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Bernstein Collocation Method for Solving Linear Differential Equations

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ABSTRACT

In this study, a new collocation method based on Bernstein polynomials defined on the interval [*a*, *b*] is introduced for approximate solutions of initial and boundary value problems involving higher order linear differential equations with variable coefficients. Error analysis of the method is demonstrated. Some numerical solutions are given to illustrate the accuracy, efficiency and implementation of the method, and the results of the proposed method are also compared with the other methods in several examples.

Key Words: Bernstein polynomial approximation, Linear differential equations, Collocation method

1. INTRODUCTION

Differential equations, which describe how quantities change across time or space, arise naturally in science, engineering, and in almost every field of study where measurements can be taken. Most realistic mathematical models cannot be solved through the traditional pencil-and-paper techniques providing an excellent means to put across the underlying theory; instead, they must be dealt with the computational methods that deliver approximate solutions.

Polynomials have played a central role in approximation theory and numerical analysis for many years. They are useful mathematical tools as they are precisely defined, calculated rapidly on a modern computer system and can represent a great variety of functions. Moreover they can be differentiated and integrated simply.

Bernstein polynomials have many useful properties such as the positivity, the continuity, recursion's relation, symmetry and unity partition of the basis set over the interval [0, 1], [1]. For this reason, they have been studied in an enormous number of publications, and are frequently used both in approximation theory and computer aided geometric design [2].

In recent years, many researchers have been interested with the Bernstein polynomials. These polynomials have been utilized for solving several equations by using various numerical methods. For example; Bernstein polynomials have been used for solving differential equations [3-8], integral equations [9, 10], partial differential equations [11-13], integro-differential equations [14]. Especially, Işık and et al. [6] have introduced a new method to solve high order linear differential equations with initial and boundary conditions. The method is numerically based on rational interpolation and Bernstein series solution depending on collocation method. Doha and et al. [4, 5] have proved new formulas about derivatives and integrals of Bernstein polynomials, and have used the Galerkin and Petrov-Galerkin methods based on Bernstein polynomials for solving high even-order differential equations. Bhatti and Bracken [3] have given solutions of linear and non-linear differential equations with linear combinations of Bernstein polynomials, and their coefficients have been determined by Galerkin method. Ordokhani and Davaei far [8] have proposed an operational matrix by an expansion of Bernstein polynomials in terms of Legendre polynomials for solving differential equations.

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The Bernstein polynomials and their basis form that can be generalized on the interval [*a*, *b*], are defined as follows:

Definition 1.1 Generalized Bernstein basis polynomials can be defined on the interval [*a*, *b*] by

$$p_{i,n}(x)=\frac{1}{(b-a)^n}\binom{n}{i}(x-a)^i(b-x)^{n-i};\quad i=0,1,\dots,n.$$

For convenience, we set $p_{i,n}(x)=0$, if $i<0$ or $i>n$.

We give the properties of the generalized Bernstein basis polynomials in the following list:

(a) Positivity property:

$p_{i,n}(x)>0$ is hold for all $i=0,1,\dots,n$ and all $x\in[a,b]$.

(b) Unity partition property:

$$\sum_{i=0}^{n}p_{i,n}(x)=\sum_{i=0}^{n-1}p_{i,n-1}(x)=\ldots=\sum_{i=0}^{1}p_{i,1}(x)=1.$$

(c) Recursion's relation property:

$$p_{i,n}(x)=\tfrac{1}{b-a}\left[(b-x)p_{i,n-1}(x)+(x-a)p_{i-1,n-1}(x)\right].$$

Definition 1.2 Let $y:[a,b]\to\mathbb{R}$ be continuous function on the interval [*a*, *b*]. Generalized Bernstein polynomials of degree *n* are defined by

$$B_n(y;x)=\sum_{i=0}^{n}y\left(a+\tfrac{b-a}{n}i\right)p_{i,n}(x).$$

Theorem 1.1 If $y\in C^k[a,b]$, for some integer $m\geq 0$, then

$$\lim_{n\to\infty}B_n^{(k)}(y;x)=y^{(k)}(x);\quad k=0,1,\dots,m$$

converges uniformly.

For more information about Bernstein polynomials defined on the interval [0, 1], see [15].

In this paper, the purpose is to approximate the solution of *m*th-order linear differential equations

$$\sum_{k=0}^{m}a_k(x)y^{(k)}(x)=f(x),\quad a\leq x\leq b \tag{1}$$

under the initial conditions

$$\sum_{k=0}^{m-1}\lambda_{jk}y^{(k)}(c)=\mu_j;\ c\in[a,b] \tag{2}$$

or boundary conditions

$$\sum_{k=0}^{m-1}\left[\alpha_{jk}y^{(k)}(a)+\beta_{jk}y^{(k)}(b)\right]=\gamma_j \tag{3}$$

for $j=0,1,\dots,m-1$, with the generalized Bernstein polynomials:

$$y^{(k)}(x)\cong B_n^{(k)}(y;x)=\sum_{i=0}^{n}y\left(a+\tfrac{(b-a)i}{n}\right)p_{i,n}^{(k)}(x). \tag{4}$$

Here $a_k(x)$ and $f(x)$ are continuous functions on the interval [*a*, *b*], α_{jk}, β_{jk}, λ_{jk}, μ_j and γ_j are known constants, and $y(x)$ is an unknown function.

The paper is organized as follows: In Section 2, some fundamental relations are given and a new relation that can be qualified as key between the generalized Bernstein basis polynomials and its derivatives is introduced. Then, Bernstein collocation method is discussed in Section 3. In Section 4, convergence of the method is analyzed and some errors are defined. In Section 5, the presented method is applied to three problems and numerical results are compared with the other methods for showing the accuracy and efficiency of the proposed method. The Section 6 is ended with the conclusions.

2. FUNDAMENTAL RELATIONS

Theorem 2.1 On the interval [*a*, *b*], any generalized Bernstein basis polynomials of degree *n* can be written as a linear combination of the generalized Bernstein basis polynomials of degree *n* + 1:

$$p_{i,n}(x)=\frac{n-i+1}{n+1}p_{i,n+1}(x)+\frac{i+1}{n+1}p_{i+1,n+1}(x).$$

Proof. By using Definition 1.1, we have

$$\begin{aligned}\frac{x-a}{b-a}p_{i,n}(x)&=\frac{x-a}{b-a}\frac{1}{(b-a)^n}\binom{n}{i}(x-a)^i(b-x)^{n-i}\\&=\frac{\binom{n}{i}/\binom{n+1}{i+1}}{(b-a)^{n+1}}\binom{n+1}{i+1}(x-a)^{i+1}(b-x)^{n+1-(i+1)}\\&=\frac{\binom{n}{i}}{\binom{n+1}{i+1}}p_{i+1,n+1}(x)=\frac{i+1}{n+1}p_{i+1,n+1}(x)\end{aligned}$$

and

$$\begin{aligned}\left(1-\frac{x-a}{b-a}\right)p_{i,n}(x)&=\frac{b-x}{b-a}p_{i,n}(x)\\&=\frac{b-x}{b-a}\frac{1}{(b-a)^n}\binom{n}{i}(x-a)^i(b-x)^{n-i}\\&=\frac{\binom{n}{i}/\binom{n+1}{i}}{(b-a)^{n+1}}\binom{n+1}{i}(x-a)^i(b-x)^{n+1-i}\\&=\frac{\binom{n}{i}}{\binom{n+1}{i}}p_{i,n+1}(x)=\frac{n-i+1}{n+1}p_{i,n+1}(x).\end{aligned}$$

By summing both sides of these expressions, we have desired result.

Theorem 2.2 The derivatives of *n*th-degree generalized Bernstein basis polynomials are given by:

$$\frac{d}{dx}p_{i,n}(x)=\frac{n}{b-a}\left[p_{i-1,n-1}(x)-p_{i,n-1}(x)\right] \tag{5}$$

for $i=0,1,\dots,n$.

Proof. By using Definition 1.1, this expression can be obtained as:

$$
\begin{aligned}
\frac{d}{dx}p_{i,n}(x) &= \frac{d}{dx}\left(\frac{1}{(b-a)^n}\binom{n}{i}(x-a)^i(b-x)^{n-i}\right)\\
&= \frac{1}{(b-a)^n}\binom{n}{i}\left[i(x-a)^{i-1}(b-x)^{n-i}-(n-i)(x-a)^i(b-x)^{n-i-1}\right]\\
&= \frac{n}{(b-a)^n}\left[\binom{n-1}{i-1}(x-a)^{i-1}(b-x)^{n-i}-\binom{n-1}{i}(x-a)^i(b-x)^{n-i-1}\right]\\
&= \frac{n}{b-a}\left[\binom{n-1}{i-1}\frac{1}{(b-a)^{n-1}}(x-a)^{i-1}(b-x)^{n-1-(i-1)}-\binom{n-1}{i}\frac{1}{(b-a)^{n-1}}(x-a)^i(b-x)^{(n-1)-i}\right]\\
&= \frac{n}{b-a}\left[p_{i-1,n-1}(x)-p_{i,n-1}(x)\right].
\end{aligned}
$$

Theorem 2.3 The first derivatives of nth-degree generalized Bernstein basis polynomials can be written as a linear combination of the generalized Bernstein basis polynomials of degree n:

$$p_{i,n}(x) = \frac{1}{b-a}\left[(n-i+1)\,p_{i-1,n}(x)+(2i-n)\,p_{i,n}(x)-(i+1)\,p_{i+1,n}(x)\right].$$

Proof. By utilizing Theorem 2.1, the following equalities can be written as

$$
\begin{aligned}
p_{i,n-1}(x) &= \frac{n-i}{n}p_{i,n}(x)+\frac{i+1}{n}p_{i+1,n}(x),\\
p_{i-1,n-1}(x) &= \frac{n-i+1}{n}p_{i-1,n}(x)+\frac{i}{n}p_{i,n}(x).
\end{aligned}
$$

Substituting these relations into the right hand side of the expression (5), the desired relation is obtained.

Theorem 2.4 There is a relation between generalized Bernstein basis polynomials matrix and their derivatives of the form

$$\mathbf{P}^{(k)}(x) = \mathbf{P}(x)\mathbf{N}^k;\ k = 1,2,\ldots,m.$$

Here the elements of $(n + 1) \times (n + 1)$ matrix $\mathbf{N} = (m_{ij})$, $i,j = 0,1,\ldots,n$ are defined by:

$$m_{ij} = \frac{1}{b-a}\begin{cases} n-i, & \text{if } \ j=i+1\\ 2i-n, & \text{if } \ j=i\\ -i, & \text{if } \ j=i-1\\ 0, & \text{otherwise}\end{cases}.$$

Proof. From Theorem 2.3 and condition $p_{i,n}(x)=0$ if $i<0$ or $i>0$, we have

$$
\begin{aligned}
p'_{0,n}(x) &= \frac{1}{b-a}\left[-np_{0,n}(x)-p_{1,n}(x)\right]\\
p'_{1,n}(x) &= \frac{1}{b-a}\left[np_{0,n}(x)+(2-n)p_{1,n}(x)-2p_{2,n}(x)\right]\\
p'_{2,n}(x) &= \frac{1}{b-a}\left[(n-1)p_{1,n}(x)+(4-n)p_{2,n}(x)-3p_{3,n}(x)\right]\\
&\ \ \vdots\\
p'_{n-1,n}(x) &= \frac{1}{b-a}\left[2p_{n-2,n}(x)+(n-2)p_{n-1,n}(x)-np_{n,n}(x)\right]\\
p'_{n,n}(x) &= \frac{1}{b-a}\left[p_{n-1,n}(x)+np_{n,n}(x)\right].
\end{aligned}
$$

Hence we obtain the matrix relation

$$\mathbf{P}'(x) = \mathbf{P}(x)\mathbf{N}$$

such that

$$
\begin{aligned}
\mathbf{P}(x) &= \left[p_{0,n}(x)\quad p_{1,n}(x)\quad \ldots\quad p_{n,n}(x)\right],\\
\mathbf{P}'(x) &= \left[p'_{0,n}(x)\quad p'_{1,n}(x)\quad \ldots\quad p'_{n,n}(x)\right],
\end{aligned}
$$

$$\mathbf{N} = \frac{1}{b-a}\begin{bmatrix} -n & n & 0 & \ldots & 0 & 0 & 0\\ -1 & 2-n & n-1 & \ldots & 0 & 0 & 0\\ 0 & -2 & 4-n & \ldots & 0 & 0 & 0\\ 0 & 0 & -3 & \ldots & 0 & 0 & 0\\ \vdots & & & \vdots & \ddots & & \vdots\\ 0 & 0 & 0 & \ldots & n-4 & 2 & 0\\ 0 & 0 & 0 & \ldots & 1-n & n-2 & 1\\ 0 & 0 & 0 & \ldots & 0 & -n & n\end{bmatrix}.$$

In a similar way, the second derivative becomes

$$\mathbf{P}''(x) = \mathbf{P}'(x)\mathbf{N} = \mathbf{P}(x)\mathbf{N}^2.$$

Thus we get derivatives of the generalized Bernstein basis polynomials in the form

$$\mathbf{P}^{(k)}(x) = \mathbf{P}^{(k-1)}(x)\mathbf{N} = \mathbf{P}(x)\mathbf{N}^k.$$

This completes the proof.

3. METHOD OF SOLUTION

The main idea of the Bernstein collocation method is to seek a solution of the problem in the form of the Bernstein polynomials. For this reason, a higher order linear differential equation with variable coefficients is satisfied by the Bernstein polynomials at the collocation points. Therefore the main matrix equation is obtained as follows:

Theorem 3.1 Let $x_i \in [a,b]$; $i = 0,1,\ldots,n$ be collocation points. General mth-order linear non-homogen differential equation (1) can be written as the matrix form

$$\sum_{k=0}^{m}\mathbf{A}_k\mathbf{P}\mathbf{N}^k\mathbf{Y} = \mathbf{F}. \tag{6}$$

Here the matrices are $\mathbf{Y}=\left[y\left(a+(b-a)i/n\right)\right]$, $\mathbf{A}_k=\mathrm{diag}\left[a_k\left(x_i\right)\right]$, $\mathbf{P}=\left[p_{j,n}\left(x_i\right)\right]$ and $\mathbf{F}=\left[f\left(x_i\right)\right]$; $i,j=0,1,\ldots,n$.

Proof. The expression (4) can be denoted by the matrix form

$$y^{(k)}(x)\cong B_n^{(k)}\left(y;x\right)=\mathbf{P}^{(k)}(x)\mathbf{Y} .$$

By utilizing Theorem 2.4, the derivatives of the unknown functions can also be written by

$$y^{(k)}(x)\cong \mathbf{P}(x)\mathbf{N}^k\mathbf{Y};\quad k=0,1,...,m. \tag{7}$$

Substituting the collocation points and relation (7) into equation (1), we obtain the linear algebraic equation system

$$\sum_{k=0}^{m}a_k\left(x_i\right)\mathbf{P}\left(x_i\right)\mathbf{N}^k\mathbf{Y}=f\left(x_i\right);\quad i=0,\ldots,n$$

such that $y^{(k)}(x_i)=B_n^{(k)}\left(y;x_i\right)$. This equation system can be denoted by the matrix form (6) and the proof is completed.

We can solve the differential equation with variable coefficients (1) under the conditions (2) or (3) as following the steps:

Step 1. The equation (6) can be written in the compact form

$$\mathbf{WY}=\mathbf{F}\text{ or }\left[\mathbf{W};\mathbf{F}\right] \tag{8}$$

so that $\mathbf{W}=\sum_{k=0}^{m}\mathbf{A}_k\mathbf{P}\mathbf{N}^k$. This matrix equation (8) corresponds to a linear algebraic system with unknown coefficients $y_0,y_1,\ldots,y_n$.

Step 2. From expression (7), matrix forms of the conditions (2) and (3) can be written respectively

$$\mathbf{V}_j=\sum_{k=0}^{m-1}\lambda_{jk}\mathbf{P}(c)\mathbf{N}^k=\begin{bmatrix}v_{j,0} & v_{j,1} & \ldots & v_{j,n}\end{bmatrix},$$

$$\mathbf{U}_j=\sum_{k=0}^{m-1}[\alpha_{jk}\mathbf{P}(a)\mathbf{N}^k+\beta_{jk}\mathbf{P}(b)\mathbf{N}^k]$$
$$=\begin{bmatrix}u_{j,0} & u_{j,1} & \ldots & u_{j,n}\end{bmatrix}$$

or implicitly

$$\mathbf{V}_j\mathbf{Y}=\mu_{\mathbf{j}}\quad\text{or}\quad[\mathbf{V}_j;\mu_j], \tag{9}$$

$$\mathbf{U}_j\mathbf{Y}=\mu_{\mathbf{j}}\quad\text{or}\quad[\mathbf{U}_j;\mu_j]. \tag{10}$$

Step 3. To obtain the solution of equation (1) under the conditions (2) or (3), we add the elements of the row matrices (9) or (10) to the end of the matrix (8). In this way, we have the new augmented matrix $\left[\tilde{\mathbf{W}};\tilde{\mathbf{F}}\right]$. Here the augmented matrix is a $(n+m+1)\times(n+1)$ rectangular matrix. This new matrix equation shortly can be denoted by $\tilde{\mathbf{W}}\mathbf{Y}=\tilde{\mathbf{F}}$.

Step 4. If $rank\left(\tilde{\mathbf{W}}\right)=rank\left[\tilde{\mathbf{W}};\tilde{\mathbf{F}}\right]=n+1$, then unknown coefficients $y_i;\ i=0,1,\ldots,n$ are uniquely determined. These kinds of systems can be solved by the Gauss Elimination, Generalized Inverse and QR factorization methods.

4. ERROR ANALYSIS

Definition 4.1 Error of approximation is denoted by $e_n(x)=y(x)-B_n\left(y;x\right)$ such that $y(x)$ is an exact solution and $B_n\left(y;x\right)$ is a Bernstein approximate solution. Then the maximum error can be defined as

$$E_n\left(y;[a,b]\right)=E_n(y)=\left\|e_n(x)\right\|_\infty=\max_{a\le x\le b}|e_n(x)|,$$

and on the collocation points, maximum and mean error can also be numerically computed respectively by

$$E_{\max}=\max_{x_i}|e_n\left(x_i\right)|,\quad E_{\text{mean}}=\tfrac{1}{n}\sum_{i=1}^{n}\left|e_n\left(x_i\right)\right|.$$

Besides, the relative error at the points x_i is the number

$$E_{\text{rel}}=\left|e_n\left(x_i\right)\right|/y\left(x_i\right)$$

such that $y(x_i)\neq 0$.

Definition 4.2 If we obtain not the exact solution y but an approximate solution y_n, one can test y_n by substituting equation (1) to see whether it is closed to y. Thus we obtain the residual error as noted below:

$$R_n(x)=\sum_{k=0}^{m}\left|a_k(x)B_n^{(k)}(y;x)-f(x)\right|.$$

Definition 4.3 Let $y(x)$ be defined on the interval $[a, b]$. The modulus of continuity of $y(x)$ on $[a, b]$, $\omega(\delta)$, is defined for $\delta>0$ by

$$\omega\left(\delta\right)=\sup_{\substack{x_1,x_2\in[a,b]\\ |x_1-x_2|\le\delta}}|y(x_1)-y(x_2)|.$$

Lemma 4.1 If $\lambda>0$, then $\omega\left(\lambda\delta\right)=\left(1+\lambda\right)\omega\left(\delta\right)$. [16]

Lemma 4.2 $y(x)$ is uniformly continuous on the interval $[a, b]$ if and only if $\lim_{\delta\to 0}\omega\left(\delta\right)=0$. [16]

Lemma 4.3 Let the generalized Bernstein basis polynomials be defined on the interval $[a, b]$. Then we have

$$\sum_{i=0}^{n}ip_{i,n}\left(x\right)=\frac{n\left(x-a\right)}{b-a},$$

$$\sum_{i=0}^{n}i^2p_{i,n}\left(x\right)=\frac{n\left(n-1\right)\left(x-a\right)^2}{\left(b-a\right)^2}+\frac{n\left(x-a\right)}{b-a}.$$

Proof. From property (b) and Definition 1.1, desired expressions can be written as respectively,

$$\sum_{i=0}^{n} ip_{i,n}(x)=\frac{1}{(b-a)^n}\sum_{i=0}^{n} i\binom{n}{i}(x-a)^i(b-x)^{n-i}$$
$$=\frac{1}{(b-a)^n}\sum_{i=1}^{n}\frac{n!}{(i-1)!(n-i)!}(x-a)^i(b-x)^{n-i}$$
$$=\frac{n(x-a)}{b-a}\sum_{i=0}^{n-1}\frac{1}{(b-a)^{n-1}}\binom{n-1}{i}(x-a)^i(b-x)^{n-1-i}$$
$$=\frac{n(x-a)}{b-a}\sum_{i=0}^{n-1}p_{i,n-1}(x)$$
$$=\frac{n(x-a)}{b-a},$$

$$\sum_{i=0}^{n} i^2 p_{i,n}(x)=\frac{1}{(b-a)^n}\sum_{i=0}^{n} i^2\binom{n}{i}(x-a)^i(b-x)^{n-i}$$
$$=\frac{n}{(b-a)^n}\left\{\sum_{i=1}^{n}\frac{(i-1)(n-1)!}{(i-1)!(n-i)!}(x-a)^i(b-x)^{n-i}+\sum_{i=1}^{n}\frac{(n-1)!}{(i-1)!(n-i)!}(x-a)^i(b-x)^{n-i}\right\}$$
$$=\frac{n}{(b-a)^n}\left\{\sum_{i=2}^{n}\frac{(n-1)!}{(i-2)!(n-i)!}(x-a)^i(b-x)^{n-i}+\sum_{i=0}^{n-1}\frac{(n-1)!}{(i)!(n-1-i)!}(x-a)^{i+1}(b-x)^{n-i-1}\right\}$$
$$=\frac{n(n-1)(x-a)^2}{(b-a)^2}\left\{\sum_{i=0}^{n-2}\binom{n-2}{i}\frac{1}{(b-a)^{n-2}}(x-a)^i(b-x)^{n-i-2}\right\}+\frac{n(x-a)}{b-a}\left\{\sum_{i=0}^{n-1}\binom{n-1}{i}\frac{1}{(b-a)^{n-1}}(x-a)^i(b-x)^{n-i-1}\right\}$$
$$=\frac{n(n-1)(x-a)^2}{(b-a)^2}\sum_{i=0}^{n-2}p_{i,n-2}(x)+\frac{n(x-a)}{b-a}\sum_{i=0}^{n-1}p_{i,n-1}(x)$$
$$=\frac{n(n-1)(x-a)^2}{(b-a)^2}+\frac{n(x-a)}{b-a}.$$

Lemma 4.4 Generalized Bernstein basis polynomials defined on the interval [a, b] have the following relation:

$$\sum_{i=0}^{n}\left(x-\left(a+\tfrac{b-a}{n}i\right)\right)^2 p_{i,n}(x)=\frac{(x-a)(b-x)}{n}.$$

Proof. From Lemma 4.3 and property (b), we obtain

$$\sum_{i=0}^{n}\left(x-\left(a+\tfrac{(b-a)i}{n}\right)\right)^2 p_{i,n}(x)=\frac{1}{n^2}\sum_{i=0}^{n}\left(n(x-a)-(b-a)i\right)^2 p_{i,n}(x)$$
$$=(x-a)^2\sum_{i=0}^{n}p_{i,n}(x)-\frac{2(x-a)(b-a)}{n}\sum_{i=0}^{n}ip_{i,n}(x)+\frac{(b-a)^2}{n^2}\sum_{i=0}^{n}i^2p_{i,n}(x)$$
$$=(x-a)^2-2(x-a)^2+\tfrac{1}{n}\left[(n-1)(x-a)^2+(x-a)(b-a)\right]$$
$$=(x-a)(b-x)/n.$$

Theorem 4.2 Let $B_n y$ be Bernstein approximate solution on the interval [a, b]. If exact solution $y(x)$ is continuous on the interval [a, b], then

$$\lim_{n\to\infty}\left\|y-B_n y\right\|_\infty=0.$$

Proof. Considering the properties of generalized Bernstein basis polynomials and modulus of continuity of $y(x)$, the absolute error can be written as

$$
\begin{aligned}
\left|y(x)-B_n(y;x)\right| &= \left|y(x)-\sum_{i=0}^{n} y\left(a+\tfrac{b-a}{n}i\right)p_{i,n}(x)\right| \\
&\le \sum_{i=0}^{n}\left|y(x)-y\left(a+\tfrac{b-a}{n}i\right)\right|p_{i,n}(x) \\
&\le \sum_{i=0}^{n}\max_{\left|x-a-\frac{(b-a)i}{n}\right|\le\delta}\left|y(x)-y\left(a+\tfrac{b-a}{n}i\right)\right|p_{i,n}(x) \\
&\le \sum_{i=0}^{n}\omega\left(\left|x-\left(a+\tfrac{b-a}{n}i\right)\right|\right)p_{i,n}(x) \\
&\le \sum_{i=0}^{n}\omega\left(n^{-1/2}\right)\left(1+\sqrt{n}\left|x-\left(a+\tfrac{b-a}{n}i\right)\right|\right)p_{i,n}(x) \\
&\le \omega\left(n^{-1/2}\right)\left\{\sum_{i=0}^{n}p_{i,n}(x)+\sqrt{n}\sum_{i=0}^{n}\left|x-\left(a+\tfrac{b-a}{n}i\right)\right|p_{i,n}(x)\right\} \\
&\le \omega\left(n^{-1/2}\right)\left\{1+\sqrt{n}\sum_{i=0}^{n}\left|x-\left(a+\tfrac{b-a}{n}i\right)\right|p_{i,n}(x)\right\}
\end{aligned}
$$

Applying Cauchy-Schwarz inequality to the right hand side of the sum expression and using Lemma 4.4, we have

$$
\begin{aligned}
\left|y(x)-B_n(y;x)\right| &\le \omega\left(n^{-1/2}\right)\left\{1+\sqrt{n}\sum_{i=0}^{n}\left|x-\left(a+\tfrac{b-a}{n}i\right)\right|\sqrt{p_{i,n}(x)}\sqrt{p_{i,n}(x)}\right\} \\
&\le \omega\left(n^{-1/2}\right)\left\{1+\sqrt{n}\sqrt{\sum_{i=0}^{n}\left(x-\left(a+\tfrac{b-a}{n}i\right)\right)^2 p_{i,n}(x)}\sqrt{\sum_{i=0}^{n}p_{i,n}(x)}\right\} \\
&\le \omega\left(n^{-1/2}\right)\left\{1+\sqrt{n}\sqrt{\frac{(x-a)(b-x)}{n}}\right\} \\
&\le \omega\left(n^{-1/2}\right)\left(1+\sqrt{(x-a)(b-x)}\right).
\end{aligned}
$$

Therefore, the maximum error is obtained as

$$
\begin{aligned}
\left\|y-B_n y\right\|_\infty &= \max_{a\le x\le b}\left|y(x)-B_n(y;x)\right| \\
&=\omega\left(n^{-1/2}\right)\left[1+\max_{a\le x\le b}\left(\sqrt{(x-a)(b-x)}\right)\right] \\
&=\omega\left(n^{-1/2}\right)\left[1+\tfrac{b-a}{2}\right].
\end{aligned}
$$

Since $y(x)$ is continuous on the interval $[a, b]$ in view of Lemma 4.2, $\omega\left(n^{-1/2}\right)\to\infty$ as $n\to\infty$, and theorem is proved.

5. NUMERICAL RESULTS

Three numerical examples are considered by using the presented method on the collocation points $x_i = a+(b-a)i/n;\ i=0,1,\ldots n$. Numerical results obtained the Bernstein collocation method, using the algorithm written in MATLAB 7.1, are compared with the other methods.

Example 5.1 Consider the

$$
\begin{aligned}
&y''+xy'-2y = x\cos x-3\sin x,\ 0\le x\le 1; \\
&y(0)=0,\ y'(0)=1
\end{aligned}
$$

initial value problem that the exact solution is $y=\sin(x)$.

Using the Bernstein collocation method, the mean errors are given in Table 1. The maximum errors are also compared with the Rational Bernstein approximation [6] in Table 2. Table 1 and Table 2 show that our method is very effective and more accurate than the other method.

Table 1. Mean errors of Example 5.1.

n	E_{mean}	n	E_{mean}
5	9.6*e*-006	15	1.3*e*-015
6	8.6*e*-007	16	3.0*e*-015
7	1.7*e*-008	17	9.5*e*-015
8	1.3*e*-009	18	4.7*e*-016
9	2.2*e*-011	19	7.7*e*-015
10	1.4*e*-012	20	2.6*e*-015

Table 2. Comparison the maximum errors for Ex. 5.1.

n	Presented Method	Rational [6]
5	2.1*e*-005	7.2*e*-003
7	3.7*e*-008	4.2*e*-004
10	3.1*e*-012	3.6*e*-008

Example 5.2 For $x \in [-1, 1]$, consider the following boundary value problem:

$$y^{(6)} + (5x+1)y = \left(185x - 25x^2 + 10x^4\right)\cos(x) + \left(270 - 36x^2\right)\sin(x)$$
$$y(-1) = 4\cos(1), \quad y(1) = -2\cos(1),$$
$$y'(-1) = \cos(1) + 4\sin(1), \quad y'(1) = \cos(1) + 2\sin(1),$$
$$y''(-1) = -16\cos(1) + 2\sin(1), \; y''(1) = 14\cos(1) - 2\sin(1).$$

The analytic solution of the above equation is $y(x) = (2x^3 - 5x + 1)\cos(x)$.

Table 3. Mean errors of Example 5.2.

n	E_{mean}	n	E_{mean}
12	4.7*e*-006	24	1.5*e*-014
14	6.1*e*-014	28	1.4*e*-014
16	4.1*e*-015	30	7.3*e*-015
18	1.5*e*-015	32	7.5*e*-015
20	4.1*e*-015	35	2.6*e*-014
22	2.4*e*-015	40	1.5*e*-014

Table 4. Comparison the maximum errors for Ex. 5.2.

n	Presented Method	Septic [17]
16	1.0*e*-013	1.2*e*-004
32	1.3*e*-013	1.6*e*-005
64	2.9*e*-010	3.8*e*-006
128	7.3*e*-008	9.5*e*-007

In Table 3, the mean errors are computed with increasing *n*. The maximum errors are compared with the Septic spline method [17] in Table 4. Tables show that the presented method converges more rapidly than the other method for especially small *n*.

Example 5.3 Consider the following boundary value problem:

$$y^{(6)} + e^{-x}y = \left(x - x^2\right)^3 e^{-x} - 720; \quad 0 \le x \le 1$$
$$y(0) = y'(0) = y''(0) = y(1) = y'(1) = y''(1) = 0.$$

The exact solution of the problem is $y(x) = x^3(1-x)^3$.

Using the presented method, the mean errors are given in Table 5. The absolute relative errors are compared with the Sinc-Galerkin method [18] in Table 6. Table 5 and Table 6 show that the proposed method has high accuracy, and better results than the other method for various points *x*.

Table 5. Mean errors of Example 5.3.

n	E_{mean}	n	E_{mean}
6	1.4*e*-014	16	7.5*e*-018
8	5.5*e*-014	18	1.9*e*-017
10	1.1*e*-014	20	1.7*e*-018
12	1.4*e*-015	22	2.7*e*-017
14	6.6*e*-017	24	3.9*e*-017
15	1.4*e*-018	25	1.3*e*-017

Table 6. Comparison of the absolute relative errors.

x	Presented Method			Sinc-Galerkin
	$n=6$	$n=15$	$n=65$	$m=65$
0.167	8.0*e*-012	6.1*e*-016	5.5*e*-012	4.5*e*-004
0.276	2.6*e*-012	5.0*e*-016	3.5*e*-012	3.2*e*-004
0.345	1.6*e*-012	4.4*e*-016	1.3*e*-012	2.8*e*-004
0.420	9.9*e*-013	9.9*e*-017	4.8*e*-012	2.6*e*-004
0.5	5.6*e*-013	6.3*e*-020	4.1*e*-012	2.5*e*-016
0.655	2.5*e*-013	1.5*e*-016	7.2*e*-013	2.8*e*-004
0.782	9.2*e*-013	1.8*e*-016	6.7*e*-012	3.7*e*-004
0.832	4.6*e*-012	3.4*e*-016	1.4*e*-011	4.5*e*-004
0.904	2.2*e*-011	7.5*e*-016	7.4*e*-011	7.3*e*-004

6. CONCLUSIONS

In this study, a collocation method based on the generalized Bernstein polynomials has been developed for the solution of higher order differential equations under the initial or boundary conditions. If $y(x)$ and its derivatives are continuous functions on bounded interval $[a, b]$, then the method can be applied to any initial or boundary value problems. In collocation method, the residual error R_n is forced to become zero at the $n+1$ collocation points $x_i = a + (b-a)i/n$; $i = 0,1,\ldots,n$ to evaluate the $n+1$ unknown constants $y(x_i)$. If these values are not equal to zero, this case is rounding error resulted from computer. Proposed method presents useful advantages as it construct the main matrix equation simply and it is applicable for algorithms based on computer. Moreover, this method has been tested on three problems, and numerical results have been compared with the other methods. Consequently, all of the reasons are revealed that the proposed method is very effective and encouraging.

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REFERENCES

[1] Farouki, R.T. and Rajan, V.T., Algorithms for polynomials in Bernstein form, ***Computer Aided Geometric Design***, 5: 1-26, (1988).

[2] Farin, G., Curves and surfaces for CAGD fifth edition, ***Academic Press***, United States of America, (2002).

[3] Bhatti, M.I. and Bracken, P., Solutions of differential equations in a Bernstein polynomial basis, ***Journal of Computational and Applied Mathematics***, 205: 272-280, (2007).

[4] Doha, E.H., Bhrawy, A.H. and Saker, M.A., On

the derivatives of Bernstein polynomials: An application for the solution of high even-order differential equations, ***Boundary Value Problems***, 2011: 1-16, (2011).

[5] Doha, E. H., Bhrawy, A. H. and Saker, M. A., Integrals of Bernstein polynomials: An application for the solution of high even-order differential equations, ***Applied Mathematics Letters***, 24: 559-565, (2011).

[6] Işık, O.R., Sezer, M. and Güney, Z., A Rational approximation based on Bernstein polynomials for high order initial and boundary value problems, ***Applied Mathematics and Computation***, 217: 9438-9450, (2011).

[7] Işık, O.R., Güney, Z. and Sezer, M., Bernstein series solutions of pantograph equations using polynomial interpolation, ***Journal of Difference Equations and Applications***, 18: 357-374, (2010).

[8] Ordokhani, Y. and Davaei far, S., Approximate solutions of differential equations by using the Bernstein polynomials, ***International Scholarly Research Network ISRN Applied Mathematics***, 2011: 1-15, (2011).

[9] Bhattacharya, S. and Mandal, B.N., Use of Bernstein polynomials in numerical solutions of Volterra integral equations, ***Applied Mathematical Sciences***, 2: 1773-1787, (2008).

[10] Shirin, A. and Islam, A.S., Numerical solutions of Fredholm integral equations using Bernstein polynomials***, Journal of Scientific Research***, 2: 264-272, (2010).

[11] Bhatta, D.D. and Bhatti, M.I, Numerical Solution of KdV equation using modified Bernstein polynomials, ***Applied Mathematics and Computation***, 174: 1255-1268, (2006).

[12] Stewart, G.W., Matrix algorithms volume I: Basic decompositions, ***SIAM***, Philadelphia, (1998).

[13] Yousefi, S.A. and Dehghan, B.M., Bernstein Ritz-Galerkin method for solving an initial-boundary value problem that combines Neumann and integral condition for the wave equation, ***Numerical Methods for Partial Differential Equations***, 26: 1236-1246, (2009).

[14] Bhattacharya, S. and Mandal, B.N., Numerical solution of a singular integro- differential equation, ***Applied Mathematics and Computation***, 195: 346-350, (2008).

[15] Lorentz, G.G., Bernstein polynomials, ***Chelsea Publishing***, New York, N.Y., (1986).

[16] Rivlin, T.J., An introduction to the approximation of functions, ***Dover Publications***, N.Y., (2003).

[17] Siddiqi, S.S. and Akram, G., Septic spline solutions of sixth-order boundary value problems, ***Journal of Computational and Applied Mathematics***, 215: 288-301, (2008).

[18] El-Gamel, M., Cannon, J.R. and Zayed, A.I., Sinc-Galerkin method for solving linear sixth-order boundary value problems, ***Mathematics and Computation***, 73: 1325-1343, (2003).