ON A GRAPH OF IDEALS OF A COMMUTATIVE RING

S. EBRAHIMI ATANI, S. DOLATI PISHHESARI, AND M. KHORAMDEL

ABSTRACT. In this paper, we introduce and investigate a new graph of a commutative ring R, denoted by $G(R)$, with all nontrivial ideals of R as vertices, and two distinct vertices I and J are adjacent if and only if $ann(I \cap J) = ann(I) + ann(J)$. In this article, the basic properties and possible structures of the graph $G(R)$ are studied and investigated as diameter, girth, clique number, cut vertex and domination number. We characterize all rings R for which $G(R)$ is planar, complete and complete r-partite. We show that, if (R, M) is a local Artinian ring, then $G(R)$ is complete if and only if $Soc(R)$ is simple. Also, it is shown that if R is a ring with $G(R)$ is r-regular, then either $G(R)$ is complete or null graph. Moreover, we show that if R is an Artinian ring, then R is a serial ring if and only if $G(R/I)$ is complete for each ideal I of R.

1. INTRODUCTION

Over the last years, there has been an explosion of interest in associating a graph to an algebraic structure. In 1988, Istvan Beck proposed the study of commutative rings by representing them as graphs [6]. Since then a huge number of works have been added to the literature, see for instance [2, 3, 4, 10, 11, 12, 13, 14, 15, 16, 17, 18]. Most properties of a ring are connected to a behavior of its ideals. Besides, ideals play crucial roles in the study of ring constructions. This is why it is interesting and useful to associate graphs to ideals of a ring, as for example in [1, 10, 24]. The benefit of studying these graphs is that one may find some results about the algebraic structures.

A well known result of Ikeda and Nakayama [21] explores that if a (not necessarily commutative) ring R is right self-injective, then $ann_l(I \cap J) = ann_l(I) + ann_l(J)$ for all right ideals I and J of R, where $ann_l(X)$ denotes the left annihilator of X. The study of rings that satisfying the aforementioned property has been initiated by Camillo, Nicholson and Yousif in [9] and called Ikeda-Nakayama rings.

In this paper, we introduce and investigate a new graph in a commutative ring R, in order to know when $ann(I \cap J) = ann(I) + ann(J)$ for ideals I and J of R.

Our main goal is to study the connection between the algebraic properties of a ring and the graph theoretic properties of the graph associated to it. We associate a graph $G(R)$ to a commutative ring R whose vertices are nonzero proper ideals of R and two vertices I and J are adjacent if and only if $ann(I \cap J) = ann(I) + ann(J)$.

We summarize the contents of this article as follows. In section 2, we show that $G(R)$ is a connected graph with $diam(G(R)) \in \{0, 1, 2\}$ and $gr(G(R)) \in \{3, \infty\}$. Also, we show that what happen for R and $G(R)$, if $gr(G(R)) = \infty$. In this section it is shown that $G(R)$ is a star graph if and only if $G(R)$ is a bipartite graph if and only if $G(R)$ contains a cut vertex. In section 3, we investigate the planar property, complete or complete r-partite property of $G(R)$. In this section maximal ideals and socle are useful instrument which help us to do our study. We prove that, if (R, M) is a local Artinian ring, then the $G(R)$ is complete if and only if $Soc(R)$ is simple. It is shown that if R is a ring with $G(R)$ is r-regular, then either $G(R)$ is complete or null graph. Moreover, we show that if R is an Artinian ring, then R is a serial ring if and only if $G(R/I)$ is complete for each ideal I of R. A complete characterizations of rings for which $G(R)$ is planar or complete r-partite are provided. It is proved that, if R is a ring, then $G(R)$ is planar if and only if one of the following holds:

(1) $R \cong F \times S$, where F is a field and (S, M) is a local ring with the only non-zero proper ideal M or $R \cong F_1 \times F_2$, where F_1, F_2 are fields.

(2) (R, M) is a local ring with the maximal ideal M and R is Ikeda-Nakayama with at most four nontrivial ideals.

(3) (R, M) is a local ring with the maximal ideal M, $M = Rx + Ry$, $M^2 = 0$, all proper ideals, different from M, must be principal and of the form Rx, Ry, or $R(x + ay)$, where a is an invertible element of R and $G(R)$ is a star graph.

(4) (R, M) is a local ring with the maximal ideal M, $M = Rx + Ry$, $M^3 = 0$ and the set of nontrivial ideals of R is equal to

$$\{M, Rx, Ry, Ry^2, R(x + y), R(x + y^2), Rx \oplus Ry^2 = Soc(R)\}.$$

Furthermore, it is shown that if R is a ring with $G(R)$ is a complete r-partite graph with part's $V_i(1 \leq i \leq r)$, then R is Artinian and one of the following statements hold:

(1) $G(R)$ is complete and R is an Ikeda-Nakayama ring.

(2) R is a local ring, $\ell(Soc(R)) \leq 2$ and if $I, J \in V_i$, then I and J are cyclic local R-modules with common maximal submodule $I \cap J$.

Among other results, we give a description of a lower bound for the clique number of $G(R)$.

In order to make this paper easier to follow, we recall in this section various notions which will be used in the sequel. Throughout this paper all rings are commutative with non-zero identity. Let R be a ring. By $\mathcal{I}(R)$, we denote the set of all nontrivial ideals of R. A ring R is said to be local if it has a unique maximal ideal M and we denote it by (R, M). For a subset X of a ring R, the annihilator of X in R is $ann(X) = \{r \in R : rx = 0 \; for \; all \; x \in X\}$ and we denote

the set of all maximal ideals of R by $max(R)$. The *socle* of ring R, denoted by $Soc(R)$, is the sum of all minimal ideals of R. Following [22], a ring R is called a *dual ring* if every ideal of R is an annihilator. Let N be an R-module. A *chain* of R-submodules of length n is a sequence N_i ($0 \leq i \leq n$) of R-submodules of N such that $0 = N_0 \subset N_1 \subset ... \subset N_n = N$. A *composition series* of N is a maximal chain, that is one in which no extra R-submodules can be inserted. Two composition series $A_0 = 0 \leq A_1 \leq ... \leq A_n = N$ and $B_0 = 0 \leq B_1 \leq ...B_t = N$ of an R-module N are said to be *isomorphic* (or *equivalent*) provided $n = t$ and there exists a permutation π of $\{1, 2, ..., n\}$ such that $A_i/A_{i-1} \cong B_{\pi(i)}/B_{\pi(i)-1}$ (isomorphic as an R-module) for all $i = 1, ..., n$. It is known that every two composition series for N are equivalent. The *length* of composition series of N is denoted by $\ell(N)$. A submodule K of an R-module M is called *essential* in M if, for every non-zero submodule L of M, we have $K \cap L \neq 0$. An R-module M is called *uniform*, if every non-zero submodule of M is essential in M. Let N, H be two submodules of R-module M. Then H is called a complement of N if H is maximal with respect to the property $H \cap N = \{0\}$. If N and H are complement of each other, then N and H are called mutual complement. An R-module N is called uniserial if its submodules are linearly ordered by inclusion. If R is uniserial as an R-module, then we call R is uniserial. Note that uniserial rings are in particular local rings. Commutative uniserial rings are also known as valuation rings. We call an R-module N serial if it is a direct sum of uniserial modules. The ring R is called serial if R is serial as an R-module.

For a graph G by $E(G)$ and $V(G)$ we denote the set of all edges and vertices, respectively. We recall that a graph is *connected* if there exists a path connecting any two distinct vertices. A graph G is said to be *totally disconnected* if it has no edges. The *distance* between two distinct vertices a and b, denoted by $d(a, b)$, is the length of the shortest path connecting them (if such a path does not exist, then $d(a, b) = \infty$). The *diameter* of a graph G, denoted by $\text{diam}(G)$, is equal to $\sup\{d(a, b) : a, b \in V(G)\}$. The *eccentricity* of a vertex a is defined as $e(a) = \max\{d(a, b) : b \in V(G)\}$ and the radius of G is given by $rad(G) = \min\{e(x) : x \in V(G)\}$. A vertex x of a connected graph G is a *cut vertex* of G if there are vertices a and b of G such that x is in every path from a to b (and $x \neq a$, $x \neq b$). Equivalently, for a connected graph G, x is a cut vertex of G if $G - \{x\}$ is not connected. The *degree* of a vertex x in a graph G is the number of edges incident with x. The degree of a vertex x is denoted by $deg(x)$. Let r be a non-negative integer. The graph G is said to be *r-regular*, if the degree of each vertex is r. If a and b are two adjacent vertices of G, then we write $a - b$. A vertex a of G is called *end vertex*, if $deg(a) = 1$. A graph is *complete* if it is connected with diameter less than or equal to one. We denote the complete graph on n vertices by K_n. The *girth* of a graph G, denoted $\text{gr}(G)$, is the length of a shortest cycle in G, provided G contains a cycle; otherwise; $\text{gr}(G) = \infty$. A *complete bipartite* graph with part sizes m and n is denoted by $K_{m,n}$. A *star graph* is a graph with a vertex adjacent to all

other vertices and has no other edges. A *clique* of a graph is its maximal complete subgraph and the number of vertices in the largest clique of graph G, denoted by $\omega(G)$, is called the *clique number* of G. An *induced subgraph* of a graph G by the set $S \subseteq V(G)$ is a subgraph H of G where vertices are adjacent in H precisely when adjacent in G. A set of vertices S in G is a *dominating set*, if $N[S] = V$. The *domination number*, $\gamma(G)$, of G is the minimum cardinality of a dominating set of G ([20]).

Lemma 1. [9, Proposition 1] *Let R be a ring and I, J two ideals of R such that $I + J = R$. Then $ann(I \cap J) = ann(I) \oplus ann(J)$.*

2. **Basic properties of G(R)**

In this section, we give some basic properties of the graph $G(R)$ which are useful in the following sections. We begin with the following useful lemma.

Lemma 2. *Let R be a ring and I, J two nontrivial ideals of R. Then the following statements hold.*
   (1) *If $I + J = R$, then I and J are adjacent in $G(R)$.*
   (2) *If $I \subseteq J$ or $J \subseteq I$, then I and J are adjacent in $G(R)$.*

Proof. (1) is clear by Lemma 1
   (2) is obvious.   □

The benefit of studying the graph $G(R)$ is that one may find some results about its known subgrapghs, for a ring R.

Remark 3. *(1) Let R be a ring. The inclusion ideal graph of a ring R, denoted by $In(R)$, is a graph whose vertices are all nontrivial ideals of R and two distinct ideals I and J are adjacent if and only if $I \subset J$ or $J \subset I$ [1]. By Lemma 2, $In(R)$ is a subgraph of $G(R)$.*
   (2) Let R be a ring. The comaximal ideal graph, denoted by $\Gamma(R)$, is a graph whose vertices are proper ideals of R that are not contained in the Jacobson radical of R, and two vertices I and J are adjacent if and only if $I + J = R$ [24]. By Lemma 2, $\Gamma(R)$ is a subgraph of $G(R)$.

Proposition 4. *Let R be a ring. Then the following statements hold.*
   (1) *$G(R)$ is a totally disconnected graph if and only if R has only one non-zero proper ideal.*
   (2) *$G(R)$ is a complete graph if and only if R is an Ikeda-Nakayama ring.*

Proof. (1) One side is clear. To prove the other side, suppose that $G(R)$ is a totally disconnected graph, with at least two vertices I, J. Since $G(R)$ is totally disconnected, $I \cap J = \{0\}$ and $I + J = R$. Hence Lemma 1 gives I and J are adjacent, a contradiction. So R has only one non-zero proper ideal.
   (2) It is clear.   □

Lemma 5. *Let R be a ring and $max(R) = \{M_i : i \in K\}$. Then the following statements hold.*

(1) For any finite subset Λ of K, $ann(\cap_{i\in\Lambda}M_i) = \sum_{i\in\Lambda} ann(M_i)$.

(2) If Λ is a finite subset of K and $\cap_{i\in\Lambda}M_i \neq 0$, then $\cap_{i\in\Lambda}M_i$ is adjacent to every other vertex in $G(R)$.

Proof. (1) It is clear by Lemma 1.

(2) Let I be a nonzero proper ideal of R and Λ a finite subset of K. We split the proof into three cases for I:

Case 1: If $I \subseteq M_i$, for each $i \in \Lambda$, then $I \subseteq \cap_{i\in\Lambda}M_i$. Hence I and $\cap_{i\in\Lambda}M_i$ are adjacent by Lemma 2.

Case 2: If $I \nsubseteq M_i$ for each $i \in \Lambda$, then $I + \cap_{i\in\Lambda}M_i = R$. Hence I and $\cap_{i\in\Lambda}M_i$ are co-maximal, so I and $\cap_{i\in\Lambda}M_i$ are adjacent by Lemma 2.

Case 3: Let $\Delta = \{i \in \Lambda : I \subseteq M_i\}$ and $\Theta = \{i \in \Lambda : I \nsubseteq M_i\}$. So $ann(I \cap (\cap_{i\in\Lambda}M_i)) = ann(I \cap (\cap_{i\in\Theta}M_i))$. Since I and $\cap_{i\in\Theta}M_i$ are co-maximal, $ann(I \cap (\cap_{i\in\Theta}M_i)) = ann(I) + ann(\cap_{i\in\Theta}M_i)$. Also for each $i \in \Delta$, $ann(M_i) \subseteq ann(I)$, hence $ann(I) = ann(I) + \Sigma_{i\in\Delta}ann(M_i)$. Therefore

$$ann(I \cap (\cap_{i\in\Lambda}M_i)) = ann(I \cap (\cap_{i\in\Theta}M_i)) =$$

$$= ann(I) + \Sigma_{i\in\Delta}ann(M_i) + ann(\cap_{i\in\Theta}M_i) = ann(I) + ann(\cap_{i\in\Lambda}M_i).$$

Hence $\cap_{i\in\Lambda}M_i$ is adjacent to every other vertex in $G(R)$. $\square$

Theorem 6. *Let R be a ring. Then the following statements hold.*

(1) If $\deg(I)$ is finite for some ideal I of R, then R is an Artinian ring.

(2) If M is a maximal ideal of R, then $deg(M)$ is finite if and only if $G(R)$ is finite.

(3) $\gamma(G(R)) = 1$ and $rad(G(R)) = 1$.

Proof. (1) It is clear by Lemma 2(2) and [19, Proposition 4.5].

(2) It is clear by Lemma 5.

(3) By Lemma 5, every maximal ideal of R is adjacent to every other vertex of R. Hence $\gamma(G(R)) = 1$ and $rad(G(R)) = 1$. $\square$

Theorem 7. *Let R be a ring. Then $G(R)$ is a connected graph and $diam(G(R)) \in \{0, 1, 2\}$.*

Proof. Let R contain more than one non-zero proper ideal. Let I, J be two non-zero proper ideals of R. If $I \in max(R)$ or $J \in max(R)$, then I and J are adjacent by Lemma 2. Hence $d(I, J) = 1$. Suppose that I, J are not maximal. By Lemma 5, for each $M \in max(R)$, M is adjacent to I, J, hence $d(I, J) \leq 2$. Hence $G(R)$ is connected and $diam(G(R)) \in \{0, 1, 2\}$. $\square$

Theorem 8. *Let R be a ring. Then $gr(G(R)) \in \{3, \infty\}$.*

Proof. If $|max(R)| \geq 3$, then $gr(G(R)) = 3$ by Lemma 5. Suppose $|max(R)| \leq 2$. We divide the proof in two cases:

Case 1: $max(R) = \{M_1, M_2\}$. If $M_1 \cap M_2 \neq 0$, then $M_1 - M_1 \cap M_2 - M_2 - M_1$ is a path in $G(R)$, which gives $gr(G(R)) = 3$. If $M_1 \cap M_2 = 0$, then $R = M_1 \oplus M_2$. Hence M_1, M_2 are the only nonzero proper ideals of R, so $gr(G(R)) = \infty$.

Case 2: Let (R, M) be a local ring. If there exist non-zero proper (non-maximal) ideals I and J of R such that $I \subsetneqq J$, then $gr(G(R)) = 3$. Suppose, for each ideal I of R, there is no ideal J of R such that $J \subseteq I$, hence each non maximal ideal of R is minimal, which gives $gr(G(R)) = \infty$. $\square$

Corollary 9. *Let R be a ring. Then $gr(G(R)) = 3$ if and only if $G(R)$ contains a cycle.*

Theorem 10. *Let R be a ring. Then the following statements are equivalent:*
(1) *$G(R)$ contains an end vertex;*
(2) *Either $R = M_1 \oplus M_2$, where $\mathcal{I}(R) = \{M_1, M_2\}$ or (R, M) is a local ring and each proper non-maximal ideal of R is minimal;*
(3) *$G(R)$ is a star graph;*
(4) *$gr(G(R)) = \infty$;*
(5) *$G(R)$ is a bipartite graph.*

Proof. $(1) \Rightarrow (2)$ Let I be an end vertex of $G(R)$. If I is a maximal ideal of R, then $|\mathcal{I}(R)| = 2$, because $deg(I) = 1$ and I is adjacent to every other vertex of $G(R)$ by Lemma 5. Suppose that I is not maximal. By Lemma 5, I is adjacent to every maximal ideal of R. Hence R is a local ring. We show for each non-maximal ideal $J \neq 0$ $(J \neq I)$ of R, J is minimal. Since I is only adjacent to the maximal ideal M of R, $I \cap J = \{0\}$, $I + J = M$ for each ideal J of R. Suppose that, there exists an ideal J of R such that J is not minimal. Hence there exists an ideal K of R such that $K \subset J$. By the above argument, $I \oplus K = M$. By using modular law, $J = K \oplus (I \cap J)$. Hence $J = K$, a contradiction. So R is a local ring and each proper non-maximal ideal of R is minimal.

$(2) \Rightarrow (3)$ If $R = M_1 \oplus M_2$, where $\mathcal{I}(R) = \{M_1, M_2\}$, then it is clear that $G(R)$ is a star graph. If (R, M) is a local ring and each proper non-maximal ideal of R is minimal, then M is adjacent to every other vertex of $G(R)$ and two non-zero non-maximal ideals I, J of R are not adjacent (because $I \cap J = \{0\}$ and $ann_l(I) = ann_l(J) = M$). Hence $G(R)$ is a star graph.

$(3) \Rightarrow (4)$ It is clear.

$(4) \Rightarrow (5)$ By Corollary 9 and the proof of Theorem 8, $G(R)$ is a star graph and so it is a bipartite graph.

$(5) \Rightarrow (1)$ By [7, Theorem 4.7], a graph is bipartite if and only if it contains no odd cycle. Hence $G(R)$ is bipartite if and only if $gr(G(R)) = \infty$ by Theorem 8 and Corollary 9. Therefore it contains an end vertex. $\square$

Theorem 11. *Let R be a ring. Then the following statements are equivalent:*
(1) *$G(R)$ contains a cut vertex;*

(2) (i) (R, M) *is a local ring.*

(ii) Each proper non-maximal ideal of R is minimal, maximal in M and $\ell(R) =$
3.

(iii) R has at least three non-trivial ideals.

(3) $G(R) = K_{1,n}$, for some $n \geq 2$.

Proof. $(1) \Rightarrow (2)$ Let $G(R)$ contain a cut vertex. By Lemma 5 each maximal ideal of R is adjacent to every other vertex. This implies that cut vertex of $G(R)$ should be a maximal ideal. Hence $|max(R)| = 1$ or 2. If $max(R) = \{M_1, M_2\}$, then $R = M_1 \oplus M_2$, because M_1, M_2 are adjacent to every other vertex of $G(R)$. Hence $\mathcal{I}(R) = \{M_1, M_2\}$, and so $G(R) = K_2$, but K_2 contains no cut vertex. This implies that $max(R) = \{M\}$. Therefore our assumption implies M is the cut vertex of $G(R)$ (because M is adjacent to every other vertex). So $G(R) \setminus \{M\}$ is disconnected. Let I, J be two vertex of $G(R) \setminus \{M\}$ such that there is no path between them. Hence $I \cap J = \{0\}$, $I + J = M$. It can not be difficult to see that I, J are mutually complemented. We show I, J are maximal submodules of M as submodules of the R-module M. If I is not maximal, then $I \subseteq L$, for some right $L \neq M$ of R. Since I is complement of J, $J \cap L \neq \{0\}$. Since $I \nsubseteq J$, $J \cap L \neq L$. If $J \cap L = J$, then $J \subseteq L$, gives $J + I \subseteq L$, a contradiction. Hence $J - J \cap L - L - I$ is a path between I, J, a contradiction, so I is a maximal submodule of M. Similarly, J is a maximal submodule of M, too, which gives I and J are minimal ideals of R. So M is a semisimple R-module with $\ell(M) = 2$. So $deg(I) = 1$, for each ideal $I \neq M$ of R. Since $G(R) \neq K_2$, R has at least three non-trivial ideals.

$(2) \Rightarrow (3)$ and $(3) \Rightarrow (1)$ are clear. $\qquad\square$

3. When is G(R) planar, complete or complete r-partite?

In this section, planar property, complete and complete r-partite property of $G(R)$ are investigated.

Lemma 12. *Let (R, M) be a local ring. If I, J are two nontrivial ideals of R which are adjacent in $G(R)$, then $I \cap J \neq 0$.*

Lemma 13. *Let (R, M) be a local Artinian ring. Then the following statements are equivalent:*

(1) $G(R)$ is complete;

(2) $Soc(R)$ is simple;

(3) R is a dual ring;

(4) R is uniform;

(5) R is an Ikeda-Nakayama ring.

Proof. $(1) \Rightarrow (2)$ As $G(R)$ is complete and R is local, we have $I \cap J \neq \{0\}$ for each nontrivial ideals I, J of R, by Lemma 12. Hence $\ell(Soc(R)) = 1$.

$(2) \Rightarrow (1)$ If $\ell(Soc(R)) = 1$, then R contains only one minimal ideal, say K. Hence $K \subseteq I \cap J$, for each ideals I, J of R. By [8, Exercise 3.2.15], $ann(ann(I)) = I$,

which gives $I = J$ if and only if $ann(I) = ann(J)$. Now, we claim that $G(R)$ is a complete graph. Let I, J be two ideals of R. Then

$$ann(ann(I) + ann(J)) = ann(ann(I)) \cap ann(ann(J)) =$$

$$= I \cap J = ann(ann(I \cap J)).$$

Hence $ann(I \cap J) = ann(I) + ann(J)$ and I, J are adjacent, which implies $G(R)$ is a complete graph.

$(2) \Leftrightarrow (3) \Leftrightarrow (4)$ See [8, Exercise 3.2.15].

$(1) \Leftrightarrow (5)$ By Proposition 4.   □

Theorem 14. *Let R be an Artinian ring. Then the following statements are equivalent:*

(1) $G(R)$ *is complete;*

(2) $R = R_1 \times R_2 \times .... \times R_n$ $(n \in \mathbb{N})$, *where (R_i, M_i) is local and $G(R_i)$ is complete.*

(3) $R = R_1 \times R_2 \times .... \times R_n$ $(n \in \mathbb{N})$, *where (R_i, M_i) is local and $Soc(R_i)$ is simple.*

(4) $R = R_1 \times R_2 \times .... \times R_n$ $(n \in \mathbb{N})$, *where (R_i, M_i) is a uniform local ring.*

Proof. $(1) \Rightarrow (2)$ Let R be an Artinian ring. By [5, Theorem 8.7], R is isomorphic to the product of local Artinian rings R_i with maximal ideals M_i. We show that $G(R_i)$ is complete. Let I, J be two nontrivial ideals of R_i, then $R_1 \times ... \times R_{i-1} \times I \times R_{i+1} \times ... \times R_n$ and $R_1 \times ... \times R_{i-1} \times J \times R_{i+1} \times ... \times R_n$ are nontrivial ideals of R. As $G(R)$ is complete, I and J are adjacent in R_i. Therefore $G(R_i)$ is complete.

$(2) \Rightarrow (1)$ Let $I = I_1 \times ... \times I_n$ and $J = J_1 \times ... \times J_n$ be two nontrivial ideals of $R_1 \times ... \times R_n$. Set $S_I = \{i : I_i \text{ is nontrivial }\}$ and $S_J = \{i : J_i \text{ is nontrivial }\}$. If $S_I \cap S_J = \emptyset$, then I and J are adjacent. If $S_I \cap S_J \neq \emptyset$, then by assumption, for each $i \in S_I \cap S_J$, I_i and J_i are adjacent in $G(R_i)$. Hence I and J are adjacent.

$(2) \Leftrightarrow (3) \Leftrightarrow (4)$ is clear by Lemma 13.   □

Theorem 15. *Let R be a ring with $G(R)$ r-regular. Then either $G(R)$ is complete or null graph.*

Proof. Suppose $G(R)$ is not null. By Theorem 6, R is an Artinian ring. Thus $R \cong R_1 \times ... \times R_n$ by [5, Theorem 8.7], where (R_i, M_i) is a local Artinian ring. Toward a contradiction, assume that $G(R)$ is not complete. Hence by Theorem 14, $G(R_i)$ is not complete for some $1 \leq i \leq n$. Thus $Soc(R_i)$ is not minimal. Let I_1 and I_2 be two minimal ideals of R_i. If H is a vertex of $G(R_i)$ that is adjacent to I_1, then $I_1 \subseteq H$, by Lemma 12. Thus $ann(H \cap (I_1 \oplus I_2)) = ann((H \cap I_2) \oplus I_1) = M_i = ann(I_1 \oplus I_2) \oplus ann(H)$. Therefore every vertex which is adjacent to I_1 is adjacent to $I_1 \oplus I_2$ too, in $G(R_i)$. Moreover, I_1 and I_2 are not adjacent in $G(R_i)$. This shows that $deg(R_1 \times ... \times R_{i-1} \times (I_1 \oplus I_2) \times R_{i+1} \times ... \times R_n) > deg(R_1 \times ... \times R_{i-1} \times I_1 \times R_{i+1} \times ... \times R_n)$, a contradiction. Therefore $G(R)$ is complete.   □

Theorem 16. *Let R be an Artinian ring. Then R is a serial ring if and only if $G(\frac{R}{I})$ is complete for each ideal I of R.*

Proof. Let R be an Artinian ring. Then $R \cong R_1 \times R_2 \times ... \times R_n$, where (R_i, M_i) is a local Artinian ring. Assume that $G(\frac{R}{I})$ is complete for each ideal I of R. Let I_i be an arbitrary ideal of R_i. As $\frac{R_1 \times R_2 \times ... \times R_n}{R_1 \times ... \times R_{i-1} \times I_i \times R_{i+1} \times ... \times R_n} \cong \frac{R_i}{I_i}$, $G(\frac{R_i}{I_i})$ is complete by our assumption. Therefore $Soc(\frac{R_i}{I_i})$ is simple, by Lemma 13. This shows that R_i is uniserial by [23, 55.1(1)]; and so R is serial.

Let R be a serial ring. Then $R = R_1 \times R_2 \times ... \times R_n$, where R_i is a uniserial ring (and so local) for each $1 \leq i \leq n$. As R_i is Artinian for each $1 \leq i \leq n$, $Soc(R_i/I_i)$ is simple for each proper ideal I_i of R_i. Therefore, $G(R_i/I_i)$ is complete by Lemma 13, for each $1 \leq i \leq n$. Hence Theorem 14 implies that $G(\frac{R}{I})$ is complete for each ideal I of R. $\square$

In the following, we characterize the rings for which their Ikeda-Nakayama graph is planar.

Lemma 17. *Let R be a ring with $G(R)$ is planar. Then R is Artinian and R has at most 2 maximal ideals.*

Proof. Since $G(R)$ is planar, R is Artinian. Let $|max(R)| \geq 3$ and $M_1, M_2, M_3 \in max(R)$. Then by Lemma 5, $\{M_1, M_2, M_3, M_1 \cap M_2, M_1 \cap M_3\}$ is a vertex set of K_5 as a subgraph of $G(R)$, a contradiction (note that $M_i \cap M_j \neq \{0\}$ for each $i \neq j$). Therefore $|max(R)| \leq 2$. $\square$

Theorem 18. *Let R be a ring with two maximal ideals. Then $G(R)$ is planar if and only if either $R \cong F \times S$, where F is a field and (S, M) is a local ring with the only non-zero proper ideal M or $R \cong F_1 \times F_2$ where F_1, F_2 are fields. In these cases, either $G(R) \cong K_2$ or $G(R) \cong K_4$ and R is an Ikeda-Nakayama ring.*

Proof. Let $G(R)$ be a planar graph. Then by Lemma 17, R is Artinian. Hence $R \cong R_1 \times R_2$, where (R_i, M_i) is a local ring for each $i = 1, 2$. If R_i is not field for each $i = 1, 2$, then $\{R_1 \times 0, R_1 \times M_2, M_1 \times R_2, 0 \times R_2, M_1 \times 0\}$ is a clique in $G(R)$. Therefore, R_1 or R_2 is field. Let R_1 be field. We show that R_2 has only one non-zero proper ideal M_2. Otherwise, assume that I is a non-zero ideal of R_2 such that $I \subset M_2$. Then $\{R_1 \times 0, 0 \times R_2, R_1 \times M_2, R_1 \times I, 0 \times I, 0 \times M_2\}$ is a clique in $G(R)$. This contradicts the planar property of $G(R)$. Thus (R_2, M_2) is a local ring with the only non-zero proper ideal M_2. Hence $G(R) \cong K_4$ and R is an Ikeda-Nakayama ring. If R_2 is a field $(M_2 = 0)$, then $G(R) \cong K_2$ and R is an Ikeda-Nakayama ring.

The converse is clear. $\square$

Lemma 19. *Let (R, M) be a local Artinian ring. Then $ann(Soc(R)) = M$ and $ann(M) = Soc(R)$.*

Proof. Since R is Artinian, $Soc(R) \neq 0$. By [23, Theorem 21.12], $M.Soc(R) = 0$, so $Soc(R) \subseteq ann(M)$. Now, let $x \in ann(M) \backslash Soc(R)$. So there exists an essential ideal I of R such that $x \notin I$. Since $I \leq^{ess} R$, $I \cap Rx \neq 0$. Let $0 \neq rx = i \in I \cap Rx$, for some $r \in R$. Since $x \in ann(M)$ and $rx \neq 0$, $r \notin M$. So r is invertible which implies $x \in I$, a contradiction. Hence $ann(M) \subseteq Soc(R)$, which gives $Soc(R) = ann(M)$. Moreover $ann(Soc(R)) = ann(ann(M)) = M$. $\square$

Lemma 20. *Let (R, M) be a local ring, with $G(R)$ planar. Then*
  (1) *$Soc(R)$ is adjacent to every other vertex of $G(R)$.*
  (2) *$\ell(Soc(R)) \leq 2$.*

Proof. (1) Let I be an ideal of R. By Lemma 17, R is Artinian. Hence $Soc(R) \cap I \neq \{0\}$. Therefore $ann(Soc(R) \cap I) = M$. This implies that $Soc(R)$ and I are adjacent.

(2) Let $Soc(R) = K_1 \oplus K_2 \oplus ... \oplus K_n$, where K_i's are minimal ideals of R. Suppose, on the contrary, $n \geq 3$. Hence $\{K_1 \oplus K_2, K_1 \oplus K_3, K_2 \oplus K_3, M, Soc(R)\}$ makes K_5 in $G(R)$, which is a contradiction. $\square$

Lemma 21. *Let (R, M) be a local ring, with $G(R)$ planar. Then every set of minimal generators for M has at most two elements.*

Proof. Let $M = Rx + Ry + Rz$. By Lemma 20, $\ell(Soc(R)) \leq 2$. If $\ell(Soc(R)) = 1$, then by Lemma 13, the vertex set $\{Rx, Ry, Rz, Rx + Ry, M\}$ makes K_5 in $G(R)$ which is a contradiction. So $\ell(Soc(R)) = 2$. We will show that $M^2 = \{0\}$. Let $x^2 \neq 0$. By Lemma 20 the vertex set $\{Rx^2, Rx, Rx + Ry, M, Soc(R)\}$ makes K_5 in $G(R)$. Since $G(R)$ is planar and $\ell(Soc(R)) = 2$, $Soc(R) = Rx$. Therefore $x^2 = 0$ ($MSoc(R) = 0$), a contradiction. So $x^2 = 0$. Similarly, $y^2 = z^2 = 0$. Let $xy \neq 0$. If $V_1 = \{Rxy, Rz, M\}$ and $V_2 = \{Rxy + Rz, R(x + y) + Rz, Rx + Rz\}$, then V_1 and V_2 are two parts of $K_{3,3}$ as a subgraph of $G(R)$, a contradiction. Hence $xy = 0$. Similarly, $xz = yz = 0$. Thus $M^2 = \{0\}$, which implies $ann(M) = M$. By Lemma 19, $ann(M) = Soc(R)$, hence $M = Soc(R)$. So $\ell(M) = 2$, which is a contradiction. So every set of minimal generators for M has at most two elements. $\square$

Lemma 22. *Let (R, M) be a local Artinian ring with $M = Rx + Ry$ and $\{x, y\}$ is the set of minimal generators for M. Then $Soc(R) \neq Rx, Ry$.*

Proof. Suppose, on the contrary, $Soc(R) = Rx$. Since R is Artinian $Rx \cap Ry \neq 0$. Let $0 \neq rx = sy \in Rx \cap Ry$. By Lemma 19, $r \notin M$. So r is invertible, which is a contradiction. Similarly, $Soc(R) \neq Ry$. $\square$

Lemma 23. *Let (R, M) be a local ring with $M = Rx + Ry$, where x, y are minimal generators of M. If $G(R)$ is a planar graph, then $M^3 = 0$.*

Proof. Suppose, $M^3 \neq 0$. As $M^3 = Rx^3 + Ry^3 + Rxy^2 + Rx^2y$, we show Rx^3, Ry^3, Rxy^2, $Rx^2y = 0$. Suppose $x^3 \neq 0$. Then $Rx^3 \subseteq Rx^2 \subseteq Rx \subseteq M$. If $Soc(R) \notin \{Rx^3, Rx^2, Rx, M\}$, we have K_5 as a subgraph of $G(R)$, a contradiction, so $Soc(R) \in \{Rx^3, Rx^2, Rx, M\}$. By Lemma 20, $\ell(Soc(R)) \leq 2$. If $\ell(Soc(R)) = 1$, then $Soc(R) = Rx^3$, hence $\{Rx^3, Rx^2, Rx, M, Ry\}$ makes a K_5 in $G(R)$, by Lemma

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13, a contradiction. If $\ell(Soc(R)) = 2$, then $Soc(R) = Rx^2$; so by Lemma 19, $x^3 = 0$, a contradiction. Similarly, one can show that $Ry^3, Rxy^2, Rx^2y = 0$, as needed. □

Lemma 24. *Let (R, M) be a local ring with $M = Rx + Ry$, where x, y are minimal generators of M. If $M^2 = \{0\}$, then $G(R)$ is a star graph.*

Proof. As $M^2 = \{0\}$, by Lemma 19, $M = Soc(R)$. So by Lemma 12 and Theorem 10, $G(R)$ is a star graph and every proper ideal, different from M, must be principal and of the form Rx, Ry, or $R(x + ay)$, where a is an invertible element of R. □

In the following theorem, G_1 denotes the next graph.

$$G_1$$
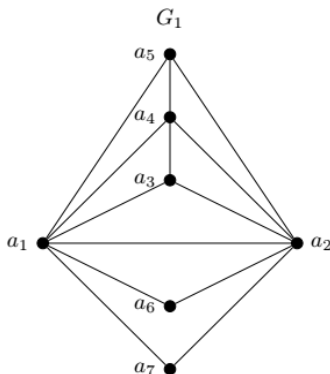
Theorem 25. *Let (R, M) be a local ring. Then $G(R)$ is planar if and only if one of the following statements holds.*

(1) R is an Ikeda-Nakayama ring and $G(R)$ is isomorphic to K_n, for some $n \leq 4$.

(2) $M^2 = 0$, $M = Rx + Ry$, every proper ideal, different from M, must be principal and of the form Rx, Ry, or $R(x + ay)$, where a is an invertible element of R and $G(R)$ is a star graph.

(3) $M^3 = 0$, $M = Rx + Ry$, the set of nontrivial ideals of R is equal to

$$\{M, Rx, Ry, Ry^2, R(x + y), R(x + y^2), Rx \oplus Ry^2 = Soc(R)\}$$

and $G(R) \cong G_1$.

Proof. Let R be a ring with $G(R)$ planar. By Lemma 21, the set of minimal generators for M has at most 2 elements. If $M = Rx$, then R is a principal ideal ring. This implies that

$$\mathcal{I}(R) = \{M^i : 1 \leq i \leq n\},$$

where n is the smallest number such that $M^n = 0$ and $n \leq 4$. Thus (1) holds.

Let $M = Rx + Ry$, where x, y are minimal generators of M and $x, y \in M \setminus M^2$. By Lemma 20, $\ell(Soc(R)) \leq 2$. If $\ell(Soc(R)) = 1$, then $G(R)$ is complete by lemma 13. As $G(R)$ is planar, $G(R) \cong K_n$ for some $n \leq 4$ and (1) holds. Now, let $\ell(Soc(R)) = 2$. By Lemma 23, $M^3 = \{0\}$. If $M^2 = \{0\}$, then $G(R)$ is a star graph

by Lemma 24; so (2) holds. Assume that $M^2 \neq \{0\}$. Therefore $M^2 \subseteq Soc(R)$. We will show that M^2 is simple. Suppose on the contrary, $M^2 = Soc(R)$. Let $Soc(R) = K_1 \oplus K_2$ for some minimal ideals K_1 and K_2 of R. If $Soc(R) \subseteq Rx, Ry$, then $V_1 = \{K_1, K_2, M\}$ and $V_2 = \{Rx, Ry, Soc(R)\}$ are two parts of $K_{3,3}$ as a subgraph of $G(R)$, a contradiction. Hence, suppose without lose of generality, $Soc(R) \nsubseteq Rx$ and $K_2 \subseteq Rx$. Then $\{M, M^2, Rx, K_2, K_1 \oplus Rx\}$ is the vertex set of K_5 as a subgraph of $G(R)$, a contradiction(note that $Soc(R) \nsubseteq Rx$ implies that $M \neq Rx \oplus K_1$). Therefore M^2 is simple. We have three cases.

Case 1: $M^2 \cap Rx = \{0\}$. Hence $x^2 = xy = 0$.

Fact 1.1: Let $\alpha, \beta \in U(R)$ ($U(R)$ denotes the set of invertible elements of R). Then $R(\alpha x + y) = R(\beta x + y)$ if and only if $\alpha - \beta \in M$.

`Proof:` If $\alpha - \beta \in M$, then $\alpha x + y = (\alpha - \beta)x + \beta x + y = \beta x + y$. Conversely, if $R(\alpha x + y) = R(\beta x + y)$, then $\alpha x + y = r\beta x + ry$ for some $r \in R$. Thus $(\alpha - r\beta)x = (1-r)y$. Clearly $1 - r \in M$ and $\alpha - r\beta \in M$. Therefore $\alpha - \beta = \alpha - r\beta + (r-1)\beta \in M$.

Fact 1.2: $|R/M| = 2$.

`Proof:` Let $|R/M| \geq 3$. Then $\alpha + M \neq 1 + M$ for some $\alpha \in U(R)$. So $1 - \alpha \notin M$ and $R(x + y) \neq R(\alpha x + y)$ by Fact 1.1. Let $V_1 = \{Ry, R(x + y), R(\alpha x + y)\}$ and $V_2 = \{M^2 = Ry^2, M, Soc(R)\}$. Then V_1 and V_2 make $K_{3,3}$ as a subgraph of $G(R)$, a contradiction. Therefore $|R/M| = 2$

Fact 1.3: $R(x + y) = R(\alpha x + y)$ for each $\alpha \in U(R)$.

`Proof:` By Fact 1.1 and Fact 1.2.

Fact 1.4: For each $\alpha, \beta \in U(R)$, $R(x + \alpha y^2) = R(x + \beta y^2)$ and $R(\alpha x + y^2) = R(\beta x + y^2)$ if and only if $\alpha - \beta \in M$.

`Proof:` It is similar to Fact 1.1.

Fact 1.5: For each $\alpha \in U(R)$, $R(\alpha x + y^2) = R(x + \alpha y^2) = R(x + y^2)$.

`Proof:` By Fact 1.2 and Fact 1.4.

Fact 1.6: $R(x + \alpha y) \in \{Rx, R(x + y^2), R(x + y)\}$ for each $\alpha \in R$.

`Proof:` If $\alpha \in M$, then $\alpha = t_1 x + t_2 y$ for some $t_1, t_2 \in R$. So $x + \alpha y = x + t_2 y^2$. If $t_2 \in M$, then $x + \alpha y = x$. If $t_2 \notin M$, then $R(x + \alpha y) = R(x + t_2 y^2) = R(x + y^2)$ by Fact 1.6. If $\alpha \notin M$, then $R(x + \alpha y) = R(\alpha^{-1}x + y) = R(x + y)$ by Fact 1.3.

Fact 1.7: $R(\alpha x + \beta y) \in \{Rx, R(x + y^2), R(x + y)\}$.

`Proof:` It is similar to the proof of Fact 1.6.

Hence

$$\mathcal{I}(R) = \{M, Rx, Ry, Ry^2, R(x + y), R(x + y^2), Rx \oplus Ry^2 = Soc(R)\}.$$

As $Rx \cap Ry = \{0\}$ and $Ry^2 \subseteq R(x + y)$, $G(R) \cong G_1$.

Case 2: $M^2 \cap Ry = \{0\}$. Similar to Case 1,

$$\mathcal{I}(R) = \{M, Rx, Ry, Rx^2, R(x + y), R(y + x^2), Ry \oplus Rx^2 = Soc(R)\}$$

and $G(R) \cong G_1$.

Case 3: $M^2 \subseteq Rx, Ry$. If $M^2 \subseteq R(x + y)$, then $V_1 = \{Rx, Ry, R(x + y)\}$ and $V_2 = \{M^2, M, Soc(R)\}$ make $K_{3,3}$ as subgraph of $G(R)$, a contradiction. Therefore $M^2 \cap R(x + y) = \{0\}$. Hence $x^2 = y^2 = -xy \neq 0$, because $M^2 \neq 0$.

Hence, we have $M = Rp + Rq$ and $M^2 \cap Rq = \{0\}$, where $p = x$ and $q = x + y$. Therefore, this case is similar to Case 2 and

$$\mathcal{I}(R) = \{M, Rp, Rq, Rp^2, R(p+q), R(q+p^2), Rq \oplus Rp^2 = Soc(R)\}.$$

It is clear that $G(R) \cong G_1$.

The converse is clear. $\square$

Theorem 26. *Let R be a ring. If $G(R)$ is a complete r-partite graph with parts V_i $(1 \leq i \leq r)$, then R is Artinian and one of the following statements hold:*

(1) *$G(R)$ is complete and R is an Ikeda-Nakayama ring.*

(2) *R is a local ring, $\ell(Soc(R)) \leq 2$ and if $I, J \in V_i$, then I and J are cyclic local R-modules with common maximal submodule $I \cap J$.*

Proof. Since $G(R)$ is complete r-partite, R is Artinian.

Suppose R is not local, so $R \cong R_1 \times ... \times R_n$, where R_is are local rings. Since $G(R)$ is not complete, there exists i such that $G(R_i)$ is not complete, by Theorem 14. Suppose, without lose of generality, $G(R_1)$ is not complete, hence, there exist ideals I_1 and J_1 of R_1 such that they are not adjacent. So $I_1 \times 0 \times ... \times 0$ and $J_1 \times 0 \times ... \times 0$ are in the same part, say V_1. Consider $J_1 \times R_2 \times 0 \times ... \times 0$ as an ideal of R. So $I_1 \times 0 \times ... \times 0$ is not adjacent to $J_1 \times R_2 \times 0... \times 0$ but $J_1 \times 0 \times ... \times 0$ is adjacent to $J_1 \times R_2 \times 0... \times 0$, which is a contradiction. Therefore $G(R)$ is complete and (1) holds.

Suppose R is local. We show that $\ell(Soc(R)) \leq 2$. Suppose, on the contrary, $\ell(Soc(R)) \geq 3$ and $K_1 \oplus K_2 \oplus K_3 \subseteq Soc(R)$, where K_i is a minimal ideal of R, for each $1 \leq i \leq 3$. It is clear that $K_1 \oplus K_2$ and K_3 are not adjacent. Let $K_1 \oplus K_2, K_3 \in V_j$. Since K_2 is adjacent to $K_1 \oplus K_2$, K_2 is adjacent to K_3, which is a contradiction.

Let $I, J \in V_i$. We show I and J are maximal in $I + J$. Let $I \subseteq L \subseteq I + J$, for some ideal L of R. We show $J \subseteq L$. If $J \nsubseteq L$, then $J \cap L \neq J$. Hence I is adjacent to $J \cap L$. Thus

$$ann(I \cap J) = ann(I \cap L \cap J) = ann(I) + ann(L \cap J)$$

$$= ann(I) + ann(L) + ann(J) = ann(I) + ann(J),$$

a contradiction. So $J \subseteq L$. Thus $I + J \subseteq L$, which gives I is maximal in $I + J$. Similarly, J is maximal in $I + J$. Since $\frac{I+J}{J} \cong \frac{I}{I \cap J}$, so $I \cap J$ is maximal in I, J. We show that I and J are local as R-modules. Suppose on the contrary, there exists $x \in I \setminus I \cap J$ such that $Rx \neq I$. Hence $J \subseteq J + Rx \subseteq J + I$, $I + J = J + Rx$ and $I = Rx + I \cap J$. Thus

$$ann(I \cap J) = ann(I \cap J) \cap ann(Rx \cap J).$$

Since Rx and J are adjacent,

$$ann(I \cap J) = ann(I \cap J) \cap (ann(Rx) + ann(J))$$

$$= ann(J) + ann(I \cap J) \cap ann(Rx) = ann(J) + ann(I \cap J + Rx)$$

$$= ann(J) + ann(I).$$

So $I \cap J$ is the only maximal submodule of I. Similarly, $I \cap J$ is the only maximal submodule of J. $\square$

Theorem 27. *Let R be a ring. If $\omega(G(R)) < \infty$, then the following statements hold.*

(1) *R is Artinian.*

(2) *$\omega(G(R)) \geq 2^{|max(R)|} + n - 3$, where $n = max\{\ell(M_i) : M_i \in max(R)\}$.*

Proof. (1) It is clear.

(2) Since $\omega(G(R))$ is finite, $max(R)$ is finite, by Lemma 5(2). Let $max(R) = \{M_1, M_2, ...M_t\}$ and $P(max(R))$ be the power set of $max(R)$. Let $T_X = \cap_{T \in X} T$, for each $X \in P(max(R))$. Then by Lemma 5, the subgraph of $G(R)$ with vertex set $\{T_X\} - \{\emptyset, J(R)\}$ is a complete subgraph of $G(R)$, say G' ($J(R)$ may be zero). So $\omega(G(R)) \geq 2^{|max(R)|} - 2$. Now, let $n = max\{\ell(M_i) : M_i \in max(R)\}$. Hence $n = \ell(M_j)$ for some $1 \leq j \leq t$ and M_j has the composition series

$$0 = N_0 \subset N_1 \subset ... \subset N_n = M_j,$$

for some submodules N_s of M_j ($0 \leq s \leq n$). Similar to the proof of Lemma 5, one can prove that for each $1 \leq s \leq n - 1$, every N_s is adjacent to every other vertex of G'. Therefore $V(G') \cup \{N_s\}_{s=1}^{n-1}$ is a clique in $G(R)$, and thus $\omega(G(R)) \geq 2^{|max(R)|} - 2 + n - 1 = 2^{|max(R)|} + n - 3$. $\square$

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Current address: Shahabaddin Ebrahimi Atani: Department of Mathematics, University of Guilan, P.O.Box 1914, Rasht, Iran.

E-mail address: ebrahimi@guilan.ac.ir

ORCID Address: http://orcid.org/0000-0003-0568-9452

Current address: Saboura Dolati Pishhesari: Department of Mathematics, University of Guilan, P.O.Box 1914, Rasht, Iran.

E-mail address: Saboura_dolati@yahoo.com

ORCID Address: http://orcid.org/0000-0001-8830-636X

Current address: Mehdi Khoramdel: Department of Mathematics, University of Guilan, P.O.Box 1914, Rasht, Iran.

E-mail address: mehdikhoramdel@gmail.com

ORCID Address: http://orcid.org/0000-0003-0663-0356