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Weighted Approximation by the q −Szász−Schurer−Beta Type Operators

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ABSTRACT

In this study, we investigate approximation properties of a Schurer type generalization of q −Szász-beta type operators. We estimate the rate of weighted approximation of these operators for functions of polynomial growth on the interval $[0,\infty)$.

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1. INTRODUCTION

In [1], based on $q-$integer and q −binomial coefficients, firstly, Lupaş introduced a q −analogue of the Bernstein operators. After then several interesting generalization about q −calculus were given in $[2]-[13]$. Recently, in [14], Dinlemez studied convergence of the q −Stancu-Szász-beta type operators. Our aim is to study on Schurer type generalization of q −Szász-beta type operators. We use without further explanation the basic notations and formulas, from the theory of q −calculus as set out in $[15]-[19]$. Let $A>0$ and f be a real valued continuous function defined on the interval $[0,\infty)$. For $0<q\leq 1, q$ −Szász-Schurer-beta type operators are defined as

$$S_{n,p,q}(f,x)=\sum_{k=0}^{\infty}s_{n,p,k}^{q}(x)\int_0^{\infty/A}b_{n,p,k}^{q}(t)f(t)d_q(t), \quad (1)$$

where

$$s_{n,p,k}^{q}(x)=\left([n+p]_q x\right)^k\frac{e^{-[n+p]_q x}}{[k]_q!}$$

and

$$b_{n,p,k}^{q}(x)=\frac{q^{k^2}x^k}{B_q(k+1,n+p)(1+x)_q^{n+p+k+1}}.$$

If we write $p=0$ in (1), then the operators $S_{n,p,q}$ are reduced to q −Szász-beta type operators studied in [10, 11] and [14]. If we write $p=0$ and $q=1$ in (1), then the operators $S_{n,p,q}$ are reduced to Szász-beta type operators given in $[20]-[23]$.

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2. MOMENT ESTIMATION

For the sake of shortness, the notation $F_s^-(m) = \prod_{i=0}^{s}[m-i]_q$ will be used throughout the article. In the following lemma, lemma is similarly to the proof given in [10,11]. So proof of the following lemma is omitted.

Lemma 1. $e_m(t) = t^m, m = 0,1,2,3$ and 4. Then, we get

(i) $S_{n,p,q}(e_0, x) = 1$,

(ii) $S_{n,p,q}(e_1, x) = \frac{[n+p]_q}{q^2 F_0^-(n+p-1)} x + \frac{1}{q F_0^-(n+p-1)}$,

(iii) $S_{n,p,q}(e_2, x) = \frac{[n+p]_q^2}{q^6 F_1^-(n+p-1)} x^2 + \frac{[2]_q^2 [n+p]_q}{q^5 F_1^-(n+p-1)} x + \frac{[2]_q}{q^3 F_1^-(n+p-1)}$,

(iv) $S_{n,p,q}(e_3, x) = \frac{[n+p]_q^3}{q^{12} F_2^-(n+p-1)} x^3 + \frac{([5]_q + q[2]_q^2)[n+p]_q^2}{q^{11} F_2^-(n+p-1)} x^2 + \frac{([2]_q^2[4]_q + q^2[2]_q)[n+p]_q}{q^9 F_2^-(n+p-1)} x + \frac{[2]_q[3]_q}{q^6 F_2^-(n+p-1)}$,

(v) $S_{n,p,q}(e_4, x) = \frac{[n+p]_q^4}{q^{20} F_3^-(n+p-1)} x^4 + \frac{([7]_q + q[5]_q + q^2[2]_q^2)[n+p]_q^3}{q^{19} F_3^-(n+p-1)} x^3 + \frac{([5]_q[6]_q + q[2]_q^2[6]_q + q^2[2]_q^2[4]_q + q^4[2]_q)[n+p]_q^2}{q^{17} F_3^-(n+p-1)} x^2 + \frac{([2]_q^2[4]_q[5]_q + q^2[2]_q[5]_q + q^3[2]_q[3]_q)[n+p]_q}{q^{14} F_3^-(n+p-1)} x + \frac{[2]_q[3]_q[4]_q}{q^{10} F_3^-(n+p-1)}$.

To obtain our main results we need to compute second and fourth moments.

Lemma 2. Let $q \in (0,1)$ and $n > 4$. Then, we have the following inequalities

(i) $S_{n,p,q}((t-x)^2, x) \le \left(\frac{2(1-q^4)}{q^6} + \frac{28}{q^6 F_0^-(n+p-1)}\right) x(x+1) + \frac{2}{q^3 F_0^-(n+p-1)}$

and

(ii) $S_{n,p,q}((t-x)^4, x) \le \left(\frac{8(q^{16}-q^{18})}{q^{20}} + \frac{12596}{q^{20}[n+p-4]_q}\right)(x^4 + x^3 + x^2 + x + 1)$.

Proof of (i). From linearity of $S_{n,p,q}$ operators and Lemma 1, we have the second moment as

$$S_{n,p,q}((t-x)^2, x) = S_{n,p,q}(t^2, x) - 2x S_{n,p,q}(t, x) + x^2 S_{n,p,q}(1, x)$$
$$= \frac{[n+p]_q^2 - 2q^4[n+p]_q[n+p-2]_q + q^6 F_1^-(n+p-1)}{q^6 F_1^-(n+p-1)} x^2 + \frac{q[2]_q^2[n+p]_q - 2q^5[n+p-2]_q}{q^6 F_1^-(n+p-1)} x + \frac{[2]_q}{q^3 F_1^-(n+p-1)} \quad (2)$$

In (2), using the inequality $[n+p-s]_q \le [n+p]_q \; for \; s > 0$ and ignoring some negative terms, we obtain

$$S_{n,p,q}((t-x)^2, x) \le \frac{(1+q^6)[n+p]_q^2 - 2q^4[n+p-2]_q^2}{q^6 F_1^-(n+p-1)} x^2 + \frac{q[2]_q^2[n+p]_q}{q^6 F_1^-(n+p-1)} x + \frac{[2]_q}{q^3 F_1^-(n+p-1)} \quad (3)$$

In (3), using the inequality

$$[n+p]_q \le [s]_q + q^s[n+p-s]_q \text{ for } s > 0, \quad (4)$$

we get

$$S_{n,p,q}((t-x)^2, x)$$
$$\le \frac{(1+q^6)([2]_q^2 + 2q^2[2]_q^2[n+p-2]_q + q^4[n+p-2]_q^2) - 2q^4[n+p-2]_q^2}{q^6 F_1^-(n+p-1)} x^2 + \frac{q[2]_q^2([2]_q + q^2[n+p-2]_q)}{q^6 F_1^-(n+p-1)} x + \frac{[2]_q}{q^3 F_1^-(n+p-1)}$$
$$\le \frac{|q^4(1+q^6) - 2q^4|[n+p-2]_q^2 + (2[2]_q q^2(1+q^6) + [2]_q^2 q^3)[n+p-2]_q) + (1+q^6)[2]_q^2 + q[2]_q^3}{q^6 F_1^-(n+p-1)} x(1+x) + \frac{[2]_q}{q^3 F_1^-(n+p-1)}$$
$$\le \left(\frac{2(1-q^4)}{q^6} + \frac{28}{q^6 F_0^-(n+p-1)}\right) x(1+x) + \frac{2}{q^3 F_0^-(n+p-1)}.$$

And the proof of (i) of the Lemma 2 is now finished.

Proof of (ii). From linearity of $S_{n,p,q}$ operators and Lemma 1, we write the fourth moment as

$$S_{n,p,q}((t-x)^4, x)$$

$$= S_{n,p,q}(t^4,x) - 4xS_{n,p,q}(t^3,x) + 6x^2S_{n,p,q}(t^2,x) - 4x^3S_{n,p,q}(t,x) + x^4S_{n,p,q}(1,x)$$
$$= \frac{C_1(n,p,q)}{q^{20}F_3^-(n+p-1)}x^4 + \frac{C_2(n,p,q)}{q^{19}F_3^-(n+p-1)} + \frac{C_3(n,p,q)}{q^{17}F_3^-(n+p-1)}x^2 + \frac{C_4(n,p,q)}{q^{14}F_3^-(n+p-1)}x + \frac{C_5(n,p,q)}{q^{10}F_3^-(n+p-1)}, \quad (5)$$

Where

$C_1(n,p,q)$
$= [n+p]_q^4 - 4q^8[n+p]_q^3[n+p-4]_q + 6q^{14}[n+p]_q^2[n+p-3]_q[n+p-4]_q$
$-4q^{18}[n+p]_q[n+p-2]_q[n+p-3]_q[n+p-4]_q + q^{20}F_3^-(n+p-1),$
$C_2(n,p,q) = \left([7]_q + q[5]_q + q^2[2]_q^2\right)[n+p]_q^3 - 4q^8\left([5]_q + q[2]_q^2\right)[n+p]_q^2[n+p-4]_q$
$+6q^{14}[2]_q^2[n+p]_q[n+p-3]_q[n+p-4]_q - 4q^{18}[n+p-2]_q[n+p-3]_q[n+p-4]_q,$
$C_3(n,p,q) = \left([5]_q[6]_q + q[2]_q^2[6]_q + q^2[2]_q^2[4]_q + q^4[2]_q\right)[n+p]_q^2$
$-4q^8\left([2]_q^2[4]_q + q^2[2]_q\right)[n+p]_q[n+p-4]_q + 6q^{14}[2]_q[n+p-3]_q[n+p-4]_q,$
$C_4(n,p,q)$
$= \left([2]_q^2[4]_q[5]_q + q^2[2]_q[5]_q + q^3[2]_q[3]_q\right)[n+p]_q^2 - 4q^8[2]_q[3]_q[n+p-4]_q$

and

$C_5(n,p,q) = [2]_q[3]_q[4]_q.$

Now, we find upper boundaries for the coefficients $C_i(n,p,q)$, $i = 1,2,3$ and 4. Again using the inequalities $[n+p+s]_q \le [n+p]_q$, (4) and ignoring some negative terms, we write

$$C_1(n,p,q) = [n+p]_q^4(1 + 6q^{14} + q^{20}) - [n+p-4]_q^4(4q^8 + 4q^{18})$$
$$\le 8(q^{16} - q^{18})[n+p-4]_q^4 + 4992[n+p-4]_q^3, \quad (6)$$
$$C_2(n,p,q) \le 40[n+p]_q^3$$
$$\le 40([n+p-4]_q^3 + 12[n+p-4]_q^2 + 48\,[n+p-4]_q + 64), \quad (7)$$
$$C_3(n,p,q) \le 84[n+p]_q^2 \le 84\left([n+p-4]_q^2 + 8[n+p-4]_q + 16\right), \quad (8)$$
$$C_4(n,p,q) \le 96[n+p]_q \le 96\left([n+p-4]_q + 4\right) \quad (9)$$
and
$$C_5(n,p,q) = 24. \quad (10)$$

Combining among (5) and (10), we get

$$S_{n,p,q}((t-x)^4,x) \le \frac{8(q^{16}-q^{18})[n+p-4]_q^4 + 4992[n+p-4]_q^3}{q^{20}[n+p-4]_q^4}x^4$$
$$+ \frac{40[n+p-4]_q^3 + 12[n+p-4]_q^2 + 48[n+p-4]_q + 64}{q^{19}[n+p-4]_q^4}x^3$$
$$+ \frac{84\left(12[n+p-4]_q^2 + 8[n+p-4]_q + 16\right)}{q^{17}[n+p-4]_q^4}x^2 + \frac{96\left([n+p-4]_q + 4\right)}{q^{14}[n+p-4]_q^4}x + \frac{24}{q^{10}[n+p-4]_q^4}$$
$$\le \left(\frac{8(q^{16}-q^{18})}{q^{20}} + \frac{12596}{q^{20}[n+p-4]_q}\right)(x^4 + x^3 + x^2 + x + 1).$$

And the proof of (ii) of Lemma 2 is now finished.

3. LOCAL APPROXIMATION

Now, $C_B[0,\infty)$ denotes the space of bounded continuous functions with the norm $\|f\|_B = sup\{|f(x)|: x \in [0,\infty)\}$. We denote the first modulus of continuity on the finite interval $[0,b], b > 0$,

$\omega_{[0,b]}(f,\delta) = \sup_{0<h\le\delta, x\in[0,b]}|f(x+h) - f(x)|.$
(11)

The Peetre's K −functional is defined by

$K_2(f,\delta) = inf\{\|f-g\|_B + \delta\|g''\|_B : g \in W_\infty^2\}, \delta > 0),$

where $W_\infty^2 = \{g \in C_B[0,\infty): g', g'' \in C_B[0,\infty)\}$.

By [18, p. 177, Theorem 2.4], there exists a positive constant C such that

$$K_2(f,\delta) \le C\omega_2(f,\sqrt{\delta}), \quad (12)$$

where $\omega_2(f,\sqrt{\delta}) = \sup_{0<h\leq\sqrt{\delta}} \sup_{x\in[0,\infty)} |f(x+2h)-2f(x+h)+f(x)|$.

Thus we are ready to give direct results. The following lemma is routine and its proof is omitted.

Lemma 3. Let

$$\bar{S}_{n,p,q}(f,x) = S_{n,p,q}(f,x) - f\left(\frac{[n+p]_q}{q^2F_0^-(n+p-1)}x + \frac{1}{qF_0^-(n+p-1)}\right) + f(x). \tag{13}$$

Then, the operators $\bar{S}_{n,p,q}$ satisfy the following assertions;

(i). $\bar{S}_{n,p,q}(1,x) = 1$,
(ii). $\bar{S}_{n,p,q}(t,x) = x$,
(iii). $\bar{S}_{n,p,q}(t-x,x) = 0$.

Lemma 4. Let $q\in(0,1)$ and $n>1$. Then for every $x\in[0,\infty)$ and $f''\in C_B[0,\infty)$, we have the following inequality

$$\left|\bar{S}_{n,p,q}(f,x) - f(x)\right| \leq \|f''\|_B \delta_{n,p,q}(x),$$

where $\delta_{n,p,q}(x) = \left(\frac{2(1-q^4)}{q^6} + \frac{28}{q^6F_0^-(n+p-1)}\right)x(1+x) + \frac{2}{q^3F_0^-(n+p-1)}$.

Proof. Using Taylor's expansion

$$f(t) = f(x) + (t-x)f'(x) + \int_x^t (t-u)f''(u)du$$

and from Lemma 3, we obtain

$$\bar{S}_{n,p,q}(f,x) - f(x) = \bar{S}_{n,p,q}\left(\int_x^t (t-u)f''(u)du, x\right).$$

Then, using the Lemma 2 (i) and the inequality

$$\left|\int_x^t (t-u)f''(u)du\right| \leq \|f''\|_B \frac{(t-x)^2}{2},$$

we get

$$\begin{aligned}
&\left|\bar{S}_{n,p,q}(f,x) - f(x)\right| \\
&\qquad \leq \frac{\|f''\|_B}{2} S_{n,p,q}((t-x)^2,x) + \frac{\|f''\|_B}{2}\left(\left(\frac{[n+p]_q}{q^2F_0^-(n+p-1)} - 1\right)x + \frac{1}{qF_0^-(n+p-1)}\right)^2 \\
&\leq \frac{\|f''\|_B}{2}\left\{\left(\frac{2(1-q^4)}{q^6} + \frac{28}{q^6F_0^-(n+p-1)}\right)x(x+1) + \frac{2}{q^3F_0^-(n+p-1)}\right\} \\
&+ \frac{\|f''\|_B}{2}\left\{\left(\frac{[n+p]_q}{q^2F_0^-(n+p-1)} - 1\right)^2 x^2 + 2\left(\frac{[n+p]_q}{q^2F_0^-(n+p-1)} - 1\right)\frac{x}{qF_0^-(n+p-1)} + \frac{1}{q^2\left(F_0^-(n+p-1)\right)^2}\right\} \\
&\leq \left\{\left(\frac{2(1-q^4)}{q^6} + \frac{28}{q^6F_0^-(n+p-1)}\right)x(x+1) + \frac{2}{q^3F_0^-(n+p-1)}\right\}\|f''\|_B
\end{aligned}$$

and the proof of the Lemma 4 is now completed.

Theorem 1. Let $(q_n)\subset(0,1)$ be a sequence such that $q_n\to1$ as $n\to\infty$. Then for every $n>2$, $x\in[0,\infty)$ and $f\in C_B[0,\infty)$, we have the inequality

$$\left|S_{n,p,q_n}(f,x) - f(x)\right| \leq C\omega_2\left(f,\sqrt{\delta_{n,p,q_n}(x)}\right) + \omega\left(f,\eta_{n,p,q_n}(x)\right),$$

where $\eta_{n,p,q_n}(x) = \left(\frac{[n+p]_{q_n}}{q_n^2[n+p-1]_{q_n}} - 1\right)x + \frac{1}{q_n[n+p-1]_{q_n}}$.

Proof. Using (13) for any $g\in W_\infty^2$, we obtain the inequality

$$\begin{aligned}
\left|S_{n,p,q_n}(f,x) - f(x)\right| &\leq \left|\bar{S}_{n,p,q_n}(f-g,x) - (f-g)(x) + \bar{S}_{n,p,q_n}(g,x) - g(x)\right| \\
&+ \left|f\left(\frac{[n+p]_{q_n}x}{q_n^2F_0^-(n+p-1)}x + \frac{1}{q_nF_0^-(n+p-1)}\right) - f(x)\right|.
\end{aligned}$$

From Lemma 3 and Lemma 4, we get

$$\left|S_{n,p,q_n}(f,x)-f(x)\right| \le 2\|f-g\|_B+\delta_{n,p,q_n}(x)\|g''\|_B + \left|f\left(\frac{[n+p]_{q_n}}{q_n^2F_0^-(n+p-1)}x+\frac{1}{q_nF_0^-(n+p-1)}\right)-f(x)\right|$$

By using (11), we have

$$\left|S_{n,p,q_n}(f,x)-f(x)\right| \le 2\|f-g\|_B+\delta_{n,p,q_n}(x)\|g''\|_B+\omega\left(f,\eta_{n,p,q_n}(x)\right). \quad (14)$$

Taking infimum over $g\in W_\infty^2$ on the right hand side of the inequality (14) and using the inequality (12), we get the desired result.

4. WEIGHTED APPROXIMATION

The weighted Korovkin−type theorems was proved by Gadzhiev [19]. We give the Gadzhiev's results in weighted spaces. Let $\rho(x)=1+x^2$. $B_\rho[0,\infty)$ denotes the set of all functions f, from $[0,\infty)$ to R, satisfying the growth condition $|f(x)|\le N_f\rho(x)$, where N_f is a constant depending only on f. $B_\rho[0,\infty)$ is a normed space with the norm $\|f\|_\rho = sup\left\{\frac{|f(x)|}{\rho(x)}: x\in R\right\}$. $C_\rho^*[0,\infty)$ denotes the subspace of continuous functions in $B_\rho[0,\infty)$ for which $\lim_{|x|\to\infty}\frac{|f(x)|}{\rho(x)}$ exists finitely.

Theorem 2 Let $(q_n)\subset(0,1)$ be a sequence such that $q_n\to1$ as $n\to\infty$. Then for $f\in C_\rho^*[0,\infty)$ and $n>1$, we have $\lim_{n\to\infty}\left\|S_{n,p,q_n}(f)-f\right\|_\rho=0$.

Proof. From Lemma 1, it is obvious that $\left\|S_{n,p,q_n}(e_0)-e_0\right\|_\rho=0$. Let $n>2$. Using Lemma we see that,

$$\begin{aligned}\left|S_{n,p,q_n}(e_1,x)-e_1(x)\right| &= \left|\frac{[n+p]_{q_n}}{q_n^2F_0^-(n+p-1)}x+\frac{1}{q_nF_0^-(n+p-1)}-x\right|\\ &= \left|\frac{[n+p]_{q_n}-q_n^2[n+p-1]_{q_n}}{q_n^2[n+p-1]_{q_n}}x+\frac{1}{q_n[n+p-1]_{q_n}}\right|\\ &=\left|\frac{1+q_n-q_n^{n+p}}{q_n^2[n+p-1]_{q_n}}x+\frac{1}{q_n[n+p-1]_{q_n}}\right|\\ &\le\left(\frac{3}{q_n^2}x+\frac{1}{q_n}\right)\frac{1}{[n+p-1]_{q_n}}\\ &\le\frac{3}{q_n^2}(x+1)\frac{1}{[n+p-1]_{q_n}}\end{aligned}$$

And we have

$$\left\|S_{n,p,q_n}(e_1)-e_1\right\|_\rho \le \sup_{x\in[0,\infty)}\frac{1+x}{1+x^2}\frac{3}{q_n^2[n+p-1]_{q_n}}.$$

Therefore,

$$\lim_{n\to\infty}\left\|S_{n,p,q_n}(e_1)-e_1\right\|_\rho=0.$$

Similarly,

$$\begin{aligned}\left|S_{n,p,q_n}(e_2,x)-e_2(x)\right| &= \left|\frac{[n+p]_{q_n}^2}{q_n^6F_1^-(n+p-1)}x^2+\frac{[2]_{q_n}^2[n+p]_{q_n}}{q_n^5F_1^-(n+p-1)}x+\frac{[2]_{q_n}}{q_n^3F_1^-(n+p-1)}-x^2\right|\\ &= \left|\left(\frac{[n+p]_{q_n}^2}{q_n^6F_1^-(n+p-1)}-1\right)x^2+\frac{[2]_q^2[n+p]_{q_n}}{q_n^5F_1^-(n+p-1)}x+\frac{[2]_{q_n}}{q_n^3F_1^-(n+p-1)}\right|.\end{aligned}$$

By the equality

$$\begin{aligned}&[n+p]_{q_n}^2-q_n^6F_1^-(n+p-1)\\ &\quad=\frac{1}{1-q_n}\left(q_n+q_n^{2(n+p)}-2q_n^{n+p+1}-2q_n^{n+p+2}-2q_n^{n+p+3}-q_n^{n+p+4}+q_n^{2(n+p)+1}\right.\\ &\left.+q_n^{2(n+p)+2}+q_n^2+q_n^3+q_n^4+q_n^5-2q_n^{n+p}+1\right)\\ &\le\frac{9-9q_n^{n+p+4}}{1-q_n}\\ &=9[n+p+4]_{q_n}.\end{aligned}$$

We get following inequality

$$\left|S_{n,p,q_n}(e_2,x)-e_2(x)\right|\le(x^2+x+1)\frac{9[n+p+4]_{q_n}}{q_n^6F_1^-(n+p-1)}.$$

Hence,

$$\left\|S_{n,p,q_n}(e_2)-e_2\right\|_{\rho}\leq \sup_{x\in[0,\infty)}\frac{1+x+x^2}{1+x^2}\frac{9[n+p+4]_{q_n}}{q_n^6F_1^-(n+p-1)}.$$

Then we have $\lim_{n\to\infty}\left\|S_{n,p,q_n}(e_2)-e_2\right\|_{\rho}=0.$

Thus, from A. D. Gadzhiev's Theorem in [25], we obtain desired result of Theorem 2.

5. RATE OF WEIGHTED APPROXIMATION

Now we want to estimate the rate of convergence for the sequence of the q −Szász-Schurer-beta operators $S_{n,p,q}$. As it is known if f is not uniformly continuous on the interval $[0,\infty)$, then the usual first modulus of continuity $w(f,\delta)$ does not tend to zero, as $\delta\to 0$. For every $f\in C_{\rho}^{*}[0,\infty)$, we would like to take a weighted modulus of continuity $\Omega(f,\delta)$ which tends to zero as $\delta\to 0$. We consider the weighted modulus of continuity $\Omega(f,\delta)$ as

$$\Omega(f,\delta)=\sup_{0<h\leq\delta,x\geq 0}\frac{|f(x+h)-f(x)|}{(1+h^2)(1+x^2)},\ \text{for each every } f\in C_{\rho}^{*}[0,\infty). \tag{15}$$

The definition and properties of the weighted modulus of $\Omega(f,\delta)$ were given by İspir in [26].
Now we will obtain the rate of convergence for the operators S_{n,p,q_n}.

Theorem 3. Let $f\in C_{\rho}^{*}[0,\infty)$ and $(q_n)\subset(0,1)$ be a sequence such that $q_n\to 1$ as $n\to\infty$, then, we have the inequality

$$\left\|S_{n,p,q_n}(f)-f\right\|_{\bar{\rho}}\leq M(n,p,q_n)\Omega\left(f,\frac{1}{q_n^3}\sqrt{1-q_n^4+\frac{14+{q_n}^3}{F_0^-(n+p-1)}}(x+1)\right),$$

where $\bar{\rho}(x)=1+x^5$ and $M(n,p,q_n)$ is a positive real number dependent on n,p and q_n for $n>4$.

Proof. From the definition of $\Omega(f,\delta)$ in (15), we get

$$|f(t)-f(x)|\leq(1+(t-x)^2)(1+x^2)\left(1+\frac{|t-x|}{\delta}\right)\Omega(f,\delta).$$

Then we yield the inequality

$$\begin{aligned}\left|S_{n,p,q_n}(f(t),x)-f(x)\right| &\leq \Omega(f,\delta)(1+x^2)S_{n,p,q_n}\left((1+(t-x)^2)\left(1+\frac{|t-x|}{\delta}\right),x\right)\\ &\leq \Omega(f,\delta)(1+x^2)\big(S_{n,p,q_n}(1+(t-x)^2,x)\\ &+S_{n,p,q_n}\left((1+(t-x)^2)\frac{|t-x|}{\delta},x\right)\Big).\end{aligned} \tag{16}$$

Applying the Cauchy−Schwarz inequality in the last term of inequality (16) we obtain

$$S_{n,p,q_n}\left((1+(t-x)^2)\frac{|t-x|}{\delta},x\right)\leq\left(S_{n,p,q_n}((1+(t-x)^2)^2,x)\right)^{\frac{1}{2}}\left(S_{n,p,q_n}\left(\frac{(t-x)^2}{\delta^2},x\right)\right)^{\frac{1}{2}}.$$

From Lemma 1 and Lemma 2, we have the following estimates.

$$\begin{aligned}S_{n,p,q_n}(1+(t-x)^2,x) &\leq 1+\left(\frac{2(1-q_n^4)}{q_n^6}+\frac{28}{q_n^6F_0^-(n+p-1)}\right)x(x+1)+\frac{2}{q_n^3F_0^-(n+p-1)}\\ &\leq\frac{2}{q_n^6}\left(\frac{1}{2}q_n^6+1-q_n^4+\frac{14}{F_0^-(n+p-1)}+\frac{q_n^3}{F_0^-(n+p-1)}\right)(x+1)^2\\ &\leq\frac{2}{q_n^6}\left(1+q_n^6-q_n^4+\frac{14}{F_0^-(n+p-1)}+\frac{q_n^3}{F_0^-(n+p-1)}\right)(x+1)^2\\ &\leq M_1(n,p,q_n)(x+1)^2,\end{aligned} \tag{17}$$

using (17) and inequality (ii) of Lemma 2 we get

$$\begin{aligned}&S_{n,p,q_n}((1+(t-x)^2)^2,x)\\ &=1+2S_{n,p,q_n}((t-x)^2,x)+S_{n,p,q_n}((t-x)^4,x)\\ &\leq 2M_1(n,p,q_n)(x+1)^2+\left(\frac{8(q_n^{16}-q_n^{18})}{q_n^{20}}+\frac{12596}{q_n^{20}F_0^-(n+p-1)}\right)(x^4+x^3+x^2+x+1)\\ &\leq M_2(n,p,q_n)(x+1)^4,\end{aligned}$$

and

$$\left(S_{n,p,q_n}\left(\frac{(t-x)^2}{\delta^2},x\right)\right)^{\frac{1}{2}} \leq \frac{1}{\delta}\sqrt{\frac{2(1-q_n^4)}{q_n^6}+\frac{28+2q_n^3}{q_n^6F_0^-(n+p-1)}}(x+1)$$
$$\leq \frac{M_3(n,p,q_n)}{\delta q_n^3}\sqrt{1-q_n^4+\frac{14+q_n{}^3}{F_0^-(n+p-1)}}(x+1)$$

Choosing $M(n,p,q_n)=\left(M_1(n,p,q_n)+\sqrt{M_2(n,p,q_n)}\right)M_3(n,p,q_n)M_4$, where $M_4=\mathrm{Sup}_{x\geq 0}\,3(1+x^2)(1+x)^2/(1+x^5)$ and $\delta=\frac{1}{q_n^3}\sqrt{1-q_n^4+\frac{14+q_n{}^3}{F_0^-(n+p-1)}}(x+1)$ and using (17), (18), (19) in (16), we obtain

$$\left|S_{n,p,q_n}(f,x)-f(x)\right| \leq (1+x^5)M(n,p,q_n)\Omega\left(f,\frac{1}{q_n^3}\sqrt{1-q_n^4+\frac{14+q_n^3}{F_0^-(n+p-1)}}\right)(x+1)$$

Therefore the proof of the Theorem 3 is comleted.

CONFLICT OF INTEREST

No conflict of interest was declared by the authors.

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