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Extended Multivariable Fourth Type Horn Functions

Duriye KORKMAZ-DUZGUN[1,*], Esra ERKUS-DUMAN[2]

[1]Department of Business Administration, Faculty of Economics and Administrative Sciences, Kafkas University, 36100, Kars, Turkey.

[2]Department of Mathematics, Faculty of Science, Gazi University, 06500, Ankara, Turkey.

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Abstract

In this paper, we define an extension of multivariable fourth kind Horn functions. Then, we obtain some generating functions for these functions. Furthermore, we get bilateral generating functions for the extended multivariable fourth kind Horn functions and extended first kind Lauricella functions. Finally, we derive various families of multilinear and multilateral generating functions for these functions and their special cases are also given.

1. INTRODUCTION

Recently, most of the scholars studying on special functions theory have a strong interest in extensions of classical gamma, beta and hypergeometric functions, which are including new extra parameter [1, 2, 3, 4, and 5].

In this study, we use the extended beta functions to define new extensions of fourth kind Horn and multivariable fourth kind Horn functions, which introduced below.

Definition 1.1. Let a function $\Theta\left(\{K_l\}_{l\in N_0};z\right)$ be analytic within the disk $|z|<R\ \ (0<R<\infty)$ and let its Taylor-Maclaurin coefficients be explicitly denoted by sequence $\{K_l\}_{l\in N_0}$. Suppose also that the function $\Theta\left(\{K_l\}_{l\in N_0};z\right)$ can be continued analytically in the right half-plane $\operatorname{Re}(z)>0$ with the asymptotic property given as follows [4]:

$$\Theta\left(\{K_l\}_{l\in N_0};z\right)=\begin{cases}\displaystyle\sum_{l=0}^{\infty}K_l\frac{z^l}{l!} & ; \quad |z|<R,\ \ (0<R<\infty),\ \ K_0=1\\ M_0 z^w\exp(z)\left[1+O\left(\dfrac{1}{z}\right)\right]; & \operatorname{Re}(z)\to\infty,\ \ M_0>0,\ \ w\in C\end{cases} \tag{1}$$

*Corresponding author, e-mail: dryekorkmaz@mail.edu.tr

for some suitable constants M_0 and w, depending essentially on the sequence $\{K_l\}_{l\in N_0}$.

By means of the function $\Theta\left(\{K_l\}_{l\in N_0};z\right)$ defined by (1), Srivastava *et al.* defined the extended gamma function $\Gamma_p^{\left(\{K_l\}_{l\in N_0}\right)}(x)$ by [4]:

$$\Gamma_p^{\left(\{K_l\}_{l\in N_0}\right)}(x):=\int_0^\infty t^{x-1}\Theta\left(\{K_l\}_{l\in N_0};-t-\frac{p}{t}\right)dt \tag{2}$$

$\left(\operatorname{Re}(x)>0\ \operatorname{Re}(p)\geq 0\right),$

if we set $K_l=\frac{(\rho)_l}{(\sigma)_l}$, $\rho=\sigma$ and $p=0$ in (2), we arrived that the classical gamma function defined as

$$\Gamma(x)=\int_0^\infty t^{x-1}e^{-t}dt,\quad \operatorname{Re}(x)>0.$$

Extended beta function $B_p^{\left(\{K_l\}_{l\in N_0}\right)}(\alpha,\beta)$ defined by [4]:

$$B_p^{\left(\{K_l\}_{l\in N_0}\right)}(\alpha,\beta):=\int_0^1 t^{\alpha-1}(1-t)^{\beta-1}\,\Theta\left(\{K_l\}_{l\in N_0};-\frac{p}{t(1-t)}\right)dt$$

$\left(\min\{\operatorname{Re}(\alpha),\operatorname{Re}(\beta)\}>0\ \operatorname{Re}(p)\geq 0\right).$

By introducing one additional parameter q with $\operatorname{Re}(q)\geq 0$, they defined,

$$B_{p,q}^{\left(\{K_l\}_{l\in N_0}\right)}(\alpha,\beta):=\int_0^1 t^{\alpha-1}(1-t)^{\beta-1}\,\Theta\left(\{K_l\}_{l\in N_0};-\frac{p}{t}-\frac{q}{(1-t)}\right)dt \tag{3}$$

$\left(\min\{\operatorname{Re}(\alpha),\operatorname{Re}(\beta)\}>0\ \min\{\operatorname{Re}(p),\operatorname{Re}(q)\}\geq 0\right),$

if we set $K_l=\frac{(\rho)_l}{(\sigma)_l}$, $\rho=\sigma$ and $p=q=0$ in (3), then (3) becomes classical beta function as follow:

$$B(\alpha,\beta):=\int_0^1 t^{\alpha-1}(1-t)^{\beta-1}dt,\quad \operatorname{Re}(\alpha)>\operatorname{Re}(\beta)>0.$$

By similar idea, they had extended hypergeometric and confluent hypergeometric functions respectively shown below [4]:

$$F_{p,q}^{\left(\{K_l\}_{l\in N_0}\right)}(\alpha,\beta;\gamma;x):=\sum_{n=0}^\infty(\alpha)_n\frac{B_{p,q}^{\left(\{K_l\}_{l\in N_0}\right)}(\beta+n,\gamma-\beta)}{B(\beta,\gamma-\beta)}\frac{x^n}{n!} \tag{4}$$

$\left(|x|<1;\ \operatorname{Re}(\gamma)>\operatorname{Re}(\beta)>0;\ \min\{\operatorname{Re}(p),\operatorname{Re}(q)\}\geq 0\right),$

and

$$\Phi_{p,q}^{\left(\{K_l\}_{l\in N_0}\right)}(\beta;\gamma;x) := \sum_{n=0}^{\infty} \frac{B_{p,q}^{\left(\{K_l\}_{l\in N_0}\right)}(\beta+n,\gamma-\beta)}{B(\beta,\gamma-\beta)} \frac{x^n}{n!} \tag{5}$$

$$\left(|x|<1;\ \operatorname{Re}(\gamma)>\operatorname{Re}(\beta)>0;\ \min\left\{\operatorname{Re}(p),\operatorname{Re}(q)\right\}\geq 0\right).$$

If we set $K_l = \frac{(\rho)_l}{(\sigma)_l}$, $\rho=\sigma$ and $p=q=0$, then (4) and (5) become classical hypergeometric and confluent hypergeometric functions as follow:

$$F(\alpha,\beta;\gamma;x) := \sum_{n=0}^{\infty} \frac{(\alpha)_n(\beta)_n}{(\gamma)_n} \frac{x^n}{n!}$$

$$\left(|x|<1;\ \operatorname{Re}(\gamma)>\operatorname{Re}(\beta)>0\right),$$

and

$$\Phi(\beta;\gamma;x) := \sum_{n=0}^{\infty} \frac{(\beta)_n}{(\gamma)_n} \frac{x^n}{n!}$$

$$\left(|x|<1;\ \operatorname{Re}(\gamma)>\operatorname{Re}(\beta)>0\right).$$

After short period by using similar method, Minjie defined [5]:

$$F_{A,\left(\{K_l\}_{l\in N_0};p,q\right)}^{(r)}(\alpha,\beta_1,...,\beta_r;\gamma_1,...,\gamma_r;x_1,...,x_r)$$
$$:= \sum_{m_1,...,m_r=0}^{\infty} (\alpha)_{m_1+...+m_r} \prod_{j=1}^{r} \frac{B_{p,q}^{\left(\{K_l\}_{l\in N_0}\right)}(\beta_j+m_j,\gamma_j-\beta_j)}{B(\beta_j,\gamma_j-\beta_j)} \frac{x_1^{m}}{m_1!}\cdots\frac{x_r^{r}}{m_r!} \tag{6}$$

where $|x_1|+...+|x_r|<1$ and $\min\left\{\operatorname{Re}(p),\operatorname{Re}(q)\right\}\geq 0$. If we set $K_l = \frac{(\rho)_l}{(\sigma)_l}$, $\rho=\sigma$, $p=q=0$, then (6) becomes the Lauricella functions defined as follow:

$$F_A^{(r)}(\alpha,\beta_1,...,\beta_r;\gamma_1,...,\gamma_r;x_1,...,x_r)$$
$$:= \sum_{m_1,...,m_r=0}^{\infty} (\alpha)_{m_1+...+m_r} \frac{(\beta_1)_{m_1}\cdots(\beta_r)_{m_r}}{(\gamma_1)_{m_1}\cdots(\gamma_r)_{m_r}} \frac{x_1^{m}}{m_1!}\cdots\frac{x_r^{r}}{m_r!},$$

where $|x_1|+...+|x_r|<1$.

In 1931, J. Horn defined following hypergeometric functions [9, 10]:

$$H_4(\alpha,\beta;\gamma_1,\gamma_2;x,y) := \sum_{m,r=0}^{\infty} \frac{(\alpha)_{2m+r}(\beta)_r}{(\gamma_1)_m(\gamma_2)_r} \frac{x^m}{m!} \frac{y^r}{r!}, \tag{7}$$

$$\left(2\sqrt{|x|}+|y|<1\right).$$

The multivariable fourth kind Horn functions defined by Exton as follow [8, 11 p. 97]:

$$
\begin{aligned}
&{}^{(k)}H_4^{(r)}\left(\alpha,\beta_{k+1},\ldots,\beta_r;\gamma_1,\ldots\gamma_r;x_1,\ldots,x_r\right) \\
&\qquad := \sum_{m_1,\ldots,m_r=0}^{\infty} \frac{(\alpha)_{2(m_1+\ldots+m_k)+m_{k+1}+\ldots+m_r}\left(\beta_{k+1}\right)_{m_{k+1}}\cdots\left(\beta_r\right)_{m_r}}{\left(\gamma_1\right)_{m_1}\cdots\left(\gamma_r\right)_{m_r}} \frac{x_1^{m_1}}{m_1!}\cdots\frac{x_n^{m_r}}{m_r!}, \qquad (8)
\end{aligned}
$$

$\left(2\left(\sqrt{|x_1|}+\ldots+\sqrt{|x_k|}\right)+\ldots+|x_r|<1\right).$

The aim of this article is to define extended multivariable fourth kind Horn functions. We obtain generating functions for these functions. Then, we derive various families of multilinear and multilateral generating functions for the extended multivariable fourth kind Horn functions.

2. GENERATING FUNCTIONS

In this section, firstly, we define extended fourth kind Horn and multivariable fourth kind Horn functions. Then we derived a class of bilateral generating functions for the extended multivariable fourth kind Horn functions.

In a similar way the previous method in introductory section, we can define the following generalizations for the fourth kind Horn and the multivariable fourth kind Horn functions.

Definition 2.1. The extended fourth kind Horn functions is

$$
H_4^{\left(\{K_l\}_{l\in N_0};p,q\right)}\left(\alpha,\beta;\gamma_1,\gamma_2;x,y;p\right) := \sum_{m,r=0}^{\infty} \frac{(\alpha)_{2m+r}}{\left(\gamma_1\right)_m} \frac{B_{p,q}^{\left(\{K_l\}_{l\in N_0}\right)}\left(\beta+r,\gamma_2-\beta\right)}{B\left(\beta,\gamma_2-\beta\right)} \frac{x^m}{m!}\frac{y^r}{r!} \qquad (9)
$$

where $2\sqrt{|x|}+|y|<1$ and $\min\left\{\mathrm{Re}(p),\mathrm{Re}(q)\right\}\geq 0$. When $K_l=\frac{(\rho)_l}{(\sigma)_l}$, $\rho=\sigma$ and $p=q=0$, equation (9) reduces to usual fourth kind Horn functions given by (7).

Definition 2.2. The extended multivariable fourth kind Horn functions is

$$
\begin{aligned}
&{}^{(k)}H_{4,\left(\{K_l\}_{l\in N_0};p,q\right)}^{(r)}\left(\alpha,\beta_{k+1},\ldots,\beta_r;\gamma_1,\ldots\gamma_r;x_1,\ldots,x_r\right) \\
&\qquad := \sum_{m_1,\ldots,m_r=0}^{\infty} \frac{(\alpha)_{2(m_1+\ldots+m_k)+m_{k+1}+\ldots+m_r}\left(\beta_{k+1}\right)_{m_{k+1}}\cdots\left(\beta_r\right)_{m_r}}{\left(\gamma_1\right)_{m_1}\cdots\left(\gamma_r\right)_{m_k}} \qquad (10) \\
&\qquad\qquad \times\prod_{J=k+1}^{r}\frac{B_{p,q}^{\left(\{K_l\}_{l\in N_0}\right)}\left(\beta_j+m_j,\gamma_j-\beta_j\right)}{B\left(\beta_j,\gamma_j-\beta_j\right)}\frac{x_1^{m_1}}{m_1!}\cdots\frac{x_r^{m_r}}{m_r!}
\end{aligned}
$$

where $2\left(\sqrt{|x_1|}+\ldots+\sqrt{|x_k|}\right)+\ldots+|x_r|<1$ and $\min\left\{\mathrm{Re}(p),\mathrm{Re}(q)\right\}\geq 0$. When $K_l=\frac{(\rho)_l}{(\sigma)_l}$ $\rho=\sigma$ and $p=q=0$, equation (10) reduces to the multivariable fourth kind Horn functions given by (8).

Theorem 2.1. We have the following generating function for the extended multivariable fourth kind Horn functions defined by (10):

$$\sum_{n=0}^{\infty}\frac{(\lambda)_n}{n!}\,{}^{(k)}H^{(r)}_{4,\left(\{K_l\}_{l\in N_0};p,q\right)}\left(\lambda+n,\beta_{k+1},...,\beta_r;\gamma_1,...,\gamma_r;x_1,...,x_r\right)t^n$$

$$=(1-t)^{-\lambda}\,{}^{(k)}H^{(r)}_{4,\left(\{K_l\}_{l\in N_0};p,q\right)}\left(\lambda,\beta_{k+1},...,\beta_r;\gamma_1,...,\gamma_r;\frac{x_1}{(1-t)^2},...,\frac{x_k}{(1-t)^2},\frac{x_{k+1}}{(1-t)},...,\frac{x_r}{(1-t)}\right) \tag{11}$$

where $\lambda\in C$ and $|t|<1$.

Proof. Let T denote the first member of assertion (11). Using (10) in (11), we have

$$T=\sum_{n=0}^{\infty}\frac{(\lambda)_n}{n!}\,{}^{(k)}H^{(r)}_{4,\left(\{K_l\}_{l\in N_0};p,q\right)}\left(\lambda+n,\beta_{k+1},...,\beta_r;\gamma_1,...,\gamma_r;x_1,...,x_r\right)t^n$$

$$T=\sum_{n=0}^{\infty}\sum_{m_1,...,m_r=0}^{\infty}(\lambda)_n\frac{(\lambda+n)_{2(m_1+...+m_k)+m_{k+1}+...+m_r}}{(\gamma_1)_{m_1}...(\gamma_k)_{m_k}}\prod_{J=k+1}^{r}\frac{B_{p,q}^{\left(\{K_l\}_{l\in N_0}\right)}\left(\beta_j+m_j,\gamma_j-\beta_j\right)}{B\left(\beta_j,\gamma_j-\beta_j\right)}\frac{x_1^{m_1}}{m_1!}...\frac{x_r^{m_r}}{m_r!}\frac{t^n}{n!}$$

$$=\sum_{m_1,...,m_r=0}^{\infty}\frac{(\lambda)_{2(m_1+...+m_k)+m_{k+1}+...+m_r}}{(\gamma_1)_{m_1}...(\gamma_k)_{m_k}}\prod_{J=k+1}^{r}\frac{B_{p,q}^{\left(\{K_l\}_{l\in N_0}\right)}\left(\beta_j+m_j,\gamma_j-\beta_j\right)}{B\left(\beta_j,\gamma_j-\beta_j\right)}\frac{x_1^{m_1}}{m_1!}...\frac{x_r^{m_r}}{m_r!}$$

$$\times\sum_{n=0}^{\infty}\frac{\left(\lambda+2\left(m_1+...+m_k\right)+m_{k+1}+...+m_r\right)_n}{n!}t^n$$

$$=(1-t)^{-\lambda}\sum_{m_1,...,m_r=0}^{\infty}\frac{(\lambda)_{2(m_1+...+m_k)+m_{k+1}+...+m_r}}{(\gamma_1)_{m_1}...(\gamma_k)_{m_k}}\prod_{J=k+1}^{r}\frac{B_{p,q}^{\left(\{K_l\}_{l\in N_0}\right)}\left(\beta_j+m_j,\gamma_j-\beta_j\right)}{B\left(\beta_j,\gamma_j-\beta_j\right)}$$

$$\times\left(\frac{x_1}{(1-t)^2}\right)^{m_1},...,\left(\frac{x_k}{(1-t)^2}\right)^{m_k},\left(\frac{x_{k+1}}{(1-t)}\right)^{m_{k+1}},...,\left(\frac{x_r}{(1-t)}\right)^{m_r}\frac{1}{m_1!}...\frac{1}{m_r!}$$

$$=(1-t)^{-\lambda}\,{}^{(k)}H^{(r)}_{4,\left(\{K_l\}_{l\in N_0};p,q\right)}\left(\lambda,\beta_{k+1},...,\beta_r;\gamma_1,...,\gamma_r;\frac{x_1}{(1-t)^2},...,\frac{x_k}{(1-t)^2},\frac{x_{k+1}}{(1-t)},...,\frac{x_r}{(1-t)}\right),$$

which completes the proof.

If we set $k=0$ in (10), we clearly see that the extended multivariable fourth kind Horn functions is a generalization of the extended first kind Lauricella functions given by (6):

$$
{}^{(0)}H^{(r)}_{4,\left(\{K_l\}_{l\in N_0};p,q\right)}\left(\begin{matrix}\alpha,\beta_{k+1},...,\beta_r\\ \gamma_1,...\gamma_r\end{matrix};x_1,...,x_r\right)=F^{(r)}_{A,\left(\{K_l\}_{l\in N_0};p,q\right)}\left(\begin{matrix}\alpha,\beta_1,...,\beta_r\\ \gamma_1,...\gamma_r\end{matrix};x_1,...,x_r\right).
$$

Corollary 2.2. In Theorem 2.1, if we take $k=0$, then we have the following generating function for the extended first kind Lauricella functions given by (6),

$$
\sum_{n=0}^{\infty}\frac{(\lambda)_n}{n!}F^{(r)}_{A,\left(\{K_l\}_{l\in N_0};p,q\right)}\left(\lambda+n,\beta_1,...,\beta_r;\gamma_1,...,\gamma_r;x_1,...,x_r\right)t^n
$$

$$
=(1-t)^{-\lambda}F^{(r)}_{A,\left(\{K_l\}_{l\in N_0};p,q\right)}\left(\lambda,\beta_1,...,\beta_r;\gamma_1,...,\gamma_r;\frac{x_1}{(1-t)},...,\frac{x_r}{(1-t)}\right)
$$

where $\lambda\in C$ and $|t|<1$.

If we set $r=2$ Corollary 2.2, we immediately have the following conclusion for the extended second kind Appell functions in [4].

Remark 2.1. We have

$$
\sum_{n=0}^{\infty}\frac{(\lambda)_n}{n!}F_2^{\left(\{K_l\}_{l\in N_0}\right)}\left(\lambda+n,\beta_1,\beta_2;\gamma_1,\gamma_2;x_1,x_2;p,q\right)t^n
$$

$$
=(1-t)^{-\lambda}F_2^{\left(\{K_l\}_{l\in N_0}\right)}\left(\lambda,\beta_1,\beta_2;\gamma_1,\gamma_2;\frac{x_1}{(1-t)},\frac{x_2}{(1-t)};p,q\right)
$$

where

$$
F_2^{\left(\{K_l\}_{l\in N_0}\right)}\left(\alpha,\beta_1,\beta_2;\gamma_1,\gamma_2;x,y;p,q\right)
$$

$$
:=\sum_{m,n=0}^{\infty}(\alpha)_{m+n}\frac{B_{p,q}^{\left(\{K_l\}_{l\in N_0}\right)}\left(\beta_1+m,\gamma_1-\beta_1\right)}{B\left(\beta_1,\gamma_1-\beta_1\right)}\frac{B_{p,q}^{\left(\{K_l\}_{l\in N_0}\right)}\left(\beta_2+n,\gamma_2-\beta_2\right)}{B\left(\beta_2,\gamma_2-\beta_2\right)}\frac{x^m}{m!}\frac{y^n}{n!}
$$

$$
\left(|x|+|y|<1,\ \min\left\{\operatorname{Re}(p),\operatorname{Re}(q)\right\}\geq 0\right).
$$

Theorem 2.3. Following generating function for the extended multivariable fourth kind Horn functions in (10) holds true:

$$
\sum_{n=0}^{\infty}\frac{(\lambda)_n}{n!}\,{}^{(k)}H^{(r)}_{4,\left(\{K_l\}_{l\in N_0};p,q\right)}\left(-n,\beta_{k+1},...,\beta_r;\gamma_1,...,\gamma_r;x_1,...,x_r\right)t^n
$$

$$
=(1-t)^{-\lambda} \tag{12}
$$

$$
\times\,{}^{(k)}H^{(r)}_{4,\left(\{K_l\}_{l\in N_0};p,q\right)}\left(\lambda,\beta_{k+1},...,\beta_r;\gamma_1,...,\gamma_r;\frac{x_1t^2}{(1-t)^2},...,\frac{x_kt^2}{(1-t)^2},\frac{-x_{k+1}t}{(1-t)},...,\frac{-x_rt}{(1-t)}\right)
$$

where $\lambda\in C$ and $|t|<1$.

Proof. Let T denote the first member of assertion (12). Then,

$$T=\sum_{n=0}^{\infty}\sum_{m_1,\dots,m_r=0}^{2(m_1+\dots+m_k)+m_{k+1}+\dots+m_r\le n}(\lambda)_n\frac{(-n)_{2(m_1+\dots+m_k)+m_{k+1}+\dots+m_r}}{(\gamma_1)_{m_1}\dots(\gamma_k)_{m_k}}$$

$$\times\prod_{J=k+1}^{r}\frac{B_{p,q}^{(\{K_l\}_{l\in N_0})}(\beta_j+m_j,\gamma_j-\beta_j)}{B(\beta_j,\gamma_j-\beta_j)}\frac{x_1^{m_1}}{m_1!}\dots\frac{x_r^{m_r}}{m_r!}\frac{t^n}{n!}$$

where we have used [10 p. 102],

$$\sum_{n=0}^{\infty}\sum_{k_1,\dots,k_r=0}^{m_1k_1+\dots+m_rk_r\le n}\Phi(k_1,\dots,k_2;n)=\sum_{n=0}^{\infty}\sum_{k_1,\dots,k_r=0}^{m_1k_1+\dots+m_rk_r\le n}\Phi(k_1,\dots,k_2;n+m_1k_1+\dots+m_rk_r) \tag{13}$$

and we get

$$T=\sum_{m_1,\dots,m_r=0}^{\infty}\frac{1}{(\gamma_1)_{m_1}\dots(\gamma_k)_{m_k}}\prod_{J=k+1}^{r}\frac{B_{p,q}^{(\{K_l\}_{l\in N_0})}(\beta_j+m_j,\gamma_j-\beta_j)}{B(\beta_j,\gamma_j-\beta_j)}$$

$$\times\frac{(x_1t^2)^{m_1}\dots(x_kt^2)^{m_k}(-x_{k+1}t)^{m_{k+1}}\dots(-x_rt)^{m_r}}{m_1!\dots m_k!m_{k+1}!\dots m_r!}\sum_{n=0}^{\infty}\frac{(\lambda)_{n+2(m_1+\dots+m_k)+m_{k+1}+\dots+m_r}}{n!}t^n$$

$$=\sum_{m_1,\dots,m_r=0}^{\infty}\frac{(\lambda)_{2(m_1+\dots+m_k)+m_{k+1}+\dots+m_r}}{(\gamma_1)_{m_1}\dots(\gamma_k)_{m_k}}\prod_{J=k+1}^{r}\frac{B_{p,q}^{(\{K_l\}_{l\in N_0})}(\beta_j+m_j,\gamma_j-\beta_j)}{B(\beta_j,\gamma_j-\beta_j)}$$

$$\times\frac{(x_1t^2)^{m_1}\dots(x_kt^2)^{m_k}(-x_{k+1}t)^{m_{k+1}}\dots(-x_rt)^{m_r}}{m_1!\dots m_k!m_{k+1}!\dots m_r!}$$

$$\times\sum_{n=0}^{\infty}\frac{(\lambda+2(m_1+\dots+m_k)+m_{k+1}+\dots+m_r)_n}{n!}t^n$$

$$=\left(1-t\right)^{-\lambda}\sum_{m_1,\dots,m_r=0}^{\infty}\frac{\left(\lambda\right)_{2\left(m_1+\dots+m_k\right)+m_{k+1}+\dots+m_r}}{\left(\gamma_1\right)_{m_1}\cdots\left(\gamma_k\right)_{m_k}}\prod_{J=k+1}^{r}\frac{B_{p,q}^{\left(\{K_l\}_{l\in N_0}\right)}\left(\beta_j+m_j,\gamma_j-\beta_j\right)}{B\left(\beta_j,\gamma_j-\beta_j\right)}$$

$$\times\frac{\left(\frac{x_1t^2}{\left(1-t\right)^2}\right)^{m_1},\dots,\left(\frac{x_kt^2}{\left(1-t\right)^2}\right)^{m_k},\left(\frac{-x_{k+1}t}{1-t}\right)^{m_{k+1}},\dots,\left(\frac{-x_rt}{1-t}\right)^{m_r}}{m_1!\dots m_r!}$$

$$=\left(1-t\right)^{-\lambda}\,{}^{(k)}H_{4,\left(\{K_l\}_{l\in N_0};p,q\right)}^{(r)}\left(\lambda,\beta_{k+1},\dots,\beta_r;\gamma_1,\dots,\gamma_r;\frac{x_1t^2}{\left(1-t\right)^2},\dots,\frac{x_kt^2}{\left(1-t\right)^2},\frac{-x_{k+1}t}{\left(1-t\right)},\dots,\frac{-x_rt}{\left(1-t\right)}\right),$$

which completes the proof.

Corollary 2.4. In Theorem 2.3, if we take $k=0$, then we have the following generating function for the extended first kind Lauricella functions:

$$\sum_{n=0}^{\infty}\frac{\left(\lambda\right)_n}{n!}F_{A,\left(\{K_l\}_{l\in N_0};p,q\right)}^{(r)}\left(-n,\beta_1,\dots,\beta_r;\gamma_1,\dots,\gamma_r;x_1,\dots,x_r\right)t^n$$

(14)

$$=\left(1-t\right)^{-\lambda}F_{A,\left(\{K_l\}_{l\in N_0};p,q\right)}^{(r)}\left(\lambda,\beta_1,\dots,\beta_r;\gamma_1,\dots,\gamma_r;\frac{-x_1t}{\left(1-t\right)},\dots,\frac{-x_rt}{\left(1-t\right)}\right)$$

where $\lambda\in C$ and $\left|t\right|<1$.

Remark 2.2. In Corollary 2.4, taking $r=2$, following generating function for the extended second kind Appell hypergeometric functions holds true:

$$\sum_{n=0}^{\infty}\frac{\left(\lambda\right)_n}{n!}F_2^{\left(\{K_l\}_{l\in N_0}\right)}\left(-n,\beta_1,\beta_2;\gamma_1,\gamma_2;x_1,x_2;p,q\right)t^n$$

$$=\left(1-t\right)^{-\lambda}F_2^{\left(\{K_l\}_{l\in N_0}\right)}\left(\lambda,\beta_1,\beta_2;\gamma_1,\gamma_2;\frac{-x_1t}{\left(1-t\right)},\frac{-x_2t}{\left(1-t\right)};p,q\right).$$

Theorem 2.5. Following generating function for the extended multivariable fourth kind Horn functions in (10) and $\left|t\right|<1$ holds true:

$$\sum_{n=0}^{\infty}{}^{(k)}H_{4,\left(\{K_l\}_{l\in N_0};p,q\right)}^{(r)}\left(-n,\beta_{k+1},\dots,\beta_r;\gamma_1,\dots,\gamma_r;x_1,\dots,x_r\right)\frac{t^n}{n!}$$

$$=e^t\;{}_0F_1\left(-;\gamma_1;x_1t^2\right)\dots{}_0F_1\left(-;\gamma_k;x_kt^2\right)$$

$$\times\Phi_{p,q}^{\left(\{K_l\}_{l\in N_0}\right)}\left(\beta_{k+1};\gamma_{k+1};-x_{k+1}t\right)\dots\Phi_{p,q}^{\left(\{K_l\}_{l\in N_0}\right)}\left(\beta_r;\gamma_r;-x_rt\right)$$

where ${}_0F_1$ is hypergeometric series and $\Phi_{p,q}^{\left(\{K_l\}_{l\in N_0}\right)}$ is the extended confluent hypergeometric function given by (5).

Proof. Let T denote the first member of assertion of Theorem 2.5. Then,

$$T=\sum_{n=0}^{\infty}\sum_{m_1,\dots,m_r=0}^{2(m_1+\dots+m_k)+m_{k+1}+\dots+m_r\le n}\frac{(-n)_{2(m_1+\dots+m_k)+m_{k+1}+\dots+m_r}}{(\gamma_1)_{m_1}\dots(\gamma_k)_{m_k}}$$

$$\times\prod_{J=k+1}^{r}\frac{B_{p,q}^{\left(\{K_l\}_{l\in N_0}\right)}\left(\beta_j+m_j,\gamma_j-\beta_j\right)}{B\left(\beta_j,\gamma_j-\beta_j\right)}\frac{x_1^{\,m_1}}{m_1!}\dots\frac{x_r^{\,m_r}}{m_r!}\frac{t^n}{n!}$$

where using (13) and we obtain,

$$T=\sum_{m_1,\dots,m_r=0}^{\infty}\frac{1}{(\gamma_1)_{m_1}\dots(\gamma_k)_{m_k}}\prod_{J=k+1}^{r}\frac{B_{p,q}^{\left(\{K_l\}_{l\in N_0}\right)}\left(\beta_j+m_j,\gamma_j-\beta_j\right)}{B\left(\beta_j,\gamma_j-\beta_j\right)}$$

$$\times\frac{\left(x_1t^2\right)^{m_1}\dots\left(x_kt^2\right)^{m_k}\left(-x_{k+1}t\right)^{m_{k+1}}\dots\left(-x_rt\right)^{m_r}}{m_1!\dots m_k!\,m_{k+1}!\dots m_r!}\sum_{n=0}^{\infty}\frac{t^n}{n!}$$

$$=e^t\ {}_0F_1\left(-;\gamma_1;x_1t^2\right)\dots{}_0F_1\left(-;\gamma_k;x_kt^2\right)$$

$$\times\Phi_{p,q}^{\left(\{K_l\}_{l\in N_0}\right)}\left(\beta_{k+1};\gamma_{k+1};-x_{k+1}t\right)\dots\Phi_{p,q}^{\left(\{K_l\}_{l\in N_0}\right)}\left(\beta_r;\gamma_r;-x_rt\right)$$

which completes the proof.

Corollary 2.6. If we take $k=0$, in Theorem 2.5, then we have the following generating function for the extended first kind Lauricella functions:

$$\sum_{n=0}^{\infty}F_{A,\left(\{K_l\}_{l\in N_0};p,q\right)}^{(r)}\left(\begin{matrix}-n,\beta_1,\dots,\beta_r\\ \gamma_1,\dots,\gamma_r\end{matrix};x_1,\dots,x_r\right)\frac{t^n}{n!}$$

$$=e^t\ \Phi_{p,q}^{\left(\{K_l\}_{l\in N_0}\right)}\left(\beta_1;\gamma_1;-x_1t\right)\dots\Phi_{p,q}^{\left(\{K_l\}_{l\in N_0}\right)}\left(\beta_r;\gamma_r;-x_rt\right)$$

where $|t|<1$.

Remark 2.3. Corollary 2.6 for $r=2$ gives following generating function for the extended second kind Appell hypergeometric functions holds true:

$$\sum_{n=0}^{\infty}F_2^{\left(\{K_l\}_{l\in N_0}\right)}\left(-n,\beta_1,\beta_2;\gamma_1,\gamma_2;x_1,x_2;p,q\right)\frac{t^n}{n!}$$

$$=e^t\ \Phi_{p,q}^{\left(\{K_l\}_{l\in N_0}\right)}\left(\beta_1;\gamma_1;-x_1t\right)\Phi_{p,q}^{\left(\{K_l\}_{l\in N_0}\right)}\left(\beta_2;\gamma_2;-x_2t\right)$$

where $\left|t\right|<1$.

Theorem 2.7 We have the following bilateral generating function for the extended multivariable fourth kind Horn functions and the extended first kind Lauricella functions:

$$\sum_{n=0}^{\infty}\frac{(\lambda)_n}{n!}\,{}^{(k)}H^{(r)}_{4,\left(\{K_l\}_{l\in N_0};p,q\right)}\left(\begin{matrix}\lambda+n,\beta_{k+1},...,\beta_r\\ \gamma_1,...,\gamma_r\end{matrix};x_1,...,x_r\right)$$

$$\times F^{(s)}_{A,\left(\{K_l\}_{l\in N_0};p,q\right)}\left(\begin{matrix}-n,\delta_1,...,\delta_s\\ \varepsilon_1,...,\varepsilon_s\end{matrix};y_1,...,y_s\right)t^n \tag{15}$$

$$=(1-t)^{-\lambda}\,{}^{(k)}H^{(r+s)}_{4,\left(\{K_l\}_{l\in N_0};p,q\right)}\left(\begin{matrix}\lambda,\beta_{k+1},...,\beta_r,\delta_1,...,\delta_s;\gamma_1,...,\gamma_r,\varepsilon_1,...,\varepsilon_s\\ \frac{x_1}{(1-t)^2},...,\frac{x_k}{(1-t)^2},\frac{x_{k+1}}{(1-t)},...,\frac{x_r}{(1-t)},\frac{-y_1t}{(1-t)},...,\frac{-y_st}{(1-t)}\end{matrix}\right).$$

Proof. Let T denote the first member of assertion (15). Then,

$$T=\sum_{n=0}^{\infty}\sum_{m_1,...,m_r=0}^{\infty}(\lambda)_n\frac{(\lambda+n)_{2(m_1+...+m_k)+m_{k+1}+...+m_r}}{(\gamma_1)_{m_1}...(\gamma_k)_{m_k}}\prod_{J=k+1}^{r}\frac{B_{p,q}^{\left(\{K_l\}_{l\in N_0}\right)}\left(\beta_j+m_j,\gamma_j-\beta_j\right)}{B\left(\beta_j,\gamma_j-\beta_j\right)}\frac{x_1^{m_1}}{m_1!}...\frac{x_n^{m_r}}{m_r!}$$

$$\times F^{(s)}_{A,\left(\{K_l\}_{l\in N_0};p,q\right)}\left(\begin{matrix}-n,\delta_1,...,\delta_s\\ \varepsilon_1,...,\varepsilon_s\end{matrix};y_1,...,y_s\right)\frac{t^n}{n!}$$

$$=\sum_{m_1,...,m_r=0}^{\infty}\frac{(\lambda)_{2(m_1+...+m_k)+m_{k+1}+...+m_r}}{(\gamma_1)_{m_1}...(\gamma_k)_{m_k}}\prod_{J=k+1}^{r}\frac{B_{p,q}^{\left(\{K_l\}_{l\in N_0}\right)}\left(\beta_j+m_j,\gamma_j-\beta_j\right)}{B\left(\beta_j,\gamma_j-\beta_j\right)}\frac{x_1^{m_1}}{m_1!}...\frac{x_r^{m_r}}{m_r!}$$

$$\times\sum_{n=0}^{\infty}\frac{\left(\lambda+2(m_1+...+m_k)+m_{k+1}+...+m_r\right)_n}{n!}F^{(s)}_{A,\left(\{K_l\}_{l\in N_0};p,q\right)}\left(\begin{matrix}-n,\delta_1,...,\delta_s\\ \varepsilon_1,...,\varepsilon_s\end{matrix};y_1,...,y_s\right)t^n.$$

Using (14) we get,

$$T=\sum_{m_1,...,m_r=0}^{\infty}\frac{(\lambda)_{2(m_1+...+m_k)+m_{k+1}+...+m_r}}{(\gamma_1)_{m_1}...(\gamma_k)_{m_k}}\prod_{J=k+1}^{r}\frac{B_{p,q}^{\left(\{K_l\}_{l\in N_0}\right)}\left(\beta_j+m_j,\gamma_j-\beta_j\right)}{B\left(\beta_j,\gamma_j-\beta_j\right)}\frac{x_1^{m_1}}{m_1!}...\frac{x_r^{m_r}}{m_r!}$$

$$\times(1-t)^{-\lambda-2(m_1+...+m_k)-m_{k+1}-...-m_r}$$

$$\times F^{(s)}_{A,\left(\{K_l\}_{l\in N_0};p,q\right)}\left(\begin{matrix}\lambda+2(m_1+...+m_k)+m_{k+1}+...+m_r,\delta_1,...,\delta_s\\ \varepsilon_1,...,\varepsilon_s\end{matrix};\frac{-y_1t}{1-t},...,\frac{-y_st}{1-t}\right)$$

$$=(1-t)^{-\lambda}\sum_{m_1,\dots,m_r=0}^{\infty}\frac{(\lambda)_{2(m_1+\dots+m_k)+m_{k+1}+\dots+m_r}}{(\gamma_1)_{m_1}\cdots(\gamma_k)_{m_k}}\prod_{J=k+1}^{r}\frac{B_{p,q}^{\left(\{K_l\}_{l\in N_0}\right)}\left(\beta_j+m_j,\gamma_j-\beta_j\right)}{B\left(\beta_j,\gamma_j-\beta_j\right)}$$

$$\times\frac{\left(\frac{x_1}{(1-t)^2}\right)^{m_1},\dots,\left(\frac{x_k}{(1-t)^2}\right)^{m_k},\left(\frac{x_{k+1}}{(1-t)}\right)^{m_{k+1}},\dots,\left(\frac{x_r}{(1-t)}\right)^{m_r}}{m_1!,\dots,m_r!}$$

$$\times\sum_{n_1,\dots,n_s=0}^{\infty}\left(\lambda+2\left(m_1+\dots+m_k\right)+m_{k+1}+\dots+m_r\right)_{n_1+\dots+n_s}$$

$$\times\prod_{J=1}^{s}\frac{B_{p,q}^{\left(\{K_l\}_{l\in N_0}\right)}\left(\delta_j+n_j,\varepsilon_j-\delta_j\right)}{B\left(\delta_j,\varepsilon_j-\delta_j\right)}\frac{\left(\frac{-y_1t}{1-t}\right)^{n_1},\dots,\left(\frac{-y_st}{1-t}\right)^{n_s}}{n_1!\dots n_s!}$$

$$=(1-t)^{-\lambda}\sum_{m_1,\dots,m_r,n_1,\dots,n_s=0}^{\infty}\frac{(\lambda)_{2(m_1+\dots+m_k)+m_{k+1}+\dots+m_r+n_1+\dots+n_s}}{(\gamma_1)_{m_1}\cdots(\gamma_k)_{m_k}}$$

$$\times\prod_{J=k+1}^{r}\frac{B_{p,q}^{\left(\{K_l\}_{l\in N_0}\right)}\left(\beta_j+m_j,\gamma_j-\beta_j\right)}{B\left(\beta_j,\gamma_j-\beta_j\right)}\prod_{J=1}^{s}\frac{B_{p,q}^{\left(\{K_l\}_{l\in N_0}\right)}\left(\delta_j+n_j,\varepsilon_j-\delta_j\right)}{B\left(\delta_j,\varepsilon_j-\delta_j\right)}$$

$$\times\frac{\left(\frac{x_1}{(1-t)^2}\right)^{m_1},\dots,\left(\frac{x_k}{(1-t)^2}\right)^{m_k},\left(\frac{x_{k+1}}{(1-t)}\right)^{m_{k+1}},\dots,\left(\frac{x_r}{(1-t)}\right)^{m_r}\left(\frac{-y_1t}{(1-t)}\right)^{n_1},\dots,\left(\frac{-y_st}{(1-t)}\right)^{n_s}}{m_1!\dots m_r!n_1!\dots n_s!}$$

$$=(1-t)^{-\lambda}\ {}^{(k)}H_{4,\left(\{K_l\}_{l\in N_0};p,q\right)}^{(r)}\left(\begin{array}{c}\lambda,\beta_{k+1},\dots,\beta_r,\delta_1,\dots,\delta_s;\gamma_1,\dots,\gamma_r,\varepsilon_1,\dots,\varepsilon_s\\ \frac{x_1}{(1-t)^2},\dots,\frac{x_k}{(1-t)^2},\frac{x_{k+1}}{(1-t)},\dots,\frac{x_r}{(1-t)},\frac{-y_1t}{(1-t)},\dots,\frac{-y_st}{(1-t)}\end{array}\right)$$

which completes the proof.

If we set $k=0$ in Theorem 2.7, we have the following bilinear generating function for the extended first kind Lauricella functions given by (6).

Corollary 2.8. We have

$$\sum_{n=0}^{\infty}\frac{(\lambda)_n}{n!}F^{(r)}_{A,\left(\{K_l\}_{l\in N_0};p,q\right)}\left(\begin{matrix}\lambda+n,\beta_1,...,\beta_r\\ \gamma_1,...,\gamma_r\end{matrix};x_1,...,x_r\right)F^{(s)}_{A,\left(\{K_l\}_{l\in N_0};p,q\right)}\left(\begin{matrix}-n,\delta_1,...,\delta_s\\ \varepsilon_1,...,\varepsilon_s\end{matrix};y_1,...,y_s\right)t^n$$

$$=(1-t)^{-\lambda}\;F^{(r+s)}_{A,\left(\{K_l\}_{l\in N_0};p,q\right)}\left(\begin{matrix}\lambda,\beta_1,...,\beta_r,\delta_1,...,\delta_s;\gamma_1,...,\gamma_r,\varepsilon_1,...,\varepsilon_s\\ \frac{x_1}{1-t},...,\frac{x_r}{1-t},\frac{-y_1t}{1-t},...,\frac{-y_st}{1-t}\end{matrix}\right).$$

If we set $r=2$ in Corollary 2.8, we have the following bilateral generating function for the extended second kind Appell functions and the extended first kind Lauricella functions.

Remark 2.4. We get

$$\sum_{n=0}^{\infty}\frac{(\lambda)_n}{n!}F_2^{\left(\{K_l\}_{l\in N_0}\right)}\left(\begin{matrix}\lambda+n,\beta_1,\beta_2\\ \gamma_1,\gamma_2\end{matrix};x_1,x_2;p,q\right)F^{(s)}_{A,\left(\{K_l\}_{l\in N_0};p,q\right)}\left(\begin{matrix}-n,\delta_1,...,\delta_s\\ \varepsilon_1,...,\varepsilon_s\end{matrix};y_1,...,y_s\right)t^n$$

$$=(1-t)^{-\lambda}\;F^{(2+s)}_{A,\left(\{K_l\}_{l\in N_0};p,q\right)}\left(\begin{matrix}\lambda,\beta_1,\beta_2,\delta_1,...,\delta_s;\gamma_1,\gamma_2,\varepsilon_1,...,\varepsilon_s\\ \frac{x_1}{1-t},\frac{x_2}{1-t},\frac{-y_1t}{1-t},...,\frac{-y_st}{1-t}\end{matrix}\right).$$

If we choose $k=1$ and $r=2$ in Definition 2.2, we clearly see that extended multivariable fourth kind Horn functions are a generalization of the extended fourth kind Horn functions,

$${}^{(0)}H^{(r)}_{4,\left(\{K_l\}_{l\in N_0};p,q\right)}\left(\begin{matrix}\alpha,\beta_{k+1},...,\beta_r\\ \gamma_1,...\gamma_r\end{matrix};x_1,...,x_r\right)=H_4^{\left(\{K_l\}_{l\in N_0};p,q\right)}\left(\begin{matrix}\alpha,\beta_1\\ \gamma_1,\gamma_2\end{matrix};x_1,x_2\right).$$

Corollary 2.9. If we taking $k=1$ and $r=2$ in Theorem 2.7, bilateral generating function can be obtained for extended fourth kind Horn and extended first kind Lauricella functions,

$$\sum_{n=0}^{\infty}\frac{(\lambda)_n}{n!}H_4^{\left(\{K_l\}_{l\in\mathbb{N}_0};p,q\right)}\left(\begin{matrix}\lambda+n,\beta_1\\ \gamma_1,\gamma_2\end{matrix};x_1,...,x_r\right)F^{(s)}_{A,\left(\{K_l\}_{l\in\mathbb{N}_0};p,q\right)}\left(\begin{matrix}-n,\delta_1,...,\delta_s\\ \varepsilon_1,...,\varepsilon_s\end{matrix};y_1,...,y_s\right)t^n$$

$$=(1-t)^{-\lambda}\;{}^{(1)}H^{(2+s)}_{4,\left(\{K_l\}_{l\in\mathbb{N}_0};p,q\right)}\left(\begin{matrix}\lambda+n,\beta_1,\delta_1,...,\delta_s;\gamma_1,\gamma_2,\varepsilon_1,...,\varepsilon_s\\ \frac{x_1}{(1-t)^2},\frac{x_2}{(1-t)},\frac{-y_1t}{(1-t)},...,\frac{-y_st}{(1-t)}\end{matrix}\right).$$

Theorem 2.10. We have the following bilateral generating function for the extended multivariable fourth kind Horn and the extended first kind Lauricella functions:

$$\sum_{n=0}^{\infty}\frac{(\lambda)_n}{n!}\,{}^{(k)}H^{(r)}_{4,\left(\{K_l\}_{l\in N_0};p,q\right)}\left(\begin{matrix}-n,\beta_{k+1},...,\beta_r\\ \gamma_1,...,\gamma_r\end{matrix};x_1,...,x_r\right)$$

$$\times F^{(s)}_{A,\left(\{K_l\}_{l\in N_0};p,q\right)}\left(\begin{matrix}\lambda+n,\delta_1,...,\delta_s\\ \varepsilon_1,...,\varepsilon_s\end{matrix};y_1,...,y_s\right)t^n$$

$$=(1-t)^{-\lambda}\,{}^{(k)}H^{(r+s)}_{4,\left(\{K_l\}_{l\in N_0};p,q\right)}\left(\begin{matrix}\lambda,\beta_{k+1},...,\beta_r,\delta_1,...,\delta_s;\gamma_1,...,\gamma_r,\varepsilon_1,...,\varepsilon_s\\ \frac{x_1t^2}{(1-t)^2},...,\frac{x_kt^2}{(1-t)^2},\frac{-x_{k+1}t}{(1-t)},...,\frac{-x_rt}{(1-t)},\frac{y_1}{(1-t)},...,\frac{y_s}{(1-t)}\end{matrix}\right).$$

Proof. In a similar manner of proof of Theorem 2.7, we complete the proof.

3. MULTILATERAL AND MULTILINEER GENERATING FUNCTIONS

In this section, we derive several families of multilinear and multilateral generating function for the extended multivariable fourth kind Horn functions defined by (10) by using the similar method considered in [6,7,12,13,14,15].

Theorem 3.1. Corresponding to an identically non-vanishing function $\Omega_\mu(y_1,...,y_s)$ of s complex variables $y_1,...,y_s$ $(s\in N)$ and of complex order μ, let

$$\Lambda_{\mu,\psi}(y_1,...,y_s;\zeta):=\sum_{k=0}^{\infty}a_k\Omega_{\mu+\psi k}(y_1,...,y_s)\zeta^k$$
$$(a_k\neq 0,\ \mu,\psi\in C)$$

and

$$\Theta^{\mu,\psi}_{n,b}(x_1,...,x_r;y_1,...,y_s;\xi):=\sum_{k=0}^{[n/b]}a_k\,\Omega_{\mu+\psi k}(y_1,...,y_s)(\lambda)_{n-bk}$$

$$\times{}^{(k)}H^{(r)}_{4,\left(\{K_l\}_{l\in N_0};p,q\right)}\left(\begin{matrix}\lambda+n-bk,\beta_{k+1},...,\beta_r\\ \gamma_1,...,\gamma_r\end{matrix};x_1,...,x_r\right)\frac{\xi^k}{(n-bk)!}$$

Then, for $b\in N$, we have,

$$\sum_{n=0}^{\infty}\Theta^{\mu,\psi}_{n,b}\left(x_1,...,x_r;y_1,...,y_s;\frac{\eta}{t^b}\right)t^n:=\Lambda_{\mu,\psi}(y_1,...,y_s;\eta)(1-t)^{-\lambda}$$

$$\times{}^{(k)}H^{(r)}_{4,\left(\{K_l\}_{l\in N_0};p,q\right)}\left(\begin{matrix}\lambda,\beta_{k+1},...,\beta_r\\ \gamma_1,...,\gamma_r\end{matrix};\frac{x_1}{(1-t)^2},...,\frac{x_k}{(1-t)^2},\frac{x_{k+1}}{(1-t)},...,\frac{x_r}{(1-t)}\right)\tag{16}$$

provided that each member of (16) exists.

Proof. For convenience, let S denote the first member of the assertion (16). Then,

$$S=\sum_{n=0}^{\infty}\sum_{k=0}^{[n/b]}a_k\,\Omega_{\mu+\psi k}\left(y_1,...,y_s\right)\left(\lambda\right)_{n-bk}$$

$$\times\,{}^{(k)}H^{(r)}_{4,\left(\{K_l\}_{l\in N_0};p,q\right)}\left(\begin{matrix}\lambda+n-bk,\beta_{k+1},...,\beta_r\\ \gamma_1,...,\gamma_r\end{matrix};x_1,...,x_r\right)\frac{\eta^k t^{n-bk}}{(n-bk)!}.$$

Replacing n by $n+bk$, we may write that

$$S=\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}a_k\,\Omega_{\mu+\psi k}\left(y_1,...,y_s\right)\left(\lambda\right)_{n}$$

$$\times\,{}^{(k)}H^{(r)}_{4,\left(\{K_l\}_{l\in N_0};p,q\right)}\left(\begin{matrix}\lambda+n,\beta_{k+1},...,\beta_r\\ \gamma_1,...,\gamma_r\end{matrix};x_1,...,x_r\right)\frac{\eta^k t^{n}}{n!}$$

$$=\sum_{n=0}^{\infty}\left(\lambda\right)_n\,{}^{(k)}H^{(r)}_{4,\left(\{K_l\}_{l\in N_0};p,q\right)}\left(\begin{matrix}\lambda+n,\beta_{k+1},...,\beta_r\\ \gamma_1,...,\gamma_r\end{matrix};x_1,...,x_r\right)\frac{t^{n}}{n!}$$

$$\times\sum_{k=0}^{\infty}a_k\,\Omega_{\mu+\psi k}\left(y_1,...,y_s\right)\eta^k$$

$$=\left(1-t\right)^{-\lambda}\,{}^{(k)}H^{(r)}_{4,\left(\{K_l\}_{l\in N_0};p,q\right)}\left(\begin{matrix}\lambda,\beta_{k+1},...,\beta_r\\ \gamma_1,...,\gamma_r\end{matrix};\frac{x_1}{(1-t)^2},...,\frac{x_k}{(1-t)^2},\frac{x_{k+1}}{(1-t)},...,\frac{x_r}{(1-t)}\right)$$

$$\times\Lambda_{\mu,\psi}\left(y_1,...,y_s;\eta\right)$$

which completes the proof.

In a similar manner, we also get the following results immediately.

Theorem 3.2. Corresponding to an identically non-vanishing function $\Omega_\mu\left(y_1,...,y_s\right)$ of s complex variables $y_1,...,y_s$ $\left(s\in N\right)$ and of complex order μ, let

$$\Lambda_{\mu,\psi}\left(y_1,...,y_s;\zeta\right):=\sum_{k=0}^{\infty}a_k\Omega_{\mu+\psi k}\left(y_1,...,y_s\right)\zeta^k$$

$$\left(a_k\neq 0,\ \mu,\psi\in C\right)$$

and

$$\Theta_{n,b}^{\mu,\psi}\left(x_1,...,x_r;y_1,...,y_s;\xi\right):=\sum_{k=0}^{[n/b]} a_k\,\Omega_{\mu+\psi k}\left(y_1,...,y_s\right)\left(\lambda\right)_{n-bk}$$

$$\times\,^{(k)}H_{4,\left(\{K_l\}_{l\in N_0};p,q\right)}^{(r)}\left(\begin{matrix}-n+bk,\beta_{k+1},...,\beta_r\\ \gamma_1,...,\gamma_r\end{matrix};x_1,...,x_r\right)\frac{\xi^k}{(n-bk)!}$$

Then, for $b\in N$, we have,

$$\sum_{n=0}^{\infty}\Theta_{n,b}^{\mu,\psi}\left(x_1,...,x_r;y_1,...,y_s;\frac{\eta}{t^b}\right)t^n:=\Lambda_{\mu,\psi}\left(y_1,...,y_s;\eta\right)(1-t)^{-\lambda}$$

$$\times\,^{(k)}H_{4,\left(\{K_l\}_{l\in N_0};p,q\right)}^{(r)}\left(\begin{matrix}\lambda,\beta_{k+1},...,\beta_r\\ \gamma_1,...,\gamma_r\end{matrix};\frac{x_1t^2}{(1-t)^2},...,\frac{x_kt^2}{(1-t)^2},\frac{-x_{k+1}t}{(1-t)},...,\frac{-x_rt}{(1-t)}\right) \tag{17}$$

provided that each member of (17) exists.

Teorem 3.3. Corresponding to an identically non-vanishing function $\Omega_\mu\left(y_1,...,y_s\right)$ of s complex variables $y_1,...,y_s$ $\left(s\in N\right)$ and of complex order μ , let

$$\Lambda_{\mu,\psi}\left(y_1,...,y_s;\zeta\right):=\sum_{k=0}^{\infty}a_k\Omega_{\mu+\psi k}\left(y_1,...,y_s\right)\zeta^k$$
$$\left(a_k\neq 0,\ \mu,\psi\in C\right)$$

and

$$\Theta_{n,b}^{\mu,\psi}\left(x_1,...,x_r;y_1,...,y_s;\xi\right):=\sum_{k=0}^{[n/b]} a_k\,\Omega_{\mu+\psi k}\left(y_1,...,y_s\right)$$

$$\times\,^{(k)}H_{4,\left(\{K_l\}_{l\in N_0};p,q\right)}^{(r)}\left(\begin{matrix}-n+bk,\beta_{k+1},...,\beta_r\\ \gamma_1,...,\gamma_r\end{matrix};x_1,...,x_r\right)\frac{\xi^k}{(n-bk)!}$$

Then, for $b\in N$, we have,

$$\sum_{n=0}^{\infty}\Theta_{n,b}^{\mu,\psi}\left(x_1,...,x_r;y_1,...,y_s;\frac{\eta}{t^b}\right)t^n:=\Lambda_{\mu,\psi}\left(y_1,...,y_s;\eta\right)$$

$$\times e^t\,{}_0F_1\left(-;\gamma_1;x_1t^2\right)...{}_0F_1\left(-;\gamma_k;x_kt^2\right) \tag{18}$$

$$\times\Phi_{p,q}^{\left(\{K_l\}_{l\in N_0}\right)}\left(\beta_{k+1},\gamma_{k+1};-x_{k+1}t\right)...\Phi_{p,q}^{\left(\{K_l\}_{l\in N_0}\right)}\left(\beta_r,\gamma_r;-x_rt\right)$$

provided that each member of (18) exists.

Furthermore, for every suitable choice of the coefficients $a_k\left(k \in IN_0\right)$, if the multivariable functions $\Omega_{\mu+\psi k}\left(y_1,...,y_r\right), r \in IN$ are expressed as an appropriate product of several simpler functions, the assertions of Theorems 3.1, 3.2 and 3.3 can be applied in order to derive various families of multilinear and multilateral generating functions for the extended multivariable fourth kind Horn functions defined by (10).

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