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Research Article

Isometry classes of planes in $(\mathbb{R}^3, d_\infty)$

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Abstract

We determine geodesics in $\mathbb{R}^n_\infty$ (i.e. $(\mathbb{R}^n, d_\infty)$) and by using this, classify planes up to isometry in $\mathbb{R}^3_\infty$.

Mathematics Subject Classification (2010). 51F99

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1. Introduction

In metric spaces, it is possible to define length of paths. Let (X, d) be a metric space and $\alpha : [0,1] \to X$ be a path. Then, the length of α is defined as

$$\sup_{\mathcal{P}} \left\{ \sum_{i=1}^{n} d(\alpha(t_{i-1}), \alpha(t_i)) \right\}$$

over all partitions $P = \{t_0 = 0, t_1, \dots, t_n = 1\}$ of $[0,1]$ and it is denoted by $L(\alpha)$. If α satisfies $L(\alpha|_{[0,t]}) = t \cdot L(\alpha)$ for all $t \in (0,1)$, then α is called a natural path. It is clear that every path has a natural reparametrization. If the path α is natural and satisfies $L(\alpha) = d(x,y)$, where $\alpha(0) = x$, $\alpha(1) = y$, then α is called a geodesic. For a metric space and any two points in it, there may not exist any geodesic between these points. For example, $S^1 = \{(x,y) \in \mathbb{R}^2 \,|\, x^2 + y^2 = 1\}$ with induced standard metric and for $a = (1,0)$ and $b = (-1,0)$, there is no path connecting these points whose length is less than π, but the distance between a and b is equal to 2 according to the standard metric. If there is at least one geodesic between any two points in a metric space, this metric space is called "geodesic space" [2, 7], or "strictly intrinsic space" according to another terminology [3]. So, S^1 is not a geodesic space with the induced standard metric, but it becomes a geodesic space with the "arc length metric". $(\mathbb{R}^n, d_p)$ with

$$d_p((x_1, x_2, \dots, x_n), (y_1, y_2 \dots, y_n)) = \sqrt[p]{|x_1 - y_1|^p + |x_2 - y_2|^p + \cdots + |x_n - y_n|^p}$$

for $1 \le p < \infty$ and $(\mathbb{R}^n, d_\infty)$ with

$$d_\infty((x_1, x_2, \dots, x_n), (y_1, y_2 \dots, y_n)) = \max_{i=1}^{n}\{|x_i - y_i|\}$$

are also geodesic spaces. The path $t \mapsto (1-t)x + ty$ is a geodesic between $x = (x_1, x_2, \dots, x_n)$ and $y = (y_1, y_2 \dots, y_n)$ in $(\mathbb{R}^n, d_p)$ for $1 \le p \le \infty$.

Let (X, d_X) and (Y, d_Y) be two metric spaces and f be a function from X onto Y. f is called an isometry if $d_Y(f(x), f(y)) = d_X(x,y)$ for all $x, y \in X$. If there is an isometry

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between two metric spaces, then they are called isometric spaces. It is clear that notion of isometry deals with not only set on the space but also metric on the space. For example, S^1 with the induced standard metric from $\mathbb{R}^2$ and the same set with the "arc length metric" are not isometric.

Aronszajn-Panitchpakdi [1] called a metric space (X, d) hyperconvex, if for any collection $(x_i)_{i\in I}$ of points in X and any collection $(r_i)_{i\in I}$ of nonnegative real numbers satisfying $d(x_i, x_j) \leq r_i + r_j$ for all $i, j \in I$, the intersection of closed balls around x_i with radius r_i is nonempty: $\bigcap_{i\in I} \bar{B}(x_i, r_i) \neq \emptyset$. ($\bar{B}(x_i, r_i) = \{x \in X |\ d(x_i, x) \leq r_i\}$). In our previous work [4], we have noted that the plane

$$L = \{(x, y, z) \mid x + y + z = 0\} \subseteq \mathbb{R}^3_\infty$$

with the induced metric is not hyperconvex; therefore, it is not isometric to the plane $\mathbb{R}^2_\infty$ because $\mathbb{R}^2_\infty$ is hyperconvex. But it is clear that the $xy-$plane in $\mathbb{R}^3_\infty$ is isometric to the plane $\mathbb{R}^2_\infty$; hence, all planes in $\mathbb{R}^3_\infty$ are not isometric to each other. Therefore, the following question arises:

Question 1.1. What are the isometry classes of planes in $\mathbb{R}^3_\infty$?

In this paper, we have answered this question without using the notion of hyperconvexity. In order to do that, first, we have determined geodesics in $\mathbb{R}^n_\infty$ and then we have achieved our main aim.

2. Geodesics in $\mathbb{R}^n_\infty$

Definition 2.1. For $p = (p_1, p_2, \ldots, p_n) \in \mathbb{R}^n_\infty$, we define

$$S_i^\varepsilon(p) = \{q = (q_1, q_2, \ldots, q_n) \in \mathbb{R}^n |\ d_\infty(p, q) = \varepsilon(q_i - p_i)\}$$

for $i = 1, 2, \ldots, n$ and $\varepsilon = \pm$ and call them the sectors at the point p (see Fig. 1 and Fig. 2).

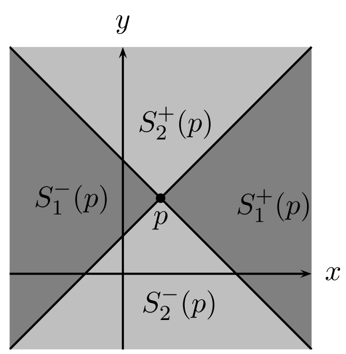

Figure 1. Sectors of a point p in $\mathbb{R}^2_\infty$.

Theorem 2.2. *Let $p = (p_1, p_2, \ldots, p_n)$, $q = (q_1, q_2, \ldots, q_n) \in \mathbb{R}^n$ be two points, $q \in S_i^\varepsilon(p)$ and $\alpha : [0, 1] \to \mathbb{R}^n$ be a natural path such that $\alpha(0) = p$ and $\alpha(1) = q$. Then α is a geodesic in $\mathbb{R}^n_\infty$ if and only if $\alpha(t') \in S_i^\varepsilon(\alpha(t))$ for all $t, t' \in [0, 1]$ such that $t < t'$.*

Proof. ($\Rightarrow$) Let $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ be a geodesic and assume that $\alpha(t') \notin S_i^\varepsilon(\alpha(t))$ for some $t < t'$. Then, we have $d_\infty(\alpha(t), \alpha(t')) > \varepsilon(\alpha_i(t') - \alpha_i(t))$. So if we take the partition $0 < t < t' < 1$ of $[0, 1]$, we have

$$\begin{aligned} d_\infty(\alpha(0), \alpha(t)) + d_\infty(\alpha(t), \alpha(t')) + d_\infty(\alpha(t'), \alpha(1)) &> \\ \varepsilon(\alpha_i(t) - \alpha_i(0)) + \varepsilon(\alpha_i(t') - \alpha_i(t)) + \varepsilon(\alpha_i(1) - \alpha_i(t')) &= \varepsilon(\alpha_i(1) - \alpha_i(0)) \\ &= \varepsilon(q_i - p_i) \\ &= d_\infty(p, q). \end{aligned}$$

This leads to the contradiction that $L(\alpha) > d_\infty(p,q)$.

($\Leftarrow$) Let $0 = t_0 < t_1 < \cdots < t_n = 1$ be an arbitrary partition of $[0,1]$. Since $\alpha(t_j) \in S_i^\varepsilon(\alpha(t_{j-1}))$ for all $j = 1,2,\ldots,n$, we have $d_\infty(\alpha(t_j),\alpha(t_{j-1})) = \varepsilon(\alpha_i(t_j) - \alpha_i(t_{j-1}))$; so,

$$\begin{aligned}\sum_{j=1}^{n} d_\infty(\alpha(t_j),\alpha(t_{j-1})) &= \sum_{j=1}^{n} \varepsilon(\alpha_i(t_j) - \alpha_i(t_{j-1})) \\ &= \varepsilon(\alpha_i(1) - \alpha_i(0)) \\ &= \varepsilon(q_i - p_i) \\ &= d_\infty(p,q).\end{aligned}$$

This implies that $L(\alpha) = d_\infty(p,q)$ and that α is a geodesic. $\square$

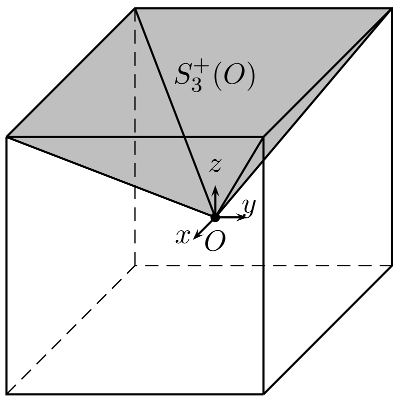

Figure 2. The sector $S_3^+(O)$ of the origin in $\mathbb{R}^3_\infty$.

Theorem 2.2 can be restated as follows: Let ε and i be such that $q \in S_i^\varepsilon(p)$, then the natural path α between p and q is a geodesic if and only if when the sectors $S_i^\varepsilon(.)$ travel on the image of α, the rest of the path is contained in the sector at every point (see Figure 3).

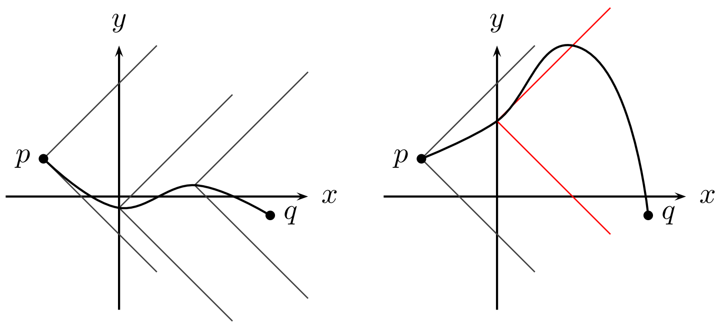

Figure 3. Two paths between p and q in $\mathbb{R}^2_\infty$ one of which (on the left) is a geodesic but the other is not.

Note that for p and q in $\mathbb{R}^2_\infty$, if the points are in a diagonal position (i.e. there is $t \in \mathbb{R}$ such that $p = q + t\cdot(1,1)$ or $p = q + t\cdot(1,-1)$), then there is only one geodesic between p and q. Of course, this is a line segment. If the points are not in a diagonal position, then there are infinitely many geodesics between these points. Likewise, for p and q in $\mathbb{R}^3_\infty$, if the points are in a cubic diagonal position (i.e. there is $t \in \mathbb{R}$ such that $p = q + t \cdot (1,1,1)$, $p = q + t \cdot (1,1,-1)$, $p = q + t \cdot (1,-1,1)$ or $p = q + t \cdot (-1,1,1)$), then there is only one geodesic (which is, still, a line segment) between p and q. If the points are not in a cubic diagonal position, then there are infinitely many geodesics between these points.

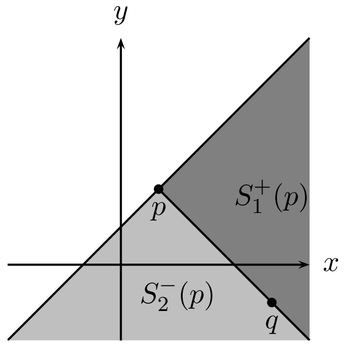

Figure 4. The points p and q are in a diagonal position.

Note that p and q are in a diagonal position in $\mathbb{R}^2_\infty$ if and only if $q \in S_1^\varepsilon(p) \cap S_2^\delta(p)$ where ε and δ are plus or minus. Likewise, p and q are in a cubic diagonal position in $\mathbb{R}^3_\infty$ if and only if $q \in S_1^\varepsilon(p) \cap S_2^\delta(p) \cap S_3^\gamma(p)$ where ε, δ and γ are plus or minus (see Figure 4 and Figure 5).

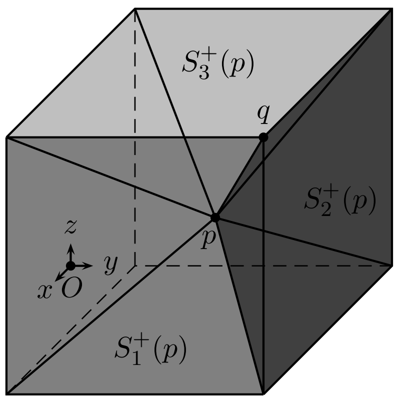

Figure 5. The points p and q are in a cubic diagonal position.

Now, let us consider a geodesic between two points in $\mathbb{R}^3_\infty$ and one belongs to the intersection of only two sectors of the other one. Theorem 2.2 implies that the image of such a geodesic must belong to the plane where these sectors intersect. So every geodesic between such points must be a planar curve (see Figure 6).

3. Planes in $\mathbb{R}^3_\infty$

Let (X, d) be a metric space, x and y be any points in X. Then denote the number of geodesics between x and y by $\tau(x, y)$. For example, let p and q be in $\mathbb{R}^3_\infty$, then $\tau(p, q) = 1$ if p and q are in a cubic diagonal position; otherwise $\tau(p, q) = \infty$.

Let (X, d) be a metric space, $x \in X$ and $\varepsilon \in \mathbb{R}^+$. We define

$$\nu(x, \varepsilon) = |\{y \in S_\varepsilon(x) \mid \tau(x, y) = 1\}|$$

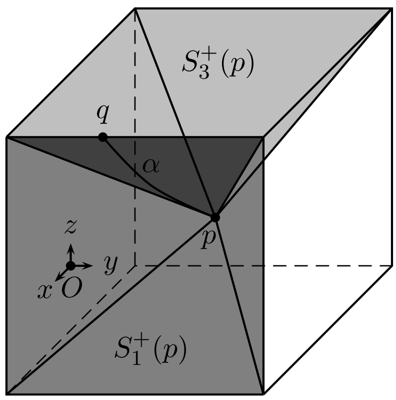

Figure 6. A geodesic α between the points p and q.

where $S_\varepsilon(x)$ is the boundary of the disc of x with radius ε:

$$S_\varepsilon(x) = \{y \in X \mid d(x, y) = \varepsilon\}.$$

One can easily prove the following two propositions:

Proposition 3.1. *Let (X, d_X), (Y, d_Y) be two metric spaces and $f : X \to Y$ be an isometry. Then we have*

$$\tau(x_1, x_2) = \tau(f(x_1), f(x_2))$$

for all $x_1, x_2 \in X$.

Proposition 3.2. *Let (X, d_X), (Y, d_Y) be two metric spaces and $f : X \to Y$ be an isometry. Then we have*

$$\nu(x, \varepsilon) = \nu(f(x), \varepsilon)$$

for all $x \in X$ and $\varepsilon \in \mathbb{R}^+$.

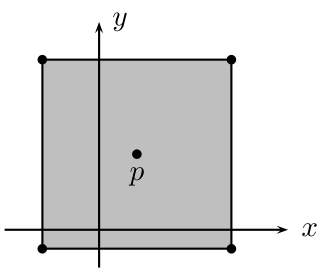

Figure 7. Four points on the boundary of the $\varepsilon-$disc of the point p which are connected to p by only one geodesic.

In the plane $\mathbb{R}^2_\infty$, for any point p in it and for any positive number ε, $\nu(p, \varepsilon) = 4$. These four points are the vertices of the boundary of the $\varepsilon-$disc of the point p which is a square (see Figure 7).

In $\mathbb{R}^3_\infty$, for any point p in it and for any positive number ε, $\nu(p, \varepsilon) = 8$. These eight points are the vertices of the boundary of the $\varepsilon-$disc of the point p which is a cube (see Figure 8).

Now we can ask a little more difficult question: Let p be an arbitrary element in the plane $\{(x, y, z) \in \mathbb{R}^3_\infty \mid x + y + z = 0\} \subseteq \mathbb{R}^3_\infty$ with induced maximum metric and ε be

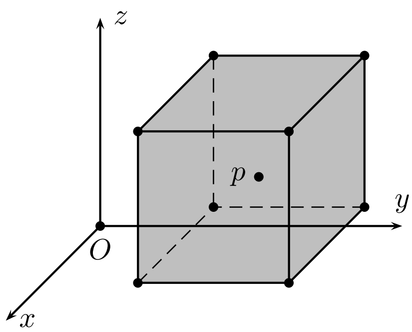

Figure 8. Eight points on the boundary of the $\varepsilon-$disc of the point p which are connected to p by only one geodesic.

any positive real number, then what is the number $\nu(p,\varepsilon)$? Maybe we must ask primarily what the boundary of the $\varepsilon-$disc of p is. Of course, it is the intersection of the plane and the $\varepsilon-$cube of p (i.e. the boundary of the $\varepsilon-$disc of p in $\mathbb{R}^3_\infty$). It is surprisingly a regular hexagon (see Figure 9).

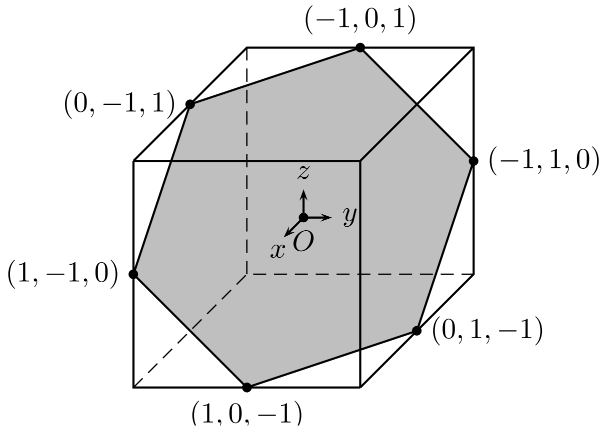

Figure 9. Disc of the origin with radius 1 in the plane $x + y + z = 0$ is a regular hexagon.

Obviously, the number $\nu(p,\varepsilon)$ is independent from p and ε. So we can take p as the origin and $\varepsilon = 1$. If q is a vertex on the hexagon, q belongs to two sectors of the origin; therefore, any geodesic between the points origin and q must be in the plane of intersection of these two sectors. (Note that all these intersection planes are $x = \pm y$, $x = \pm z$ and $y = \pm z$). Since the intersection of the former plane and the latter plane $x + y + z = 0$ is the line passing through the points q and the origin, there is only one geodesic between these points in the plane $x + y + z = 0$ which is the line segment. If q is a point on the hexagon and it is not the vertex, then q is contained by only one sector of the origin; thus, there are infinitely many geodesics between the points q and the origin in the plane $x + y + z = 0$. Hence, we have $\nu(p,\varepsilon) = 6$ for all points p in the plane $x + y + z = 0$ and positive real numbers ε. Then, Proposition 3.2 implies the following corollary:

Corollary 3.3. *The plane $x + y + z = 0$ in the $\mathbb{R}^3_\infty$ is not isometric to the plane $\mathbb{R}^2_\infty$.*

Note that the plane $ax+by+cz=d$ is isometric to the plane $ax+by+cz=0$ (to see this, consider the map $(x,y,z)\mapsto(x,y,z-\frac{d}{c})$); therefore, in order to classify all planes up to isometry in $\mathbb{R}^3_\infty$, it is enough to deal with the planes passing through the origin.

Theorem 3.4. *The plane $ax+by+cz=0$ in $\mathbb{R}^3_\infty$ is not isometric to the plane $\mathbb{R}^2_\infty$ if and only if the number $|a|$, $|b|$ and $|c|$ are the edges of a non-degenerate triangle i.e. the inequalities $|a|,|b|,|c|\neq 0$, $|a|+|b|>|c|$, $|a|+|c|>|b|$ and $|b|+|c|>|a|$ hold.*

Proof. ($\Rightarrow$) Suppose that the numbers $|a|$, $|b|$ and $|c|$ are not the edges of a non-degenerate triangle. If two of these numbers are equal to zero, then the plane is the $xy-$plane, the $xz-$plane or the $yz-$plane and they are obviously isometric to the plane $\mathbb{R}^2_\infty$. Now, consider the case where one of these numbers is equal to zero. Without loss of generality, we may assume that $c=0$ and $|a|\geq|b|$. Then, all points on the plane are in the form of $(x,\frac{-a}{b}x,z)$ and the mapping $(x,\frac{-a}{b}x,z)\mapsto(\frac{-a}{b}x,z)$ is an isometry from the plane to $\mathbb{R}^2_\infty$ because

$$|x_1-x_2|\leq\left|\frac{a}{b}\right|\cdot|x_1-x_2|$$

for all $x_1,x_2\in\mathbb{R}$.

Now, let $a\neq0$, $b\neq0$, $c\neq0$ and $|a|+|b|\leq|c|$. Then, all points on the plane are in the form of $(x,y,-\frac{ax+by}{c})$ and the mapping $(x,y,-\frac{ax+by}{c})\mapsto(x,y)$ is an isometry from the plane to $\mathbb{R}^2_\infty$ because

$$\begin{aligned}\left|\frac{ax_1+by_1}{c}-\frac{ax_2+by_2}{c}\right| &= \left|\frac{a(x_1-x_2)+b(y_1-y_2)}{c}\right|\\ &\leq \left|\frac{a}{c}\right|\cdot|x_1-x_2|+\left|\frac{b}{c}\right|\cdot|y_1-y_2|\\ &\leq \left(\left|\frac{a}{c}\right|+\left|\frac{b}{c}\right|\right)\cdot\max\{\,|x_1-x_2|,\,|y_1-y_2|\,\}\\ &\leq \max\{\,|x_1-x_2|,\,|y_1-y_2|\,\}\end{aligned}$$

for all $x_1,x_2,y_1,y_2\in\mathbb{R}$.

($\Leftarrow$) Let the number $|a|$, $|b|$ and $|c|$ be the edges of a non-degenerate triangle. Then, each face of the $1-$cube centered at the origin (i.e. $S_1(O)$ in $\mathbb{R}^3_\infty$) intersects the plane $ax+by+cz=0$ along a line segment. The reason for this is the following: For example, if we try to solve the equations $ax+by+cz=0$ and $x=1$ together, we can find infinitely many solutions for $y,z\in[-1,1]$ when $|b|+|c|>|a|$. This implies that the intersection of the plane and the square $x=1$ and $y,z\in[-1,1]$ is a line segment. As a result, intersection of the $1-$cube centered at the origin and the plane $ax+by+cz=0$ is a hexagon which is the boundary of a $1-$disc at the origin in the plane; hence, the number $\nu(O,1)=6$ and it is not isometric to the $\mathbb{R}^2_\infty$. □

Remark 3.5. Notice that Theorem 3.4 actually can be obtained using L. Nachbin's Theorem (Theorem 3 in [6]) and A. Moezzi's or M. Pavon's Theorem (Corollary 1.114 in [5] or Theorem 2.2 in [8]). But here we have given a more elementary proof for it.

Note that the intersection of a cube and any plane passing through the center of the cube is a tetragon or a hexagon. Actually, Theorem 3.4 says that if the intersection is a tetragon, then the plane is isometric to $\mathbb{R}^2_\infty$ and if the intersection is a hexagon, then the plane is not isometric to $\mathbb{R}^2_\infty$. Hence, all planes which have the tetragonal intersection with the $1-$cube centered at the origin are isometric to each other. Is it true for the others? That is, are all planes in $\mathbb{R}^3_\infty$ which are not isometric to $\mathbb{R}^2_\infty$ isometric to each other? Note that every tetragon which is the intersection of a cube and a plane passing through the center of the cube has equal side lengths (see Figure 10). But the situation of the planes which have hexagonal intersection is different. For example, $x+y+z=0$

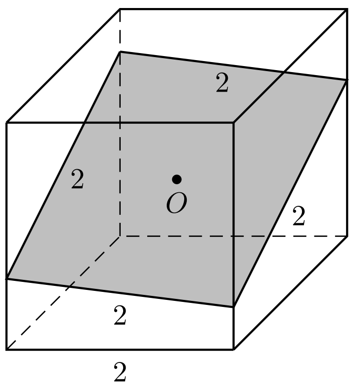

Figure 10. All edges of the tetragon above have the same length which is equal to length of any edge of the cube in $\mathbb{R}^3_\infty$.

and $2x + 2y + 3z = 0$ have different hexagons (see Figure 11). So, one can easily see that these two planes are not isometric. Actually, in order to determine the isometry classes of planes in $\mathbb{R}^3_\infty$, it is enough to determine the isometry classes of their hexagons (1−disc of the origin).

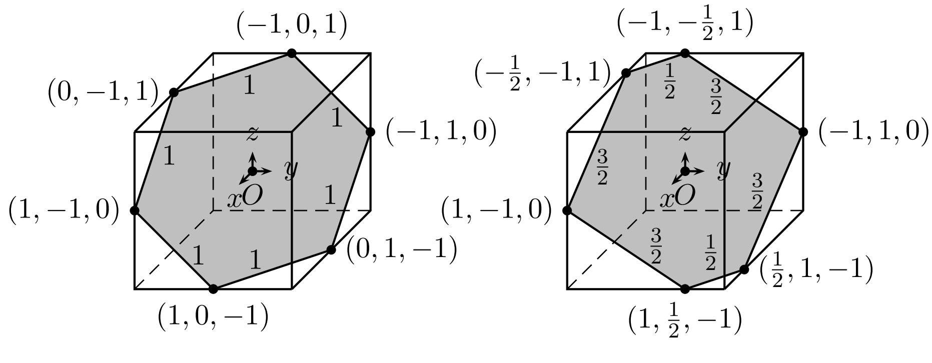

Figure 11. Two different planes (the left is $x + y + z = 0$ and the right is $2x + 2y + 3z = 0$) and their two different hexagons.

Lemma 3.6. *Let* $ax+by+cz = 0$ *be any plane in* $\mathbb{R}^3_\infty$. *Then the planes* $\pm ax \pm by \pm cz = 0$, $\pm ax \pm bz \pm cy = 0$, $\pm ay \pm bx \pm cz = 0$, $\pm az \pm bx \pm cy = 0$, $\pm ay \pm bz \pm cx = 0$ *and* $\pm az \pm by \pm cx = 0$ *are isometric to the plane* $ax + by + cz = 0$.

For the proof, we can consider the example of the map $(x, y, z) \mapsto (z, -y, x)$. This map is obviously an isometry from the plane $ax + by + cz = 0$ to the plane $az - by + cx = 0$. Indeed, all the mappings $(x, y, z) \mapsto (\pm w_1, \pm w_2, \pm w_3)$, where $\{w_1, w_2, w_3\} = \{x, y, z\}$, give us the needed isometries and more. This is because, for instance, $(x,y,z) \mapsto (-x,-y,-z)$ is an isometry from $ax+by+cz=0$ to itself. Note that there are exactly 48 such maps and these are actually all the elements of isometry group of the cube. For example, $(x,y,z) \mapsto (x,-z,y)$ is $\frac{\pi}{2}$ counter clockwise rotation around the axis x and $(x,y,z) \mapsto (x,-y,z)$ is the reflection with respect to the $xz-$plane. Denote this group by G and let X be the set of all planes passing through the origin in $\mathbb{R}^3_\infty$. Actually, X can be thought as the set of all hexagons (on the $1-$cube of $\mathbb{R}^3_\infty$) passing through the origin. Then, G acts on the X: Let the element $(x,y,z) \mapsto (w_1,w_2,w_3)$ be denoted by (w_1,w_2,w_3) and defined by

$$(w_1,w_2,w_3) \cdot (ax+by+cz=0) := aw_1+bw_2+w_3=0$$

where $\{w_1,w_2,w_3\} = \{\pm x, \pm y, \pm z\}$.

Thus, Lemma 3.6 can be restated as follows: A plane passing through the origin (in $\mathbb{R}^3_\infty$) is isometric to every plane which belongs to its orbit (according to group action above).

Let $ax+by+cz=0$ be an element of X that is not isometric to $\mathbb{R}^2_\infty$. Consider the case where $|a|, |b|$ and $|c|$ are distinct numbers. Then, the orbit of this plane has exactly 24 elements because its stabilizer contains only two elements: The identity and $(-x,-y,-z)$. What are these 24 isometric planes or their hexagons? Note that a hexagon can be determined by only one edge because the plane containing this edge and the origin is unique and the hexagon is the intersection of this plane and the cube. An example of the edges of 24 isometric planes is in Figure 12. It is easy to see that if only two of the numbers $|a|, |b|$ and $|c|$ are equal, then the orbit of this plane has exactly 12 elements and if $|a|=|b|=|c|$; that is, the plane is $x+y+z=0$, then its orbit has exactly 4 elements.

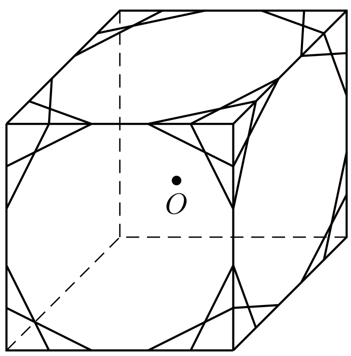

Figure 12. The edges of the 24 isometric hexagons (of the 24 isometric planes).

Theorem 3.7. *Let $|a|, |b|$ and $|c|$ be the edges of a non-degenerate triangle. Then the plane $ax+by+cz=0$ is isometric to only planes passing through the origin which belong to its orbit.*

Proof. Suppose that there exist two isometric planes such that one does not belong to the orbit of other one. Let us consider two hexagons which are the intersections of two planes and the cube, which is boundary of the disc of the origin with radius one. It is easy to see that all such hexagons have at least two edges whose lengths are less than or equal to one (according to the maximum metric). So we can assume that $A=(1,a,1)$ and $B_1=(b_1,1,1)$ are two vertexes of the first hexagon and $0 \le a \le b_1 < 1$ because one of the elements of orbit of this hexagon satisfies that. Note that $|AB_1| = \max\{1-a, 1-b_1\} =$

$1-a$. Since these two hexagons are isometric, the second hexagon has an edge whose length is equal to $1-a$. We can assume it is $|AB_2|$ with $B_2 = (b_2, 1, 1)$ and $a \le b_2 < 1$. Obviously $b_1 \ne b_2$, so let us assume $b_1 < b_2$. The equations of these planes are

$$(a-1)x + (b_1 - 1)y + (1 - ab_1)z = 0,$$

$$(a-1)x + (b_2 - 1)y + (1 - ab_2)z = 0.$$

Let us $D_1 = (1, -1, d_1)$ and $C_1 = (-1, 1, c_1)$ be vertices of the first hexagon and $D_2 = (1, -1, d_2)$ and $C_2 = (-1, 1, c_2)$ are vertices of the second one (see Figure 13). Since those two hexagons are isometric, either the equations $|AD_1| = |AD_2|$ and $|B_1C_1| = |B_2C_2|$ or the equations $|AD_1| = |B_2C_2|$ and $|AD_2| = |B_1C_1|$ must hold.

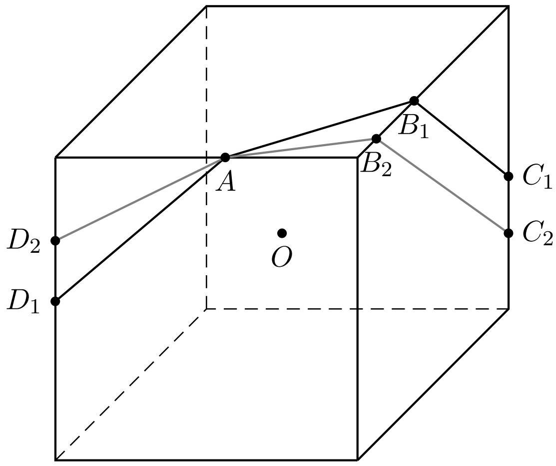

Figure 13. The first plane is the plane passing through A, B_1 and the origin O and the second one is the plane passing through A, B_2 and O.

By the equation of the first plane, we obtain

$$d_1 = \frac{b_1 - a}{1 - ab_1} \ge 0 \ \Rightarrow \ a + 1 \ge 1 - d_1 \ \Rightarrow \ |AD_1| = a + 1.$$

Similarly, we can get $|AD_2| = a + 1$. So, the equation $|B_1C_1| = |B_2C_2|$ must hold. By the equations of planes, we obtain

$$c_1 = -\frac{b_1 - a}{1 - ab_1}, \quad c_2 = -\frac{b_2 - a}{1 - ab_2}.$$

Since $a \le b_1 < b_2$, we get $c_2 < c_1 \le 0$ and this implies

$$|B_1C_1| = max\{b_1 + 1, 1 - c_1\} < max\{b_2 + 1, 1 - c_2\} = |B_2C_2|$$

and this contradiction completes the proof. □

Notice that Theorem 3.7 says that the number of the isometry classes of planes passing through the origin in $\mathbb{R}^3_\infty$ can be identified by the number of the similarity classes of triangles in the Euclidean plane. If the numbers $|a|$, $|b|$ and $|c|$ are not lengths of edges of a non-degenerate triangle, then the plane $ax + by + cz = 0$ is isometric to the $\mathbb{R}^2_\infty$; so, all such planes form one isometry class. If the numbers $|a|, |b|, |c|$ and $|a'|, |b'|, |c'|$ are lengths of edges of two non-degenerate triangles, then these triangles are similar if and only if the planes $ax + by + cz = 0$ and $a'x + b'y + c'z = 0$ are isometric. As a result, we get the following corollary:

Corollary 3.8. *Let* $ax + by + cz = d$ *be a plane in* $\mathbb{R}^3_\infty$.

i) *If the numbers* $|a|, |b|$ *and* $|c|$ *are not lengths of edges of a non-degenerate triangle, then this plane is isometric to the* $\mathbb{R}^2_\infty$*; hence, all such planes are isometric to each other.*

ii) *If the numbers* $|a|, |b|$ *and* $|c|$ *are lengths of edges of a non-degenerate triangle, then this plane is isometric to only planes whose equations can be written as* $aw_1 + bw_2 + cw_3 = D$ *where* $\{w_1, w_2, w_3\} = \{\pm x, \pm y, \pm z\}$ *and* D *is any real number.*

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