma^2\mu^2-64\gamma^3\mu^2+8\gamma\beta^2-32\gamma\mu^2\\ +4\beta^2\gamma^2-16\gamma^4\mu^2-\lambda^4\gamma^4-5\lambda^4\gamma^2-4\lambda^4\gamma^3-2\lambda^4\gamma+\gamma^4+4\gamma^3+6\gamma^2+4\beta^2\end{array}}}{6\theta}t$.

When $\lambda^2 - 4\mu < 0$, we obtain the trigonometric solutions respectively:

$$u_5(x,t) = \frac{4\gamma^2\mu + 8\mu + 12\gamma\mu - 2\beta - \beta\gamma - 3\theta c}{3\theta(\gamma+1)} + \frac{2(\gamma+2)\sqrt{\mu}}{\theta}\left(\frac{c_1\sin(\sqrt{\mu}\xi) + c_2\cos(\sqrt{\mu}\xi)}{c_1\cos(\sqrt{\mu}\xi) + c_2\sin(\sqrt{\mu}\xi)}\right)^2, \tag{62}$$

where $\xi = x - \dfrac{\gamma^2+2\gamma+1-2\beta-\sqrt{\begin{array}{c}6\gamma^2+4\gamma+4\gamma^3-8\beta\gamma+1-4\beta+\gamma^4-4\gamma^2\beta+4\beta^2-24\alpha\gamma\theta+8\gamma\beta^2\\-16\gamma^4\mu^2-32\gamma\mu^2-12\alpha\theta\gamma^2-64\gamma^3\mu^2-80\gamma^2\mu^2-12\alpha\theta+4\beta^2\gamma^2\end{array}}}{6\theta}t,$

$$u_6(x,t) = \frac{4\gamma^2\mu+8\mu+12\gamma\mu-2\beta-\beta\gamma-3\theta c}{3\theta(\gamma+1)} + \frac{2(\gamma+2)\sqrt{\mu}}{\theta}\left(\frac{c_1\sin(\sqrt{\mu}\xi)+c_2\cos(\sqrt{\mu}\xi)}{c_1\cos(\sqrt{\mu}\xi)+c_2\sin(\sqrt{\mu}\xi)}\right)^2, \tag{63}$$

where $\xi = x - \dfrac{\gamma^2+2\gamma+1-2\beta+\sqrt{\begin{array}{c}6\gamma^2+4\gamma+4\gamma^3-8\beta\gamma+1-4\beta+\gamma^4-4\gamma^2\beta+4\beta^2-24\alpha\gamma\theta+8\gamma\beta^2\\-16\gamma^4\mu^2-32\gamma\mu^2-12\alpha\theta\gamma^2-64\gamma^3\mu^2-80\gamma^2\mu^2-12\alpha\theta+4\beta^2\gamma^2\end{array}}}{6\theta}t,$

$$\begin{aligned} u_7(x,t) &= \frac{-2\lambda^2\gamma^2+8\gamma^2\mu-2\beta\gamma-6\lambda^2\gamma+24\gamma\mu-4\beta+16\mu-4\lambda^2-6\theta c}{6\theta(\gamma+1)} \\ &+ \frac{(\gamma+2)(4\mu-\lambda^2)}{2\theta}\left(\frac{-c_1\sin\left(\frac{\sqrt{4\mu-\lambda^2}}{2}\xi\right)+c_2\cos\left(\frac{\sqrt{4\mu-\lambda^2}}{2}\xi\right)}{c_1\cos\left(\frac{\sqrt{4\mu-\lambda^2}}{2}\xi\right)+c_2\sin\left(\frac{\sqrt{4\mu-\lambda^2}}{2}\xi\right)}\right)^2 \end{aligned} \tag{64}$$

where $\xi = x - \dfrac{\gamma^2+2\gamma+1-2\beta-\sqrt{\begin{array}{c}1+4\gamma-4\beta-4\gamma^2\beta+8\mu\lambda^2\gamma^4+16\gamma\mu\lambda^2+40\mu\gamma^2\lambda^2-24\alpha\theta\gamma+32\mu\lambda^2\gamma^3\\-12\alpha\theta\gamma^2-8\beta\gamma-12\alpha\theta-80\gamma^2\mu^2-64\gamma^3\mu^2+8\gamma\beta^2-32\gamma\mu^2\\+4\beta^2\gamma^2-16\gamma^4\mu^2-\lambda^4\gamma^4-5\lambda^4\gamma^2-4\lambda^4\gamma^3-2\lambda^4\gamma+\gamma^4+4\gamma^3+6\gamma^2+4\beta^2\end{array}}}{6\theta}t$

$$\begin{aligned} u_8(x,t) &= \frac{-2\lambda^2\gamma^2+8\gamma^2\mu-2\beta\gamma-6\lambda^2\gamma+24\gamma\mu-4\beta+16\mu-4\lambda^2-6\theta c}{6\theta(\gamma+1)} \\ &+ \frac{(\gamma+2)(4\mu-\lambda^2)}{2\theta}\left(\frac{-c_1\sin\left(\frac{\sqrt{4\mu-\lambda^2}}{2}\xi\right)+c_2\cos\left(\frac{\sqrt{4\mu-\lambda^2}}{2}\xi\right)}{c_1\cos\left(\frac{\sqrt{4\mu-\lambda^2}}{2}\xi\right)+c_2\sin\left(\frac{\sqrt{4\mu-\lambda^2}}{2}\xi\right)}\right)^2 \end{aligned} \tag{65}$$

where $\xi = x - \dfrac{\gamma^2+2\gamma+1-2\beta+\sqrt{\begin{array}{c}1+4\gamma-4\beta-4\gamma^2\beta+8\mu\lambda^2\gamma^4+16\gamma\mu\lambda^2+40\mu\gamma^2\lambda^2-24\alpha\theta\gamma+32\mu\lambda^2\gamma^3\\-12\alpha\theta\gamma^2-8\beta\gamma-12\alpha\theta-80\gamma^2\mu^2-64\gamma^3\mu^2+8\gamma\beta^2-32\gamma\mu^2\\+4\beta^2\gamma^2-16\gamma^4\mu^2-\lambda^4\gamma^4-5\lambda^4\gamma^2-4\lambda^4\gamma^3-2\lambda^4\gamma+\gamma^4+4\gamma^3+6\gamma^2+4\beta^2\end{array}}}{6\theta}t$

When $\lambda^2-4\mu=0$, we obtain the rational solution:

$$u_9(x,t) = \frac{-2\beta-\beta\gamma-3\theta c}{3\theta(\gamma+1)} + \frac{2(\gamma+2)}{\theta}\left(\frac{c_2}{c_1+c_2\xi}\right)^2 \tag{66}$$

where $\xi = x - \dfrac{\gamma^2+2\gamma+1-2\beta-\sqrt{\begin{array}{c}6\gamma^2+4\gamma+4\gamma^3-8\beta\gamma+1-4\beta+\gamma^4-4\gamma^2\beta+4\beta^2\\-24\alpha\gamma\theta+8\gamma\beta^2-12\alpha\theta\gamma^2-12\alpha\theta+4\beta^2\gamma^2\end{array}}}{6\theta}t,$

$$u_{10}(x,t) = \frac{-2\beta-\beta\gamma-3\theta c}{3\theta(\gamma+1)} + \frac{2(\gamma+2)}{\theta}\left(\frac{c_2}{c_1+c_2\xi}\right)^2 \tag{67}$$

where $\xi = x - \dfrac{\gamma^2+2\gamma+1-2\beta+\sqrt{\begin{array}{c}6\gamma^2+4\gamma+4\gamma^3-8\beta\gamma+1-4\beta+\gamma^4-4\gamma^2\beta+4\beta^2\\-24\alpha\gamma\theta+8\gamma\beta^2-12\alpha\theta\gamma^2-12\alpha\theta+4\beta^2\gamma^2\end{array}}}{6\theta}t,$

$$u_{11}(x,t)=\frac{-2\lambda^2\gamma^2+8\gamma^2\mu-2\beta\gamma-6\lambda^2\gamma+24\gamma\mu-4\beta+16\mu-4\lambda^2-6\theta c}{6\theta(\gamma+1)}+\frac{2(\gamma+2)}{\theta}\left(\frac{c_2}{c_1+c_2\xi}\right)^2 \tag{68}$$

where $\xi=x-\frac{\gamma^2+2\gamma+1-2\beta-\sqrt{\begin{array}{c}1+4\gamma-4\beta-4\gamma^2\beta+8\mu\lambda^2\gamma^4+16\gamma\mu\lambda^2+40\mu\gamma^2\lambda^2-24\alpha\theta\gamma+32\mu\lambda^2\gamma^3\\-12\alpha\theta\gamma^2-8\beta\gamma-12\alpha\theta-80\gamma^2\mu^2-64\gamma^3\mu^2+8\gamma\beta^2-32\gamma\mu^2\\+4\beta^2\gamma^2-16\gamma^4\mu^2-\lambda^4\gamma^4-5\lambda^4\gamma^2-4\lambda^4\gamma^3-2\lambda^4\gamma+\gamma^4+4\gamma^3+6\gamma^2+4\beta^2\end{array}}}{6\theta}t,$

$$u_{12}(x,t)=\frac{-2\lambda^2\gamma^2+8\gamma^2\mu-2\beta\gamma-6\lambda^2\gamma+24\gamma\mu-4\beta+16\mu-4\lambda^2-6\theta c}{6\theta(\gamma+1)}+\frac{2(\gamma+2)}{\theta}\left(\frac{c_2}{c_1+c_2\xi}\right)^2 \tag{69}$$

where $\xi=x-\frac{\gamma^2+2\gamma+1-2\beta+\sqrt{\begin{array}{c}1+4\gamma-4\beta-4\gamma^2\beta+8\mu\lambda^2\gamma^4+16\gamma\mu\lambda^2+40\mu\gamma^2\lambda^2-24\alpha\theta\gamma+32\mu\lambda^2\gamma^3\\-12\alpha\theta\gamma^2-8\beta\gamma-12\alpha\theta-80\gamma^2\mu^2-64\gamma^3\mu^2+8\gamma\beta^2-32\gamma\mu^2\\+4\beta^2\gamma^2-16\gamma^4\mu^2-\lambda^4\gamma^4-5\lambda^4\gamma^2-4\lambda^4\gamma^3-2\lambda^4\gamma+\gamma^4+4\gamma^3+6\gamma^2+4\beta^2\end{array}}}{6\theta}t.$

7 Conclusions

In this paper, we have seen that the $\left(\frac{G'}{G}\right)$ expansion method is successfully used to obtain abundant travelling wave solutions not only for the class of nonlinear evolution equations but also for the a class of nonlinear pseudoparabolic equations. We have seen that three types of travelling wave solutions were successfully found, in terms of hyperbolic, trigonometric and rational functions. It will be more important to seek solutions of higher-order nonlinear equations which can be reduced to ODEs of the order greater than 2. We have noted that this method changes the given difficult problems into simple problems which can be solved easily. The method yields a general solution with free parameters which can be identified by the above conditions in section 2. Moreover,some numerical methods like the Adomian decomposition method and homotopy perturbation method depend on the initial conditions and obtain a solution in a series which converges to the exact solution of the problem. However,it is obtained by the $\left(\frac{G'}{G}\right)$ expansion method a general solution without approximation and there is no need to apply the initial and boundary conditions at the outset. The $\left(\frac{G'}{G}\right)$ expansion method is also a standard, direct and computerizable method, which allows us to solve complicated and tedious algebraic calculation. The solution procedure can be easily implemented in Mathematica or Maple.

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