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Area of a Triangle in Terms of the m-Generalized Taxicab Distance

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Abstract

In this paper, we give three area formulas for a triangle in the m-generalized taxicab plane in terms of the m-generalized taxicab distance. The two of them are m-generalized taxicab versions of the standard area formula for a triangle, and the other one is an m-generalized taxicab version of the well-known Heron's formula.

Keywords: *Taxicab distance, m-generalized taxicab distance, area, Heron's formula.*
2010 Mathematics Subject Classification: *51K05, 51K99, 51N20.*

1. Introduction

Taxicab geometry was introduced by Menger [11], and developed by Krause [10], using the taxicab metric which is the special case of the well-known l_p-metric (also known as Minkowski distance) for $p=1$. In this geometry, circles are squares with each diagonal is parallel to a coordinate axis. Afterwards, in [15] Lawrance J. Wallen defined *the* (*slightly*) *generalized taxicab metric*, in which circles are rhombuses with each diagonal is also parallel to a coordinate axis. Finally, *m-generalized taxicab metric* is defined in [3], for any rhombus (so, any square) to be a circle instead of rhombuses having each diagonal parallel to a coordinate axis. In the last case, for any real number m and positive real numbers u and v, *the m-generalized taxicab distance* between points $P_1=(x_1,y_1)$ and $P_2=(x_2,y_2)$ in $\mathbb{R}^2$ is defined by

$$d_{T_g(m)}(P_1,P_2)=\left(u\left|(x_1-x_2)+m(y_1-y_2)\right|+v\left|m(x_1-x_2)-(y_1-y_2)\right|\right)/(1+m^2)^{1/2}. \tag{1.1}$$

In addition, as a special case of $d_{T_g(m)}$ for $u=v=1$,

$$d_{T(m)}(P_1,P_2)=\left(\left|(x_1-x_2)+m(y_1-y_2)\right|+\left|m(x_1-x_2)-(y_1-y_2)\right|\right)/(1+m^2)^{1/2} \tag{1.2}$$

is called the *m-taxicab distance* between points P_1 and P_2, while the well-known Euclidean distance between P_1 and P_2 is

$$d_E(P_1,P_2)=[(x_1-x_2)^2+(y_1-y_2)^2]^{1/2}. \tag{1.3}$$

The m-generalized taxicab unit circle is a rhombus with diagonals having slopes of m and $-1/m$, and with vertices $A_1=\left(\frac{1}{uk},\frac{m}{uk}\right)$, $A_2=\left(\frac{-m}{vk},\frac{1}{vk}\right)$, $A_3=\left(\frac{-1}{uk},\frac{-m}{uk}\right)$ and $A_4=\left(\frac{m}{vk},\frac{-1}{vk}\right)$, where $k=(1+m^2)^{1/2}$; if $u=v$, then m-generalized taxicab unit circle is a square with vertices A_1,A_2,A_3 and A_4. The m-generalized taxicab distance between two points is invariant under all translations. In addition, if $u\neq v$, then the m-generalized taxicab distance between two points is invariant under rotations of π radian around a point and reflections in lines parallel to the lines with slope m and $\frac{-1}{m}$; if $u=v$, then rotations of $\pi/2$, π and $3\pi/2$ radians around a point, and reflections in lines parallel to the lines with slope m, $\frac{-1}{m}$, $\frac{1+m}{1-m}$ or $\frac{m-1}{1+m}$ (see [3], [4] and [6]).

Since the distance function is different from that of Euclidean geometry, it is interesting to study the m-generalized taxicab analogues of topics that include the distance concept in Euclidean geometry. In this paper, we give area formulas for a triangle in the m-generalized taxicab plane in terms of the m-generalized taxicab distance. One can see from Figure 1 that there are triangles whose m-generalized taxicab lengths of corresponding sides are the same, while areas of these triangles are different, in the m-generalized taxicab plane. So, how can one compute the area of a triangle in the m-generalized taxicab plane? In this study, we present three formulas to compute the area of a triangle in the m-generalized taxicab plane. Henceforth, we use $u'=u/(1+m^2)^{1/2}$ and $v'=v/(1+m^2)^{1/2}$ to shorten phrases.

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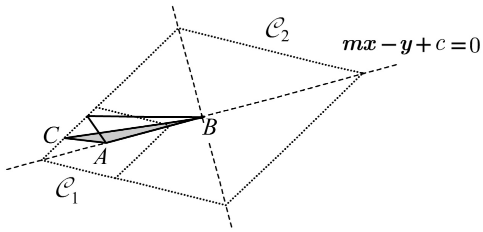

Figure 1. Let A and B be two distinct points on a line parallel to $mx-y=0$. Let $\mathscr{C}_1$ and $\mathscr{C}_2$ be m-generalized taxicab circles with center A and B, radius b and $b+c$, respectively. As point $C\in\mathscr{C}_1\cap\mathscr{C}_2$ changes, the area of triangle ABC also changes, while $d_{T_g(m)}(B,C)$, $d_{T_g(m)}(A,C)$ and $d_{T_g(m)}(A,B)$ are invariant.

2. The m-generalized taxicab version of standard area formula

It is well-known that the standard area formula for triangle ABC is $\mathscr{A}=\mathbf{ah}/2$, where $\mathbf{a}=d_E(B,C)$ and $\mathbf{h}=d_E(A,BC)$ or $\mathbf{h}=d_E(A,H)$ where H is the orthogonal projection of the point A on the line BC. Here, we give two m-generalized taxicab versions of this formula in terms of the m-generalized taxicab distance, depending on choice of $h=d_{T_g(m)}(A,H)$ or $h'=d_{T_g(m)}(A,BC)$. The following equation given in [3], which relates the Euclidean distance to the m-generalized taxicab distance between two points in the Cartesian coordinate plane, plays an important role in the first m-generalized taxicab version of the area formula.

Proposition 2.1. *For any two points A and B in $\mathbb{R}^2$ that do not lie on a vertical line, if n is the slope of the line through A and B, then*

$$d_E(A,B)=\mu(n)d_{T_g(m)}(A,B) \tag{2.1}$$

where $\mu(n)=(1+n^2)^{1/2}/(u'\,|1+mn|+v'\,|m-n|)$. *If A and B lie on a vertical line, then*

$$d_E(A,B)=[1/(u'\,|m|+v')]d_{T_g(m)}(A,B). \tag{2.2}$$

Notice that $\mu(m)=\frac{1}{u}$ and if $m\neq 0$, then $\mu(-1/m)=\frac{1}{v}$. Therefore, if l_A is the line through A with slope m, and l_B is the line through B and perpendicular to the line l_A, then

$$d_{T_g(m)}(A,B)=ud_E(A,l_B)+vd_E(B,l_A).$$

In addition, for any non-zero real number n, if $u=v$ then $\mu(n)=\mu(-1/n)$.

The following theorem gives the first m-generalized taxicab version of the standard area formula of a triangle.

Theorem 2.1. *Let ABC be a triangle with area $\mathscr{A}$ in the m-generalized taxicab plane, let H be orthogonal projection of the point A on the line BC, let n be the slope of the line BC, and let $a=d_{T_g(m)}(B,C)$ and $h=d_{T_g(m)}(A,H)$.*

(i) *If BC is parallel to a coordinate axis, then*

$$\mathscr{A}=ah/2(u'\,|m|+v')(u'+v'\,|m|). \tag{2.3}$$

(ii) *If BC is not parallel to any coordinate axis, then*

$$\mathscr{A}=[\mu(n)\mu(-1/n)]\,ah/2. \tag{2.4}$$

Proof. Let $\mathbf{a}=d_E(B,C)$ and $\mathbf{h}=d_E(A,H)$. Then, $\mathscr{A}=\mathbf{ah}/2$.
(i) If BC is parallel to x-axis, then AH is parallel to y-axis and

$$\mathbf{a}=[1/(u'+v'\,|m|)]a \text{ and } \mathbf{h}=[1/(u'\,|m|+v')]h.$$

If BC is parallel to y-axis, then AH is parallel to x-axis and

$$\mathbf{a}=[1/(u'\,|m|+v')]a \text{ and } \mathbf{h}=[1/(u'+v'\,|m|)]h.$$

Hence, we get

$$\mathscr{A}=ah/2(u'\,|m|+v')(u'+v'\,|m|).$$

(ii) Let BC not be parallel to any coordinate axis, and let n be the slope of the line BC. Then, the slope of the line AH is $(-1/n)$. Therefore $\mathbf{a}=\mu(n)a$ and $\mathbf{h}=\mu(-1/n)h$, hence

$$\mathscr{A}=[\mu(n)\mu(-1/n)]\,ah/2.$$

□

In the m-generalized taxicab plane, m-generalized taxicab distance from a point P to a line l is naturally defined by

$$d_{T_g(m)}(P,l)=\min_{Q\in l}\{d_{T_g(m)}(P,Q)\}. \tag{2.5}$$

In the following proposition, we give a formula for $d_{T_g(m)}(P,l)$, similar to the Euclidean geometry.

Proposition 2.2. *Given a point $P=(x_0,y_0)$ and a line $l: ax+by+c=0$ in the m-generalized taxicab plane. The m-generalized taxicab distance from the point P to the line l can be calculated by the following formula:*

$$d_{T_g(m)}(P,l)=(1+m^2)^{1/2}\,|ax_0+by_0+c|\,/\max\left\{\tfrac{|a+bm|}{u},\tfrac{|am-b|}{v}\right\}. \tag{2.6}$$

Proof. It is clear that if P is on line l, then equation holds. Let P not be on line l. To find the minimum m-generalized taxicab distance from the point P which is off the line l, let us define *tangent line* to an m-generalized taxicab circle with center P and radius r, as *a line whose m-generalized taxicab distance from P is equal to* r, being natural analogue to the Euclidean geometry. Then, we expand an m-generalized taxicab circle with center P until the line l becomes a tangent to the m-generalized taxicab circle (see Figure 2). It is clear to see that a line can only be a tangent to an m-generalized taxicab circle at one vertex or two vertices (that is, at one edge). Since corresponding vertices of expanding m-generalized taxicab circle are on line through P and parallel to line $mx-y=0$ or $x+my=0$, if l is a tangent to the m-generalized taxicab circle with center P, then $P_1=\left(\frac{bmx_0-by_0-c}{a+bm},\frac{-amx_0+ay_0-cm}{a+bm}\right)$ or $P_2=\left(\frac{bx_0+bmy_0+cm}{b-am},\frac{-ax_0-amy_0-c}{b-am}\right)$ is a tangent point, which are intersection points of the line l and $mx-y=0$ or $x+my=0$, respectively (see Figure 2). Therefore, $d_{T_g(m)}(P,l)=\min\{d_{T_g(m)}(P,P_1),d_{T_g(m)}(P,P_2)\}$.

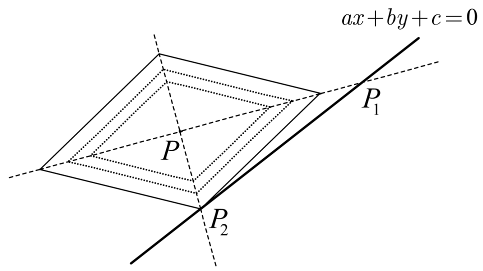

Figure 2

Since $d_{T_g(m)}(P,P_1)=\frac{(1+m^2)^{1/2}|ax_0+by_0+c|}{|a+bm|/u}$ and $d_{T_g(m)}(P,P_2)=\frac{(1+m^2)^{1/2}|ax_0+by_0+c|}{|am-b|/v}$, one gets

$$d_{T_g(m)}(P,l)=(1+m^2)^{1/2}\,|ax_0+by_0+c|\,/\max\left\{\tfrac{|a+bm|}{u},\tfrac{|am-b|}{v}\right\}.$$

□

The following equation, which relates the Euclidean distance to the m-generalized taxicab distance from a point to a line in the Cartesian coordinate plane, plays an important role in the second m-generalized taxicab version of the area formula.

Proposition 2.3. *Given a point P and a line l which is not vertical in the Cartesian plane, if n is the slope of the line l, then*

$$d_E(P,l)=\tau(n)d_{T_g(m)}(P,l) \tag{2.7}$$

where $\tau(n)=\max\left\{\frac{|m-n|}{u},\frac{|mn+1|}{v}\right\}/\left[(1+n^2)(1+m^2)\right]^{1/2}$. *If l is vertical, then* $d_E(P,l)=\left[\max\left\{\frac{1}{u},\frac{|m|}{v}\right\}/(1+m^2)^{1/2}\right]d_{T_g(m)}(P,l)$.

Proof. Let $P=(x_0,y_0)$ be a point, and $l: ax+by+c=0$ be a line with slope of n, in the Cartesian plane. If l is not a vertical line, then $b\neq 0$ and $n=-\frac{a}{b}$. Then, one gets

$$d_E(P,l)=|ax_0+by_0+c|\,/\,|b|\,(1+n^2)^{1/2} \text{ and } d_{T_g(m)}(P,l)=(1+m^2)^{1/2}\,|ax_0+by_0+c|\,/\,|b|\max\left\{\tfrac{|m-n|}{u},\tfrac{|mn+1|}{v}\right\}.$$

Therefore, $d_E(P,l)=\tau(n)d_{T_g(m)}(P,l)$ where $\tau(n)=\max\left\{\frac{|m-n|}{u},\frac{|mn+1|}{v}\right\}/\left[(1+n^2)(1+m^2)\right]^{1/2}$. If l is a vertical line, then $b=0$ and $a\neq 0$. Therefore, one gets that

$$d_E(P,l)=|ax_0+c|\,/\,|a| \text{ and } d_{T_g(m)}(P,l)=(1+m^2)^{1/2}\,|ax_0+c|\,/\,|a|\max\left\{\tfrac{1}{u},\tfrac{|m|}{v}\right\}.$$

Hence one has

$$d_E(P,l)=\left[\max\left\{\tfrac{1}{u},\tfrac{|m|}{v}\right\}/(1+m^2)^{1/2}\right]d_{T_g(m)}(P,l).$$

□

Notice that $\tau(m)=\frac{1}{v}$, and if $m\neq 0$, then $\tau(-\frac{1}{m})=\frac{1}{u}$. The following theorem gives another m-generalized taxicab version of the standard area formula of a triangle:

Theorem 2.2. *Let ABC be a triangle with area $\mathscr{A}$ in the m-generalized taxicab plane, n be the slope of the line BC, and let $a = d_{T_g(m)}(B,C)$ and $h' = d_{T_g(m)}(A,BC)$. Then*

$$\mathscr{A} = \frac{\max\left\{\frac{|m-n|}{u}, \frac{|mn+1|}{v}\right\} ah'}{2(u|mn+1| + v|m-n|)}. \tag{2.8}$$

If BC is vertical, then

$$\mathscr{A} = \frac{\max\left\{\frac{1}{u}, \frac{|m|}{v}\right\} ah'}{2(u|m| + v)}. \tag{2.9}$$

Proof. Let $\mathbf{a} = d_E(B,C)$ and $\mathbf{h} = d_E(A,BC)$. Then, $\mathscr{A} = \mathbf{ah}/2$. Let BC not be vertical, and n be the slope of the line BC. By Proposition 2.1 and Proposition 2.3, $\mathbf{a} = \mu(n)a$ and $\mathbf{h} = \tau(n)h'$, hence one has

$$\mathscr{A} = [\mu(n)\tau(n)]\, ah'/2 = \max\left\{\frac{|m-n|}{u}, \frac{|mn+1|}{v}\right\} ah'/2(u|mn+1| + v|m-n|).$$

If BC is vertical, then $\mathbf{a} = [1/(u'|m| + v')]a$ and $\mathbf{h} = [\max\left\{\frac{1}{u}, \frac{|m|}{v}\right\}/(1+m^2)^{1/2}]h'$. Hence, one has

$$\mathscr{A} = \max\left\{\frac{1}{u}, \frac{|m|}{v}\right\} ah'/2(u|m| + v).$$

□

The following corollary follows from Theorem 2.1 and Theorem 2.2.

Corollary 2.1. *Let ABC be a triangle with area $\mathscr{A}$ in the m-generalized taxicab plane, and let $a = d_{T_g(m)}(B,C)$, $h = d_{T_g(m)}(A,H)$, and $h' = d_{T_g(m)}(A,BC)$. If BC is parallel to $mx - y = 0$ or $x + my = 0$, then $h = h'$ and $\mathscr{A} = ah/2uv$.*

Proof. If BC is parallel to $mx - y = 0$ or $x + my = 0$, then $n = m$ and $n = -1/m$, respectively, and Equation (2.4) and Equation (2.8) gives $\mathscr{A} = ah/2uv = ah'/2uv$, so $h = h'$. □

3. The *m*-generalized taxicab version of Heron's formula

It is well-known that if ABC is a triangle with the area $\mathscr{A}$ in the Euclidean plane, and $\mathbf{a} = d_E(B,C)$, $\mathbf{b} = d_E(A,C)$, $\mathbf{c} = d_E(A,B)$, and $\mathbf{p} = (\mathbf{a} + \mathbf{b} + \mathbf{c})/2$, then

$$\mathscr{A} = [\mathbf{p}(\mathbf{p}-\mathbf{a})(\mathbf{p}-\mathbf{b})(\mathbf{p}-\mathbf{c})]^{1/2},$$

which is known as *Heron's formula*. In this section, we give an *m*-generalized taxicab version of this formula in terms of *m*-generalized taxicab distance, similar to the one given in [14]. We need following modified definitions given in [14] to give an *m*-generalized taxicab version of Heron's formula:

Definition 3.1. *Let ABC be any triangle in the m-generalized taxicab plane. Clearly, there exists a pair of lines passing through every vertex of the triangle, each of which is parallel to lines $mx - y = 0$ or $x + my = 0$. A line l is called* ***m-base line*** *of ABC if and only if*
(1) *l passes through a vertex,*
(2) *l is parallel to lines $mx - y = 0$ or $x + my = 0$,*
(3) *l intersects the opposite side (as a line segment) of the vertex in* (1).

Clearly, at least one of vertices of the triangle always has one or two m-base lines. Such a vertex of the triangle is called an ***m-basic vertex****. An* ***m-base segment*** *is a line segment on an m-base line, which is bounded by an m-basic vertex and its opposite side.*

Now, we give the *m*-generalized taxicab version of Heron's formula:

Theorem 3.2. *Let ABC be a triangle, and $a = d_{T_g(m)}(B,C)$, $b = d_{T_g(m)}(A,C)$, $c = d_{T_g(m)}(A,B)$, $p = (a+b+c)/2$, and let α denote the m-generalized taxicab length of a m-base segment of the triangle. Then the area $\mathscr{A}$ of the triangle is*

$$\mathscr{A} = \begin{cases} \frac{1}{2uv}\alpha(p - (\alpha + \alpha')) & \text{, if there exists only one m-base line passing through the m-basic vertex} \\ \frac{1}{2uv}\alpha(p - (\alpha + \alpha' + \alpha'')) & \text{, if there exist two m-base lines passing through the m-basic vertex} \end{cases} \tag{3.1}$$

where $\alpha' = d_{T_g(m)}(D,H)$, $\alpha'' = d_{T_g(m)}(\text{basic vertex}, H')$,
D is intersection point of the m-base line and the opposite side,
H is point of orthogonal projection of one of the remaining two vertices on the m-base line which is an endpoint of the m-base segment or not on the m-base segment,
H' is point of orthogonal projection of the third vertex on the same m-base line which is an endpoint of the m-base segment or not on the m-base segment.

Proof. Let ABC be a triangle with m-basic vertex C, without loss of generality. Let H'' be the point of orthogonal projection of one of the remaining two vertices which is on the m-base segment. Two cases are:

(i) Let ABC has only one m-base line passing through C. Figure 3 and Figure 4 represent all such triangles. Let $h = d_{T_g(m)}(A,H)$, $h' = d_{T_g(m)}(B,H'')$, $c_A = d_{T_g(m)}(A,D)$, and $c_B = d_{T_g(m)}(B,D)$. Since $c_A + \alpha = b$ and $c_B + a = \alpha + 2h'$, one gets $h' = p - b$. We also have $h = b - (\alpha + \alpha')$. Therefore, $h + h' = p - (\alpha + \alpha')$. Besides, $\mathscr{A} = \frac{1}{2uv}\alpha(h+h')$ by Corollary 2.1. Hence, $\mathscr{A} = \frac{1}{2uv}\alpha(p - (\alpha + \alpha'))$.

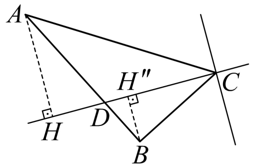

Figure 3

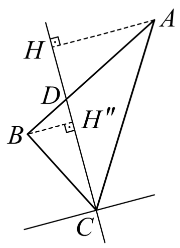

Figure 4

(ii) Let ABC has two m-base lines passing through C. Figure 5 represents all such triangles. Choose an m-base line to determine the point D. Let $h = d_{T_g(m)}(B,H)$ and $h' = d_{T_g(m)}(A,H')$. Since $a = h + \alpha + \alpha'$, $b = h' + \alpha''$, and $a + b = c$ one gets $h + h' = a + b - (\alpha + \alpha' + \alpha'') = p - (\alpha + \alpha' + \alpha'')$. Besides, $\mathscr{A} = \frac{1}{2uv}\alpha(h+h')$ by Corollary 2.1. Hence, $\mathscr{A} = \frac{1}{2uv}\alpha(p - (\alpha + \alpha' + \alpha''))$.

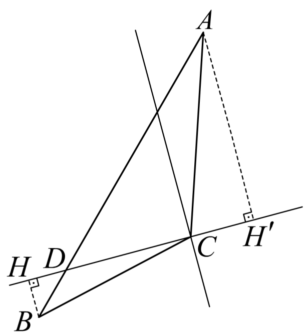

Figure 5

□

The following two corollaries give the m-generalized taxicab versions of Heron's formula for some special cases:

Corollary 3.1. *If one side of a triangle ABC, say BC, is parallel to one of lines $mx - y = 0$ or $x + my = 0$ and none of the angles B and C is an obtuse angle, then for the area $\mathscr{A}$ of ABC,*

$$\mathscr{A} = \frac{1}{2uv}a(p-a). \tag{3.2}$$

Proof. Let ABC be a triangle with BC is parallel to one of lines $mx - y = 0$ or $x + my = 0$ and none of the angles B and C is an obtuse angle. Then, there is only one m-base line passing through B or C, so B and C are m-basic vertices and BC is the m-base segment. Then, $\alpha = a$, $\alpha' = 0$, hence we have $\mathscr{A} = \frac{1}{2uv}a(p-a)$. □

Corollary 3.2. *If one side of a triangle ABC, say BC, is parallel to one of lines $mx - y = 0$ or $x + my = 0$ and one of the angles B and C is not an acute angle, then for the area $\mathscr{A}$ of ABC,*

$$\mathscr{A} = \frac{1}{2uv}a(p-(a+\alpha'')) \tag{3.3}$$

where $\alpha'' = d_{T_g(m)}(\text{basic vertex}, H')$ and H' is the point of orthogonal projection of A on the same m-base line which is an endpoint of the m-base segment or not on the m -base segment.

Proof. Let ABC be a triangle with BC is parallel to one of lines $mx - y = 0$ or $x + my = 0$ and one of the angles B and C, let us say C, is not an acute angle. Then, there are two m-base lines passing through C, so C is m-basic vertex and BC is an m-base segment. Then, $\alpha = a$, $\alpha' = 0$, hence we have $\mathscr{A} = \frac{1}{2uv}a(p-(a+\alpha''))$. □

Note that since the generalized taxicab and so the taxicab distances are special cases of the m-generalized taxicab distance, conclusions given here are also true for the generalized taxicab and so the taxicab geometry.

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