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Research Article

Compactness of Fractional Maximal Operator in Weighted and Variable Exponent Spaces

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Abstract

We have studied necessary and sufficiency conditions for the weighted fractional maximal operator to be compactness from $L^{p(.)}(0,l)$ to $L^{q(.)}(0,l)$.

Keywords: Maximal operator, variable exponent, weight functions

Ağırlıklı ve Değişken Üslü Uzayda Kesirli Maksimal operatörünün Kompaktlığı

Öz

$L^{p(.)}(0,l)$'den $L^{q(.)}(0,l)$'e ağırlıklı kesirli maksimal operatörünün kompaktlığı için gerekli ve yeterli şartlar üzerinde çalışıldı.

Anahtar Kelimeler: Maksimal operator, Değişken üslü, Ağırlık fonksiyonları

1. Introduction

In this paper, we will prove compactness of weighted the fractional maximal operator in weighted and variable exponent spaces. Operator theory were worked by very mathematicien (Akin L. 2018; Akin and Zeren, 2017). Recently a considerable number of research has been carried out on the study of generalized Orlicz-Lebesgue spaces $L^{p(x)}$, Orlicz-Sobolev spaces $W^{1,p(x)}$, and the boundedness of different integral operators. In this connection, we refer to the monographs (Cruz-Uribe and Fiorenza, 2013; Diening et al.., 2011; Edmunds et al., 2002; Musielak, 1983; Musielak and Orlicz, 1959). Some necessary and sufficient conditions that assume log-regularity of exponent functions are proved in (Diening and Samko, 2007; Cruz-Uribe and Mamedov, 2012; Mamedov and Harman, 2010; Mamedov and Zeren, 2012; Mamedov and Harman, 2009). The compactness problem of main integral operators in variable exponent Lebesgue spaces are studied in the recent works (Edmunds et al., 2005; Edmunds et al., 1995; Samko, 2010; Mamedov et al., 2017). However, excluding the boundedness the compactness problem was little studied for the fractional maximal operator. We note that, there has been no condition of necessary and sufficient character established for the fractional maximal operator yet. We establish compactness of weighted the fractional maximal operator M_a necessary and sufficient conditions on measurable almost everywhere positive functions (weights) $v(.)$

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and $w(.)$, $0 < a < n$, and an open set $\Omega \subset \mathbb{R}^n$

$$M_{a,v,w}f(x) =$$

$$\sup_{B \ni x} \frac{1}{|B|^{1-a/n}} v(x) \int_{B\cap\Omega} f(y)w(y)dy$$

from $L^{p(x)}(0,l)$ to $L^{q(x)}(0,l)$ with $p(x)$ and $q(x)$ are measurable functions on finite interval $(0,l)$, where the supremum is taken over all balls B which contain x. When $a = 0$, $M_0 = M$ is the Hardy-Littlewood maximal operator.

2. Auxillary Statements and Assertions

Let $p:(0,l) \to (1,\infty)$ be a measurable function on finite interval $(0,l)$ We define the space $L^{p(.)}(0,l)$ consisting of all measurable functions $f:(0,l) \to R$ such that the modular

$$\rho_{p(.)}(f) = \int_0^l |f(x)|^{p(x)} dx$$

is finite.

If $p^+ = ess\sup_{x\in(0,l)} p(x) < \infty$, then

$$\|f\|_{L^{p(.)}(0,l)} = \inf\left\{\lambda > 0 : \rho_{p(.)}\left(\frac{f}{\lambda}\right) < \infty\right\}$$

defines a norm on $L^{p(.)}(0,l)$ (Mamedov *et al*, 2017). Denote by χ_E the characteristic function of the set $E \subset R^n$. The weight functions v and w are assumed to be measurable and having non-negative finite values almost everywhere in $(0,l)$ and assume the following summability conditions

$$\frac{1}{p(x)} + \frac{1}{p'(x)} = 1$$

p' is conjugate of p,

$w(x)^{p'(x)} \in L^1(B(0,a))$,
$v(x)^{q(x)} \in L^1(a,l)$

For any $a > 0$. We use the following notations

$$V(x) = \int_x^l v(y)^{q(y)} dy, \quad W(x) = \int_0^x w(y)^{p'(y)} dy.$$

We use the following general assertions on compact operators.

Theorem 2.1 Mamedov *et al* (2017); Dunford and Schwartz (1958) let $T \in L(X,Y)$ be a compact operator and (u_n) be a weakly sequence in E. The compact $T \in L(X,Y)$ operator every weakly convergent (u_n) sequence converter to strongly convergence sequence.

Theorem 2.2 Rudin (1991) suppose X,Y are Banach space. If $\{T_n\}: X \to Y$ is a sequence of compact operators in $L(X,Y)$ and $\|T_n - T\|_{X\to Y} \to 0$ for some $T \in L(X,Y)$, then T is compact.

We also need for the following corollary.

Corollary 2.3 Okpoti *et al* (2008) for s,α,β positive numbers and f,g measurable functions positive a.e. in $(0,l)$. Denote:

$$F(x) = \int_x^l f(t)dt, \quad G(x) = \int_0^x g(t)dt$$

and

$$B_1(x) = F^{\alpha}(x)G^{\beta}(x),$$

$$B_4(x,s) = \left(\int_0^x f(t)G^{\frac{\beta+s}{\alpha}}(t)dt\right)^{\alpha} G^{-s}(x).$$

Then the numbers

$$B_1 = \sup_{0<x<t} B_1(x)$$

$$B_4 = \sup_{0<x<t} B_4(x)$$

are mutually equivalent. The constants in the equivalence relation can depend on s, α, β.

Let π be the class of measurable functions $f:(0,l) \to R$ that have finite values on $(0,l)$ and let Λ_0 be the class of measurable functions $f:(0,l) \to R$ such that

$$\lim_{x\to 0} \sup \left| f(x) - f(0) \right| \ln \frac{1}{W(x)} < \infty$$

(1)

Then the following result follows from (Mamedov and Zeren, 2012)

Theorem 2.4 Mamedov and Zeren (2012) let $p, q \in \Lambda_0 \cap \pi$ and $f(x) \geq 0$ be a measurable functions such that

$p^- > 1, q(0) \geq p(0)$. $p'(0)$ is

conjugate of $p(0)$

Then the inequality

$$\left\| v(.) H_{v,w} f(.) \right\|_{L^{q(.)}(0,l)} \leq C \left\| f(.) \right\|_{L^{p(.)}(0,l)} \qquad (2)$$

holds if and only if

$$B = \sup_{0<t<l} B(t) = \sup_{0<t<l} V(t)^{\frac{1}{q(0)}} . W(t)^{\frac{1}{p'(0)}} < \infty \qquad (3)$$

Remark 2.5 It is not difficult to see from the proof given in Samko, (2010) that $C \to O(B)$ as $B \to 0$. See inequalities (2), C is constant and see (Diening et al, 2011) for B.

We also need for the following lemma..

Lemma 2.6 Mamedov *et al* (2017). Let the function $W(x)$ be as above. c is a constant. Then it follows that

$$W(t)^{-\frac{1}{p(s)}} \geq \frac{1}{c} . W(t)^{-\frac{1}{p(x)}},$$

for $0 < s < x < t$.

Lemma 2.7 Mamedov *et al* (2017). For the function $W(x)$ and constant c it follows that

$$W(t)^{-\frac{q(x)}{p(x)}} \geq \frac{1}{c} . W(t)^{-\frac{q(0)}{p(0)}},$$

for $0 < x < t < l$.

Lemma 2.8 Mamedov *et al* (2017). For the function $W(x)$ and constant c it follows that

$$W(x)^{q(x)} \geq \frac{1}{c} . W(x)^{q(0)}.$$

3. Main Result

Theorem 3.1. Let $p, q \in \Lambda_0 \cap \pi$ $, B = \sup_{0<x<l} B(x)$ and $f(x) \geq 0$ be a measurable functions such that $p^- > 1$,

$(p^- = ess \inf_{x\in(0,l)} p(x))$,

$q(0) \geq p(0) > 1$. p' is conjugate of p. Then $M_{a,v,w} f(x)$ is compact from $L^{p(.)}(0,l)$ to $L^{q(.)}(0,l)$ if and only if

$$\lim_{t\to 0} \sup V(t)^{1/q(0)} W(t)^{1/p'(0)} = 0 \quad \text{and}$$

$$\sup_{t\in(0,\infty)} V(t)^{1/q(0)} W(t)^{1/p'(0)} < \infty \qquad (4)$$

Proof. Sufficiency. Following Cruz-Uribe and Mamedov (2012), set

$K_1 f(x) =$

$$\chi_{(0,\delta)}(x) \sup_{B\ni x} \frac{1}{|B|^{1-a/n}} v(x) \int_0^x f(t) w(t) dt.$$

$K_2 f(x) =$

$$\chi_{(\delta,1)}(x)\sup_{B\ni x}\frac{1}{|B|^{1-a/n}}v(x)\int_0^{\delta}f(t)w(t)dt.$$

$K_3f(x) =$

$$\chi_{(\delta,1)}(x)\sup_{B\ni x}\frac{1}{|B|^{1-a/n}}v(x)\int_{\delta}^{x}f(t)w(t)dt.$$

$a\in(0,1)$ and $B(t)=V(t)^{\frac{1}{q(0)}}W(t)^{\frac{1}{p'(0)}}$.

Then

$M_{a,v,w}f(x)=\sum_{i=1}^{3}K_if(x)$

$=K_1f(x)+K_2f(x)+K_3f(x)$

Taking into account Lemma 2 from Edmunds et al. (1994) we find that K_3 is a norm limit of a sequence of finite rank operators, while K_2 is a finite rank operator. Now using Theorem 2.4, Remark 2.5 and the condition $\lim_{t\to 0}B(t)=0$ it follows that

$$\|K_1f(x)\|_{q(.),(0,l)} = \left\|\sup_{B\ni x}\frac{1}{|B|^{1-a/n}}v(x)\int_0^{x}f(t)w(t)dt\right\|_{q(.),(0,l)} \le C\|f\|_{q(.),(0,l)}$$

Hence

$$\|M_{a,v,w}f-K_2-K_3\|_{L^{p(.)}\to L^{q(.)}} = \|K_1\|_{L^{p(.)}\to L^{q(.)}} = O\left(\min_{0<t<\delta}B(t)\right)\to 0$$

$as\ t\to 0.$

This completes the proof of the sufficiency part of Theorem 3.1.

Necessity. Let the test function

$$f_t(x)=\left(\int_0^t w(\theta)^{p'(\theta)}\,d\theta\right)^{-\frac{1}{p(s)}}\cdot\chi_{(0,t)}(x)w(x)^{p'(x)-1}$$

into the inequality (2).

It follows that

$$\rho_p(f_t)=\int_0^1\left(\left(\int_0^t w(\theta)^{p'(\theta)}\,d\theta\right)^{-\frac{1}{p(s)}}\cdot\chi_{(0,t)}(x)w(x)^{p'(x)-1}\right)^{p(x)}dx$$

$$\le\left(\int_0^t w(x)^{p'(x)}dx\right)\left(\int_0^t w(\theta)^{p'(\theta)}\,d\theta\right)^{-1}$$

$\le 1.$

Therefore, $\rho_p(f_t)\le 1$. From the elementary properties of the variable exponent norms it follows that Cruz-Uribe and Fiorenza (2013); Diening et al. (2011)

$$\|f_t(x)\|_{L^{p(.)}(0,1)}\le 1.$$

Therefore, by Hölder's inequality we have

$$\left|\int_0^l f_t(x)\varphi(x)dx\right| \le k(p)\|f_t(.)\|_{L^{p(.)}(0,l)}\|\chi_{(0,l)}(.)\varphi(.)\|_{L^{p'(.)}(0,l)}\to 0$$

as $t\to 0$ for all $\varphi\in L^{p'(.)}(0,l)$. Hence f_t converges weakly to 0. Hence, by compactness hypothesis of the operator $M_{a,v,w}f$, and Theorem 2.1 it follows that the sequence $\{M_{a,v,w}f_t\}$ converges to 0 in the norm of $L^{q(.)}(0,l)$. Therefore,

$$\rho_p(M_{a,v,w}f_t) \to 0 \quad as \quad t \to 0 \tag{5}$$

On the other hand,

$$\rho_p(M_{a,v,w}f_t) = \int_0^1 \left(\begin{array}{c} \sup_{B\ni x} \frac{1}{|B|^{(1-\alpha/n)/q(x)}} v(x) \\ \cdot \int_0^x \left(\int_0^t w(\theta)^{p'(\theta)}\, d\theta\right)^{-\frac{1}{p(s)}} \\ \cdot \chi_{(0,t)}(s) w(s)^{p'(s)-1} w(s) ds \end{array} \right)^{q(x)} dx$$

$$\geq \int_0^1 \sup_{B\ni x} \frac{1}{|B|^{1-\alpha/n}} v(x)^{q(x)} \cdot \left(\int_0^x \left(\int_0^t w(\theta)^{p'(\theta)}\, d\theta \right)^{-\frac{1}{p(s)}} \cdot w(s)^{p'(s)} ds \right)^{q(x)} dx \tag{6}$$

(by Lemma 2.6),

$$\geq (2C)^{-q^+} \int_0^t V(x)^{q(x)} W(x)^{q(x)} W(t)^{-\frac{q(x)}{p(x)}} dx$$

(by Lemma 2.7),

$$\geq \frac{2^{-q^+}}{C_1} W(t)^{-\frac{q(0)}{p(0)}} \int_0^t V(x)^{q(x)} W(x)^{q(x)} dx$$

(by Lemma 2.8),

$$\geq \frac{1}{C_2} \left[W(t)^{-\frac{1}{p(0)}} \left(\int_0^t V(x)^{q(x)} W(x)^{q(0)} dx \right)^{\frac{1}{q(0)}} \right]^{q(0)}$$

(by Corollary 2.3),

$$\geq C_3 \left[V(t)^{\frac{1}{q(0)}} W(t)^{\frac{1}{p'(0)}} \right]^{q(0)}$$

from this inequality and (5) it follows that

$$V(t)^{\frac{1}{q(0)}} W(t)^{\frac{1}{p'(0)}} \to 0 \text{ as } t \to 0 \ .$$

The necessity of Theorem 3.1 has been proved.

Notice, in the proof of inequality (6), we have applied Corollary 2.3 under the settings:

$$F(t) = V(t) = \int_t^1 v(x)^{q(x)} dx\,,$$

$$G(t) = W(t) = \int_0^t w(x)^{p'(x)} dx$$

and

$$\alpha = \frac{1}{q(0)},\ \beta = \frac{1}{p'(0)},\ s = \frac{1}{p(0)},\ f(t) = v(x)^{q(x)},\ g(t) = w(x)^{p'(x)}.$$

Then

$$\left(\int_0^t f(x) G(x)^{\frac{\beta+s}{\alpha}} dx \right)^{\alpha} G(t)^{-s} = \left(\int_0^t v(x)^{q(x)} W(x)^{q(0)} dx \right)^{\frac{1}{q(0)}} W(t)^{-\frac{1}{p(0)}}$$

(by Corollary 2.3),

$$\geq C F(t)^{\alpha} G(t)^{\beta} = C V(t)^{\frac{1}{q(0)}} W(t)^{\frac{1}{p'(0)}}$$

This completes the proof of Theorem 3.1.

Conclusion. We have shown the necessary and sufficient conditions for the compactness of thefractional maximal operator in the weighted and variable exponential space.

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