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Numerical solution of modified regularized long wave equation by using cubic trigonometric B-spline functions

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Abstract

In this study, the Modified Regularized Long Wave (MRLW) equation is solved numerically. The method used for the numerical solution of MRLW equation includes the space discretization with the Galerkin finite element method based on cubic trigonometric B-spline, and also the time discretization with the Crank-Nicolson method. We tried to obtain a more accurate method with the help of trigonometric B-spline for the numerical solution of the MRLW equation than the existing numerical methods in the first test problem. Then, the interaction problem of the two positive solitary waves of the MRLW equation is considered, and the conservation constants are compared with the existing ones to see the correctness of the method.

Keywords: *Cubic trigonemetric B-splines, Galerkin method, modified regularized long wave equation.*

Değiştirilmiş düzenli uzun dalga denkleminin kübik trigonometrik B-spline fonksiyonları kullanılarak nümerik çözümü

Özet

Bu çalışmada, Modified Regularized Long Wave (MRLW) denklemi sayısal olarak çözülmüştür. MRLW denkleminin sayısal çözümü için kullanılan yöntem, kübik trigonometrik B-spline'a dayalı Galerkin sonlu eleman yöntemi ile konum ayrıştırmasını ve ayrıca Crank-Nicolson yöntemiyle zaman ayrıştırmasını içerir. İlk test probleminde MRLW denkleminin sayısal çözümü için trigonometrik B-spline yardımıyla mevcut

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sayısal metotlardan daha doğru bir yöntem elde etmeye çalıştık. Daha sonra, MRLW denkleminin iki pozitif solitary dalganın etkileşimi problemi göz önüne alınmış ve korunum sabitleri, yöntemin doğruluğunu görmek için mevcut çalışmalarla karşılaştırılmıştır.

Anahtar kelimeler: *Galerkin metodu, kübik trigonometrik B-spline, değiştirilmiş düzenli uzun dalga denklemi.*

1. Introduction

The numerical solution of the MRLW equation in the form

$$u_t + u_x + \varepsilon u^2 u_x - \mu\, u_{xxt} = 0, \tag{1}$$

where ε and μ are positive real parameters and the subscripts x and t denote differentiations with respect to space and time, is discussed. The space variable x of the problem is defined over the interval $[\alpha, \beta]$ for the numerical treatment.

The following boundary conditions will be taken into account over the space region

$$\begin{array}{l} u(\alpha,t) = u(\beta,t) = 0, \\ u_x(\alpha,t) = u_x(\beta,t) = 0, \end{array} \quad t \in (0,T] \tag{2}$$

and the following initial condition

$$u(x,0) = f(x) \tag{3}$$

will be described in the test problems section.

The solution of the equation investigating the numerical solution is composed of solitary waves that protect its shape against the collision and it is also important to explain many physical phenomena. Thus, the equation has been often taken up by numerical analysts and different solution methods have been developed. The finite difference [1, 2], the homotopy perturbation [3], the Adomian decomposition [4], the expilicit multisymplectic [5], the He's variational iteration [6], the meshless [7], the homotopy anlysis [8], the Galerkin linear finite element [9], the second-order Fourier pseudospectral [10], the explicit multistep Galerkin finite element [11], the mixed Galerkin finite element [12], the split least-squares mixed finite element [13], the compact conservative [14] and the moving least square collocation [15] methods are some of those methods. In addition to those ones, there are also some methods that are obtained by using various B-spline functions. A collocation method is applied to solve numerically the MRLW equation using cubic B-splines in the reference [16]. A comparison of quadratic, cubic, quartic and quintic splines for solving the MRLW equation with collocation scheme is presented by Raslan and Hassan [17]. The numerical solution of MRLW equation is found using collocation method based on quadratic B-spline functions [18]. To obtain solitary wave solutions for the MRLW equation the collocation method using with quintic B-splines is used in the paper [19]. A sextic B-spline collocation algorithm have been developed for solving numerically MRLW equation [20]. The septic and quintic B-spline collocation methods, the subdomain finite element method based on quartic B-splines, the cubic B-spline Galerkin and Petrov-Galerkin methods are implemented to find the numerical solution of the MRLW equation [21-25]. A numerical technique including quartic B-splines for solution of the MRLW equation has been produced by Fazal-i-Haq *et all* [26]. Soliman obtaines the solution of the same equation by quartic B-spline collocation method [27]. Again, an another collocation method is offered for the numerical solution of the MRLW equation in the study [28]. Mittal and Rohila study the numerical solution of the MRLW by the fourth order numerical method based on cubic B-spline functions in the study [29].

Our aim in this study is to obtain a more accurate numerical method for obtaining the numerical solution of the MRLW equation. So, the Galerkin finite element and the Crank Nicolson methods have been used to get the fully integrated form of the MRLW equation by using the cubic trigonometric B-spline function. By the proposed method, two test problem as the investigation of the motion of single solitary wave and the observation of the interaction of the two solitary waves have been studied.

2. Application of the method

During computational studies, time step Δt and space step h is used over the discretization of the space-time plane. The exact solution of the unknown function is

$$u(x_p, t_q) = u_p^q, p = 0,1, \dots, N; q = 0,1,2, \dots$$

where $x_p = \alpha + ph, t_q = q\Delta\, t$ and the notation U_p^q is designated the numerical value of u_p^q.

2.1. *Time discretization*

With the application of the Crank Nicolson method to the MRLW equation for the time discretization, we have

$$\begin{aligned} u^{q+1} + \frac{\Delta t}{2}(u_x)^{q+1} + \frac{\Delta t}{2}\varepsilon u^{q+1}u^{q+1}(u_x)^{q+1} - \mu(u_{xx})^{q+1} \\ = u^q - \mu(\, u_{xx})^q - \frac{\Delta t}{2}(\, u_x)^q - \frac{\Delta t}{2}\varepsilon u^q u^q (u_x)^q. \end{aligned} \tag{4}$$

2.2. *Space discretization*

For the space discretization, the region $[\alpha, \beta]$ is splitted into uniformly sized N finite elements by the nodes $x_p, p = 0, \dots, N$ with $h = x_{p+1} - x_p$.

The cubic trigonometric B-spline functions are defined at the knots as

$$T_p\,(x) = \frac{1}{\theta}\begin{cases} \varphi^3(x_{p-2}) & ,[x_{p-2},x_{p-1}] \\ -\varphi^2(x_p-2)\varphi(x_p)-\varphi(x_{p-2})\varphi(x_{p+1})\varphi(x_{p-1}) & \\ -\varphi(x_{p+2})\varphi^2(x_{p-1}) & ,[x_{p-1},x_p] \\ \varphi(x_{p-2})\varphi^2(x_{p+1})+\varphi(x_{p+2})\varphi(x_{p-1})\varphi(x_{p+1}) & \\ +\varphi^2(x_{p+2})\varphi(x_p) & ,[x_p,x_{p+1}] \\ -\varphi^3(\,x_{p+2}) & ,[x_{p+1},x_{p+2}] \\ 0 & otherwise \end{cases} \tag{5}$$

where

$$\theta = sin\,(\frac{h}{2})sin\,(h)sin\,(\frac{3h}{2}),$$
$$\varphi(x_p)\ = sin\ (\frac{x-x_p}{2})\,.$$

The global approximation to the analytical solution of the problem can be defined by using cubic trigonometric B-splines as

$$u(x,t) \approx U(x,t) = \sum_{p=-1}^{N+1} T_p(x)\delta_p(t) \tag{6}$$

where the coefficients δ_p are the unknown parameters to be calculated from boundary conditions and the cubic trigonometric B-spline Galerkin form of the MRLW equation. $T_p, (\mathrm{p} = -1, \dots, \mathrm{N}+1)$ and their first derivatives with respect to space variable x vanish outside the interval $[x_{p-1}, x_{p+2}]$.

As can be seen from the cubic trigonometric B-spline (5), each element $[x_p, x_{p+1}]$ contains 3 splines, so an approach to exact solution $u(x,t)$ can be written as

$$U(x,t) = \sum_{j=p-1}^{p+2} T_{pj}(x)\delta_j(t) = T_{p-1}\delta_{p-1} + T_p\delta_p + T_{p+1}\delta_{p+1} + T_{p+2}\delta_{p+2} \tag{7}$$

where $\boldsymbol{\delta}^e = (\delta_{p-1}, \delta_p, \delta_{p+1}, \delta_{p+2})$ are element parameters and trigonometric cubic B-splines $\boldsymbol{T}^e = (T_{p-1}, T_p, T_{p+1}, T_{p+2})$ are element shape functions.

By Eqs. (5) and (7), the values of $U_p = U(x_p, t)$ and its first and second derivatives at points $x = x_p$ can be written as

$$U_p = \sin^2\left(\frac{h}{2}\right)\csc(h)\csc\left(\frac{3h}{2}\right)(\,\delta_{p-1}+\delta_{p+1}) + \frac{2}{1+2cos(h)}\delta_p, \tag{8}$$
$$U_p' = \frac{3}{4}\csc\left(\frac{3h}{2}\right)(-\delta_{p-1}+\delta_{p+1}), \tag{9}$$
$$U_p'' = \frac{3\left(3\cos^2\left(\frac{h}{2}\right)-1\right)}{4\sin(h)\sin\left(\frac{3h}{2}\right)}(\delta_{p-1}+\delta_{p+1}) - \frac{3\cot^2\left(\frac{h}{2}\right)}{2+4\cos(h)}\delta_p. \tag{10}$$

Applying Galerkin method to Eq. (4) with weight function $W(x)$ and then integrating by parts lead to the weak form

$$\int_{\alpha}^{\beta} W(x)\left(U^{q+1}+\frac{\Delta t}{2}(U_x)^{q+1}+\frac{\Delta t}{2}\varepsilon U^{q+1}\,U^{q+1}(U_x)^{q+1}\right)dx+ \\ \mu\int_{\alpha}^{\beta} W_x(\,x)(U_x)^{q+1}\;dx=\int_{\alpha}^{\beta} W(x)\left(U^{q}-\frac{\Delta t}{2}(U_x)^{q}-\frac{\Delta t}{2}\varepsilon U^{q}U^{q}(U_x)^{q}\right)dx+ \\ \mu\int_{\alpha}^{\beta} W_x(x)(U_x)^{q}\,dx. \tag{11}$$

With substituting the weight function by the cubic trigonometric B-spline in the Eq. (11) and adding the expression (7) in it, the following approximation is obtained over the element $[x_p, x_{p+1}]$

$$\sum_{j=p-1}^{p+2}\left\{\left(\int_{x_p}^{x_{p+1}} T_iT_j dx\right)\delta_j^{q+1}+\mu\left(\int_{x_p}^{x_{p+1}} T_i'T_j' dx\right)\delta_j^{q+1}+ \\ \frac{\Delta t}{2}\left(\int_{x_p}^{x_{p+1}} T_iT_j' dx\right)\delta_j^{q+1}+ \\ \frac{\Delta t}{2}\varepsilon\sum_{k=p-1}^{p+2}\sum_{l=p-1}^{p+2}\left(\int_{x_p}^{x_{p+1}} T_i\left(T_k\delta_k^{q+1}\right)\left(T_l\delta_l^{q+1}\right)T_j' dx\right)\delta_j^{q+1}\right\}- \\ \sum_{j=p-1}^{p+2}\left\{\left(\int_{x_p}^{x_{p+1}} T_iT_j dx\right)\delta_j^{q}+\mu\left(\int_{x_p}^{x_{p+1}} T_i'T_j' dx\right)\delta_j^{q}-\frac{\Delta t}{2}\left(\int_{x_p}^{x_{p+1}} T_iT_j' dx\right)\delta_j^{q}- \\ \frac{\Delta t}{2}\varepsilon\sum_{k=p-1}^{p+2}\sum_{l=p-1}^{p+2}\left(\int_{x_p}^{x_{p+1}} T_i\left(T_k\delta_k^{q}\right)\left(T_l\delta_l^{q}\right)T_j' dx\right)\delta_j^{q}\right\}, \tag{12}$$

where i, j, k and l take only the values $p-1, p, p+1, p+2$.

(12) can be written in the matrix form as

$$\left[\boldsymbol{A}^e+\mu\boldsymbol{D}^e+\frac{\Delta t}{2}\boldsymbol{B}^e+\frac{\Delta t}{2}\varepsilon\boldsymbol{C}^e((\boldsymbol{\delta}^e)^{q+1})\right](\boldsymbol{\delta}^e)^{q+1} \\ -\left[\boldsymbol{A}^e+\mu\boldsymbol{D}^e-\frac{\Delta t}{2}\boldsymbol{B}^e-\frac{\Delta t}{2}\varepsilon\boldsymbol{C}^e((\boldsymbol{\delta}^e)^{q})\right](\boldsymbol{\delta}^e)^{q} \tag{13}$$

together with the following expressions:

$$A_{ij}^e=\int_{x_p}^{x_{p+1}} T_iT_j dx\,,B_{ij}^e=\int_{x_p}^{x_{p+1}} T_iT_j' dx\,,$$

$$C_{ij}^e((\delta^e)^{q+1})=\int_{x_p}^{x_{p+1}} T_i\left(T_k\delta_k^{q+1}\right)\left(T_l\delta_l^{q+1}\right)T_j' dx$$

$$D_{ij}^e=\int_{x_p}^{x_{p+1}} T_i'T_j' dx\,,(\boldsymbol{\delta}^e)^{q+1}=\left(\delta_{p-1}^{q+1},\delta_p^{q+1},\delta_{p+1}^{q+1},\delta_{p+2}^{q+1}\right)^T$$

By combining all elements, the nonlinear matrix equation is achieved

$$\left[\boldsymbol{A}+\mu\boldsymbol{D}+\frac{\Delta t}{2}\boldsymbol{B}+\frac{\Delta t}{2}\varepsilon\boldsymbol{C}(\boldsymbol{\delta}^{q+1})\right]\boldsymbol{\delta}^{q+1}=\left[\boldsymbol{A}+\mu\boldsymbol{D}-\frac{\Delta t}{2}\boldsymbol{B}-\frac{\Delta t}{2}\varepsilon\boldsymbol{C}(\boldsymbol{\delta}^{q})\right]\boldsymbol{\delta}^{q} \tag{14}$$

where global element parameters

$\boldsymbol{\delta} = (\delta_{-1}, \delta_0, \dots, \delta_N, \delta_{N+1})^T$

and $\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C}, \boldsymbol{D}$ are derived from the corresponding element matrices $\boldsymbol{A}^e$, $\boldsymbol{B}^e, \boldsymbol{C}^e$ and $\boldsymbol{D}^e$.

By obtaining the initial vector $\delta^0 = (\delta_{-1}^0, \dots, \delta_N^0, \delta_{N+1}^0)$ using the boundary and initial conditions, the unknowns δ^{q+1} can be calculating using the recurrence relation (14), repeatedly. Since the obtained system (14) is an implicit system, we have tried to increase the accuracy of the algorithm by using an inner iteration with the following algorithm:

i. $Set\ error = 1\ and\ \delta_p^* = \delta_p^{q+1}\ in\ \boldsymbol{C}(\boldsymbol{\delta}^{q+1})\ and\ taking\ \delta_p^* = \delta_p^q\ then$
ii. $While\ error > 10^{-10}\ do$ iii and iv
iii. $Find\ U_p^{q+1}$
iv. $Find\ \max_p \left|U_p^{q+1} - U_p^*\right|\ and\ set\ \ \delta_p^* = \delta_p^{q+1}$
v. $Stop\ and\ go\ to\ next\ time\ step$

3. Stability analysis

A Von Neumann stability analysis has been performed for the present method. A typical row member of the linearized equation corresponding to Eq. (14) can be given by

$$\begin{aligned}&\gamma_1\delta_{p-3}^{q+1} + \gamma_2\delta_{p-2}^{q+1} + \gamma_3\delta_{p-1}^{q+1} + \gamma_4\delta_p^{q+1} + \gamma_5\delta_{p+1}^{q+1} + \gamma_6\delta_{p+2}^{q+1} + \gamma_7\delta_{p+3}^{q+1} = \\ &\gamma_7\delta_{p-3}^{q} + \gamma_5\delta_{p-2}^{q} + \gamma_5\delta_{p-1}^{q} + \gamma_4\delta_p^{q} + \gamma_3\delta_{p+1}^{q} + \gamma_2\delta_{p+2}^{q} + \gamma_1\delta_{p+3}^{q},\end{aligned} \tag{15}$$

where the parameters $\gamma_i, i = 1, \dots, 7$ are determined from system (14), in which these values are not prefered to document here due to being too long.

Substituting the Fourier mode $\delta_p^q = \hat{\delta}^q e^{ipkh}, i = \sqrt{-1}$ into Eq. (15) which becomes

$$\hat{\delta}^{q+1} = \rho\hat{\delta}^q.$$

Here, the growth factor ρ is determined as

$$\rho = \frac{a + ib}{a - ib}$$

where

$$\begin{aligned}a = \tfrac{1}{96\theta^2}\Bigg(&\Big((576\mu - 4864)\sin\left(\tfrac{h}{2}\right) + (-864\mu h - 3456h)\cos(kh)\Big)\cos\left(\tfrac{h}{2}\right)^7 + \\ &\Big((1440\mu + 6272)\cos(kh)\sin\left(\tfrac{h}{2}\right) + 3456h + 864\mu h\Big)\cos\left(\tfrac{h}{2}\right)^6 + \Big(\big((-1008\mu - \\ &704)\cos(2kh) - 2880\mu + 5888\big)\sin\left(\tfrac{h}{2}\right) + (1080\mu h + 5088h)\cos(kh)\Big)\cos\left(\tfrac{h}{2}\right)^5 + \\ &\Big((-5920 - 936\mu)\cos(kh)\sin\left(\tfrac{h}{2}\right) + (960h + 432\mu h)\cos(2kh) - 864\mu h - \\ &4224h\Big)\cos\left(\tfrac{h}{2}\right)^4 + \Big(\big((1152\mu - 512)\cos(2kh) + 2232\mu - 2720\big)\sin\left(\tfrac{h}{2}\right) + (-918\mu h -\end{aligned}$$

$$3672h)\cos(kh) + (-54\mu h - 24h)\cos(3kh)\Big)\cos\left(\frac{h}{2}\right)^3 + \Big(\big((522\mu + 1352)\cos(kh) + (126\mu + 88)\cos(3kh)\big)\sin\left(\frac{1}{2}h\right) + (-720h - 324\mu h)\cos(2kh) + 1248h + 216\mu h\Big)\cos\left(\frac{h}{2}\right)^2 + \Big(\big((-468 * \mu + 496)\cos(2kh) - 468\mu + 496\big)\sin\left(\frac{h}{2}\right) + (1140h + 297\mu h)\cos(kh) + (27\mu h - 36h)\cos(3kh)\Big)\cos\left(\frac{h}{2}\right) + \big((-216\mu + 96)\cos(kh) + (32 - 72\mu)\cos(3kh)\big)\sin\left(\frac{h}{2}\right) + (120h + 54\mu h)\cos(2kh) + 120h + 54\mu h\Big),$$

$$b = \frac{-3\Delta t(\varepsilon\lambda + 1)}{32\theta^2}\Big(96\sin(kh)\cos\left(\frac{h}{2}\right)^7 + \Big(96\sin(kh)\sin\left(\frac{h}{2}\right)h - 48\sin(2kh)\Big)\cos\left(\frac{h}{2}\right)^6 - 168\sin(kh) * \cos\left(\frac{h}{2}\right)^5 + \Big(72\sin(2kh) - 136\sin(kh)\sin\left(\frac{h}{2}\right)h\Big)\cos\left(\frac{h}{2}\right)^4 + \Big(6\sin(3kh) - 32\sin(2kh)\sin\left(\frac{h}{2}\right)h + 78\sin(kh)\Big)\cos\left(\frac{h}{2}\right)^3 + \Big((54\sin(kh)h + 2\sin(3kh)h)\sin\left(\frac{h}{2}\right) - 24\sin(2kh)\Big)\cos\left(\frac{h}{2}\right)^2 + \Big(-6\sin(kh) - 6\sin(3kh) + 20\sin(2kh)\sin\left(\frac{h}{2}\right)h\Big)\cos\left(\frac{h}{2}\right) + (\sin(kh)h + \sin(3kh)h)\sin\left(\frac{h}{2}\right)\Big).$$

Since the magnitude of the growth factor is $|\rho| = 1$, the proposed method is an unconditionally stable method.

4. Test problems

The error norm

$$L_\infty = \max_p \left|\mathrm{u_p} - \mathrm{U_p}\right|$$

is used to see the accuracy of the proposed method. The order of convergence is measured by the following formulas

$$\text{order} = \frac{log\left|(L_\infty)_{h_i}/(L_\infty)_{h_{i-1}}\right|}{log|h_i/h_{i-1}|}, \text{order} = \frac{log\left|(L_\infty)_{\Delta t_i}/(L_\infty)_{\Delta t_{i-1}}\right|}{log|\Delta t_i/\Delta t_{i-1}|}.$$

4.1. First test problem

For the investigation of a motion of single solitary wave for the MRLW equation, let take one of the solutions of the MRLW equation as follows together with the initial condition

$$u(x,t) = \sqrt{\frac{6c}{\varepsilon}}\operatorname{sech}(k[x - \tilde{x}_0 - (c+1)t]), \quad (16)$$

$$u(x,0) = \sqrt{\frac{6c}{\varepsilon}}\operatorname{sech}(k[x - \tilde{x}_0]), \quad (17)$$

where $1 + c$ is the wave velocity, $\sqrt{\frac{6c}{\varepsilon}}$ is amplitude of the solitary wave, $\tilde{x}_0$ is peak position of the initially centered wave and $k = \sqrt{\frac{c}{\mu(c+1)}}$. This solution propagates towards the right across the interval $[\alpha, \beta]$ over the up to the time T without change of shape at a steady velocity ν (see Fig. 1). The Eq. (16) satisfies three conservation laws matching to the following given integrals [30]:

$$
\begin{aligned}
I_1 &= \int_{-\infty}^{\infty} u dx \approx \int_{\alpha}^{\beta} U dx\,, \\
I_2 &= \int_{-\infty}^{\infty} (u^2 + \mu(u_x)^2) dx \approx \int_{\alpha}^{\beta} (U^2 + \mu(U_x)^2) dx\,, \\
I_3 &= \int_{-\infty}^{\infty} \left(u^4 - 6\,\frac{\mu}{\varepsilon}(u_x)^2\right) dx \approx \int_{\alpha}^{\beta} \left(U^4 - 6\,\frac{\mu}{\varepsilon}(U_x)^2\right) dx.
\end{aligned} \tag{18}
$$

The trapezoidal rule is used in the calculation of the integrals there.

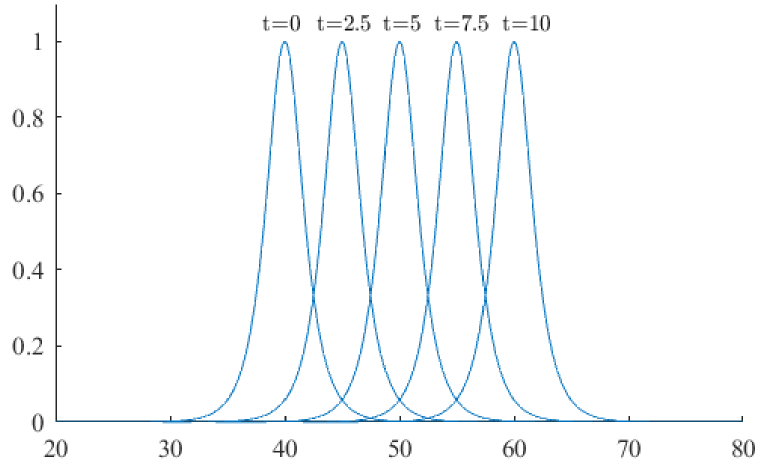

Figure 1. $U(x, t)$ at various time with $h = 0.2, \Delta t = 0.025$.

The invariants (18) for the MRLW equation can be determined analytically using the initial condition (17) as

$$
\begin{aligned}
I_1 &= \frac{\pi A}{k}, \\
I_2 &= \frac{2A^2}{k} + \frac{2\mu k A^2}{3}, \\
I_3 &= \frac{4A^2}{3k\varepsilon}(A^2\varepsilon - 3\mu k^2).
\end{aligned}
$$

The parameters been in single solitary wave simulation are taken as $\varepsilon = 6, \mu = 1, \tilde{x}_0 = 40$ and the amplitude $A = 1$ in the solution domain $[0,100]$ and the time period $[0,10]$.

Absolute error for the first test problem is seen in Fig. 2 at time $t = 10$. By comparison is made with the results given by the other finite elements methods based on various B-spline functions in Table 1, the present method seems to be better. According to the Tables 2 and 3, when the value of the time and space steps are reduced, the error norms decrease for the proposed method. It can also be seen that the order of convergence for Crank-Nicolson method is almost two in Table 2 and for Galerkin method based on cubic trigonometric B-spline function is almost four in Table 3.

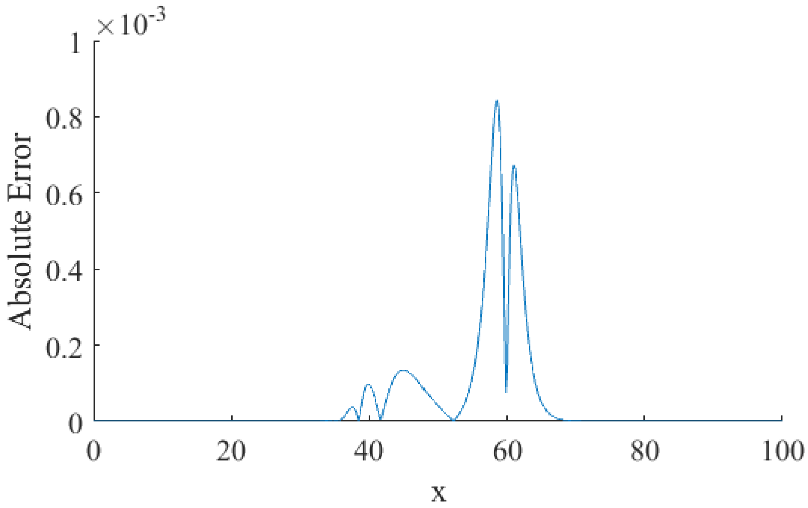

Figure 2. Absolute error at time $t = 10$ with $h = 0.2, \Delta t = 0.025$

Table 1. Error norm L_∞ and invariants at time $t = 10$ $(0 \leq x \leq 100, h = 0.2, \Delta t = 0.025,\ \varepsilon = 6,\ \mu = 1, c = 1)$.

	$L_\infty \times 10^3$	I_1	I_2	I_3
present$(t = 0)$	0	4.44288294	3.29889722	1.41514799
present	0.85	4.44288308	3.29983306	1.41421337
[16]	5.44	4.44288	3.29983	1.41420
[18]	1.25	4.445176	3.302476	1.417411
[21]	1.08	4.4431919	3.3003022	1.4146930
[28]	9.06	4.4428821	3.2997861	1.4141511
exact		4.44288294	3.29983165	1.41421356

Table 2. Rate of convergence for $h = 0.05, \varepsilon = 6,\ \mu = 1, c = 1, t = 10, 0 \leq x \leq 100$.

Δt	L_∞	order
0.2	0.044635	
0.1	0.013084	1.770405
0.05	0.003351	1.965127
0.025	0.000841	1.993839
0.01	0.000135	1.998888
0.005	0.000034	1.999298

Table 3. Rate of convergence for $\Delta t = 0.0005, \varepsilon = 6, \mu = 1, c = 1, t = 10, 0 \leq x \leq 100$.

h	L_∞	order
2	0.0970894	
1	0.0052516	4.2084889
0. 5	0.0002417	4.4411820
0. 2	0.0000058	4.0774062
0.1	0.0000006	3.2251087

4.2. Second test problem

As a second test problem, the problem of the interaction of two positive solitary waves is discussed for the MRLW equation. For this, the following initial condition given by the linear sum of two separate solitary waves at different amplitudes is used.

$$u(x,0) = A_1 \operatorname{sech}(k_1[x - x_1]) + A_2 \operatorname{sech}(k_2[x - x_2]) \tag{19}$$

where $A_i = \sqrt{\frac{6c_i}{\varepsilon}}, k_i = \sqrt{\frac{c_i}{\mu(c_i+1)}}, i = 1,2$, and x_i, c_i are arbitrary constants.

For our computational works, the parameters are chosen as $\varepsilon = 6, \mu = 1, c_1 = 4, c_2 = 1, x_1 = 25, x_2 = 55, \Delta t = 0.02$ and $h = 0.2$ in the space domain $[0,150]$ and the time period $0 \leq t \leq 20$. With these parameters, two singular waves which peak positions are $x = 20$ and 50 occurs as seen in Fig. 3. As the amplitude of the first solitary wave is small compared to the second one, the collision of those takes place around time $t = 10$. It is seen in Fig. 3 that these waves, which are separated from each other, then maintain their amplitudes.

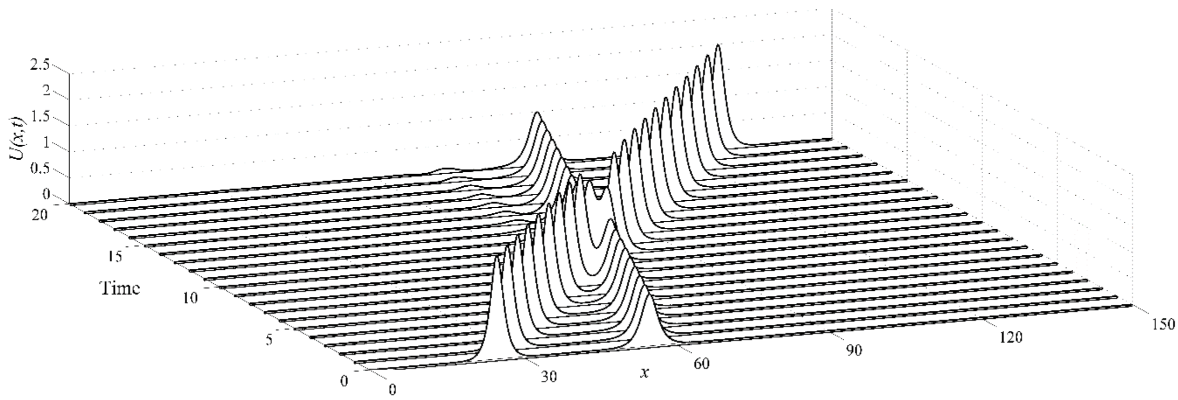

Figure 3. Interaction of two solitary waves.

The analytical invariants can be found using the initial condition (19) and the integrals (18) as

$$I_1 = \frac{\pi}{k_1 k_2}(k_2 A_1 + k_1 A_2),$$
$$I_2 = \frac{2}{k_1 k_2}(k_2 A_1^2 + k_1 A_2^2) + \frac{2\,\mu}{3 k_1 k_2}(k_1^2 k_2 A_1^2 + k_1 k_2^2 A_2^2),$$

$$I_3 = \frac{4}{3k_1k_2\varepsilon}\left(\varepsilon k_1 A_2^4 - 3\,\mu k_1 k_2^2 A_2^2 + \varepsilon k_2 A_1^4 - 3\mu k_1^2 k_2 A_1^2\right).$$

In Table 4, the comparison of analytical invariants is given. Accordingly, it can be said that the invariants are in harmony with the exact values.

5. Conclusion

The MRLW equation has been solved numerically by the method includes the space discretization with the Galerkin finite element method based on cubic trigonometric B-spline, and also the time discretization with the Crank-Nicolson method. The proposed method has tested on the propagation of single solitary wave and the interaction of two solitary waves. As a result, when the successful results of the proposed method are taken into consideration, high accuracy can be obtained if numerical solutions of equations with similar characteristics are obtained with this method.

Table 4. Comparison of invariants for the interaction of two solitary waves.

	Present			[26]		
t	I_1	I_2	I_3	I_1	I_2	I_3
0	11.4677	14.6292	22.8805	11.4677	14.6293	22.8804
4	11.4680	14.6312	22.8883	11.4677	14.6192	22.8403
8	11.4684	14.6361	22.9019	11.4677	14.6068	22.7879
12	11.4686	14.6343	22.9007	11.4677	14.6031	22.7758
16	11.4689	14.6363	22.9084	11.4677	14.5930	22.7361
20	11.4692	14.6382	22.9162	11.4677	14.5831	22.6965
exact	11.4677	14.6292	22.8805			

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