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Coefficient Bounds for a Subclass of m-fold Symmetric Bi-univalent Functions Involving Hadamard Product and Differential Operator

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Abstract: In this study, we construct a new subclass of m-fold symmetric bi-univalent functions using by Hadamard product and generalized Salagean differential operator in the open unit disk $U = \{z \in \mathbb{C} : |z| < 1\}$. We establish upper bounds for the coefficients $|a_{m+1}|$ and $|a_{2m+1}|$ belonging to this new class. The results presented here generalize some of the earlier studies.

Keywords: Bi-univalent functions, Coefficient estimates, m-fold symmetric functions.

1 Introduction

Let A be the family of analytic functions, normalized by the conditions $f(0) = f'(0) - 1 = 0$ and having the following form

$$f(z) = z + a_2 z^2 + a_3 z^3 + \cdots \tag{1}$$

in the open unit disk D. We also denote by S the subclass of functions in A which are univalent in U (see for details [4]).

According to the *Koebe-One Quarter Theorem* [4], it provides that the image of U under every univalent function $f \in A$ contains a disk of radius $1/4$. Thus every univalent function $f \in A$ has an inverse f^{-1} satisfying $f^{-1}(f(z)) = z$ and $f\left(f^{-1}(w)\right) = w$ $\left(|w| < r_0(f)\,,\ r_0(f) \geq \frac{1}{4}\right)$, where

$$F(w) = f^{-1}(w) = w \ - a_2 w^2 + \left(2a_2^2 - a_3\right) w^3 - \left(5a_2^3 - 5a_2 a_3 + a_4\right) w^4 + \cdots . \tag{2}$$

A function $f \in A$ is said to be bi-univalent in U if both f and f^{-1} are univalent in U. Let Σ denote the class of bi-univalent functions in U given by (1). The detailed information about the class of Σ was given in the references [2], [6], [7] and [10].

The Hadamard product or convolution of two functions $f(z) = z + \sum\limits_{k=2}^{\infty} a_k z^k \in A$ and $g(z) = z + \sum\limits_{k=2}^{\infty} b_k z^k \in A$, denoted by $f * g$, is defined by

$$(f * g)(z) = z + \sum_{k=2}^{\infty} a_k b_k z^k \quad (z \in U).$$

For $\delta \geq 1$ and $f \in A$, Al-Obodi [1] introduced the following differential operator:

$$\begin{aligned} &D_\delta^0 f(z) = f(z), \\ &D_\delta^1 f(z) = (1-\delta) f(z) + \delta z f'(z) = D_\delta f(z), \\ &\vdots \\ &D_\delta^n f(z) = (1-\delta) D_\delta^{n-1} f(z) + \delta z \left(D_\delta^{n-1} f(z)\right)' = D\left(D_\delta^n f(z)\right) \quad \left(z \in U,\ n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}\right). \end{aligned} \tag{3}$$

If f is given by (1), we see that

$$D_\delta^n f(z) = z + \sum_{k=2}^{\infty} \left[1 + (k-1)\delta\right]^n a_k z^k$$

with $D_\delta^n f(0) = 0$. It is worthy mentioning that when $\delta = 1$ in (3), we have the differential operator of Salagean [9].

Let m be a positive integer. A domain E is said to be m-fold symmetric if a rotation of E about the origin through an angle $2\pi/m$ carries E on itself. It follows that, a function f analytic in U is said to be m-fold symmetric if

$$f(e^{2\pi i/m}z) = e^{2\pi i/m}f(z).$$

A function is said to be *m*-fold symmetric if it has the following normalized form:

$$f(z) = z + \sum_{k=1}^{\infty} a_{mk+1}z^{mk+1} \qquad (z \in U, \;\; m \in \mathbb{N}). \tag{4}$$

Let S_m the class of *m*-fold symmetric univalent functions in U, which are normalized by the series expansion (4). In fact, the functions in the class S are *one*-fold symmetric. Analogous to the concept of *m*-fold symmetric univalent functions, we here introduced the concept of *m*-fold symmetric bi-univalent functions. Each function $f \in \Sigma$ generates an *m*-fold symmetric bi-univalent function for each integer $m \in \mathbb{N}$. The normalized form of f is given as in (4) and the series expansion for f^{-1}, which has been recently proven by Srivastava et al. [9], is given as follows:

$$\begin{aligned} F(w) = f^{-1}(w) = \;\; & w \; - a_{m+1}w^{m+1} + \left[\left(m+1\right)a_{m+1}^2 - a_{2m+1}\right)\right] w^{2m+1} \\ & - \left[\tfrac{1}{2}(m+1)(3m+2)a_{m+1}^3 - (3m+2)a_{m+1}a_{2m+1} + a_{3m+1}\right] w^{3m+1} + \cdots . \end{aligned}$$

We denote by Σ_m the class of *m*-fold symmetric bi-univalent functions in U. For $m = 1$, the formula (4) coincides with the formula (2) of the class Σ. Some examples of *m*-fold symmetric bi-univalent functions are given as follows:

$$\left(\frac{z^m}{1-z^m}\right)^{\frac{1}{m}}, \;\; \left[-\log(1-z^m)\right]^{\frac{1}{m}}, \;\; \left[\frac{1}{2}\log\left(\frac{1+z^m}{1-z^m}\right)^{\frac{1}{m}}\right].$$

The coefficient problem for m-fold symmetric analytic bi-univalent functions is one of the favourite subjects of Geometric Function Theory in these days, (see, e.g., [3], [5], [11], [12]).

Here, the aim of this study is to determine upper coefficients bounds $|a_{m+1}|$ and $|a_{2m+1}|$ belonging to the newly defined subclass.

Firstly, in order to derive our main results, we require the following lemma.

Lemma 1. *(See [8]) If a function $p \in P$ is given by*

$$p(z) = 1 + c_1 z + c_2 z^2 + \cdots \quad (z \in U),$$

then $|c_i|$ for each $i \in \mathbb{N}$, where the Caratheodory class P is the family of all functions p analytic in U for which $\Re(p(z)) > 0$ and $p(0) = 1$.

2 Coefficient bounds for the functions class $\Sigma_m^{t,n,\delta}(\tau, \alpha, \lambda)$

Definition 1. *A function f given by (4) is said to be in the class*

$$\Sigma_m^{t,n,\delta}(\tau, \alpha, \lambda) \quad (\tau \in \mathbb{C}\backslash\{0\}, 0 < \alpha \le 1, \lambda > 0, t, n \in \mathbb{N}_0, t > n, \delta \ge 1, z, w \in U)$$

if the following conditions are satisfied:

$$f \in \Sigma_m, \quad \left|\arg\left(1 + \frac{1}{\tau}\left[(1-\alpha)\frac{D_\delta^n(f*g)(z)}{D_\delta^t(f*h)(z)} + \alpha\frac{(D_\delta^n(f*g)(z))'}{\left(D_\delta^t(f*h)(z)\right)'} - 1\right]\right)\right| < \frac{\alpha\pi}{2} \tag{5}$$

and

$$\left|\arg\left(1 + \frac{1}{\tau}\left[(1-\alpha)\frac{D_\delta^n(F*g)(w)}{D_\delta^t(F*h)(w)} + \alpha\frac{(D_\delta^n(F*g)(w))'}{\left(D_\delta^t(F*h)(w)\right)'} - 1\right]\right)\right| < \frac{\alpha\pi}{2}, \tag{6}$$

where $g(z) = z + \sum\limits_{k=1}^{\infty} g_{mk+1}z^{mk+1}$, $h(z) = z + \sum\limits_{k=1}^{\infty} h_{mk+1}z^{mk+1}$ and the function F is extension of f^{-1} to U.

We start by finding the estimates on the coefficients $|a_{m+1}|$ and $|a_{2m+1}|$ for the functions in the $\Sigma_m^{t,n,\delta}(\tau, \alpha, \lambda)$.

Theorem 1. *Let the function f given by (4) be in the class $\Sigma_m^{t,n,\delta}(\tau,\alpha,\lambda)$. Then*

$$|a_{m+1}| \le \frac{2\,|\tau|\,\lambda}{\sqrt{|A|}}$$

and

$$|a_{2m+1}| \le \frac{2\,|\tau|\,\lambda}{(1+2m\alpha)\,|(1+2m\delta)^n g_{2m+1} - (1+2m\delta)^t h_{2m+1}|} + \frac{2(m+1)\tau^2\lambda^2}{|A|},$$

where

$$A = \tau\lambda(1+m)(1+2m\alpha)\left[(1+2m\delta)^n g_{2m+1} - (1+2m\delta)^t h_{2m+1}\right]$$
$$-2\tau\lambda(1+2m\alpha+m^2\alpha)\left[(1+m\delta)^{t+n} h_{m+1} g_{m+1} - (1+m\delta)^{2t} h_{m+1}^2\right] - (\lambda-1)(1+m\alpha)^2\left[(1+m\delta)^n g_{m+1} - (1+m\delta)^t h_{m+1}\right]^2.$$

Proof: Suppose that $\Sigma_m^{t,n,\delta}(\tau,\alpha,\lambda)$. From the conditions (5) and (6), we can write

$$1 + \frac{1}{\tau}\left[(1-\alpha)\frac{D_\delta^n (f*g)(z)}{D_\delta^t (f*h)(z)} + \alpha\frac{\left(D_\delta^n (f*g)(z)\right)'}{\left(D_\delta^t (f*h)(z)\right)'} - 1\right] = [p(z)]^\lambda, \tag{7}$$

$$1 + \frac{1}{\tau}\left[(1-\alpha)\frac{D_\delta^n (F*g)(w)}{D_\delta^t (F*h)(w)} + \alpha\frac{\left(D_\delta^n (F*g)(w)\right)'}{\left(D_\delta^t (F*h)(w)\right)'} - 1\right] = [q(w)]^\lambda, \tag{8}$$

where $F = f^{\,-1}$, p, q in P and have the following forms

$$p(z) = 1 + p_m z^m + p_{2m} z^{2m} + \cdots,$$

$$q(w) = 1 + q_m w^m + q_{2m} w^{2m} + \cdots.$$

Clearly, we deduce that

$$[p(z)]^\lambda = 1 + \lambda p_m z^m + \left(\lambda p_{2m} + \frac{\lambda(\lambda-1)}{2} p_m^2\right) z^{2m} + \cdots,$$

$$[q(w)]^\lambda = 1 + \lambda q_m w^m + \left(\lambda q_{2m} + \frac{\lambda(\lambda-1)}{2} q_m^2\right) w^{2m} + \cdots.$$

Additionaly,

$$1 + \tfrac{1}{\tau}\left[(1-\alpha)\tfrac{D_\delta^n (f*g)(z)}{D_\delta^t (f*h)(z)} + \alpha\tfrac{\left(D_\delta^n (f*g)(z)\right)'}{\left(D_\delta^t (f*h)(z)\right)'} - 1\right] = 1 + \frac{(1+m\alpha)}{\tau}\left[(1+m\delta)^n g_{m+1} - (1+m\delta)^t h_{m+1}\right] a_{m+1} z^m +$$
$$\frac{\left\{(1+2m\alpha)\left[(1+2m\delta)^n g_{2m+1} - (1+2m\delta)^t h_{2m+1}\right] a_{2m+1} - (1+2m\alpha+m^2\alpha)\left[(1+m\delta)^{t+n} h_{m+1} g_{m+1} - (1+m\delta)^{2t} h_{m+1}^2\right] a_{m+1}^2\right\}}{\tau} z^{2m} + \cdots$$

and

$$1 + \tfrac{1}{\tau}\left[(1-\alpha)\tfrac{D_\delta^n (F*g)(w)}{D_\delta^t (F*h)(w)} + \alpha\tfrac{\left(D_\delta^n (F*g)(w)\right)'}{\left(D_\delta^t (F*h)(w)\right)'} - 1\right] = 1 - \frac{(1+m\alpha)}{\tau}\left[(1+m\delta)^n g_{m+1} - (1+m\delta)^t h_{m+1}\right] a_{m+1} w^m +$$
$$\frac{\left\{(1+2m\alpha)\left[(1+2m\delta)^n g_{2m+1} - (1+2m\delta)^t h_{2m+1}\right]\left[(1+m)a_{m+1}^2 - a_{2m+1}\right] - (1+2m\alpha+m^2\alpha)\left[(1+m\delta)^{t+n} h_{m+1} g_{m+1} - (1+m\delta)^{2t} h_{m+1}^2\right] a_{m+1}^2\right\}}{\tau} w^{2m} \cdots$$

Now, equating the coefficients in (7) and (8), we have

$$(1+m\alpha)\left[(1+m\delta)^n g_{m+1} - (1+m\delta)^t h_{m+1}\right] = \tau\lambda p_m, \tag{9}$$

$$(1+2m\alpha)\left[(1+2m\delta)^n g_{2m+1} - (1+2m\delta)^t h_{2m+1}\right] a_{2m+1}$$
$$-(1+2m\alpha+m^2\alpha)\left[(1+m\delta)^{t+n} h_{m+1} g_{m+1} - (1+m\delta)^{2t} h_{m+1}^2\right] a_{m+1}^2 = \tau\left(\lambda p_{2m} + \tfrac{\lambda(\lambda-1)}{2} p_m^2\right), \tag{10}$$

$$m(1-\lambda)\left[2a_{2m+1} - (\lambda m+1)a_{m+1}^2\right] = \tau\left(\lambda p_{2m} + \frac{\lambda(\lambda-1)}{2} p_m^2\right)$$

and

$$-(1+m\alpha)\left[(1+m\delta)^n g_{m+1} - (1+m\delta)^t h_{m+1}\right] = \tau\lambda q_m, \tag{11}$$

$$
\begin{aligned}
&(1+2m\alpha)\left[(1+2m\delta)^n g_{2m+1}-(1+2m\delta)^t h_{2m+1}\right]\left[(1+m)a_{m+1}^2-a_{2m+1}\right] \\
&-(1+2m\alpha+m^2\alpha)\left[(1+m\delta)^{t+n}h_{m+1}g_{m+1}-(1+m\delta)^{2t}h_{m+1}^2\right]a_{m+1}^2=\tau\left(\lambda q_{2m}+\tfrac{\lambda(\lambda-1)}{2}q_m^2\right).
\end{aligned}
\tag{12}
$$

From (9) and (11), we obtain

$$p_m=-q_m, \tag{13}$$

$$2(1+m\alpha)^2\left[(1+m\delta)^n g_{m+1}-(1+m\delta)^t h_{m+1}\right]a_{m+1}^2=\tau^2\lambda^2(p_m^2+q_m^2). \tag{14}$$

Next, by adding Eqs. (10) and (12), we obtain

$$
\begin{aligned}
&\{(1+m)(1+2m\alpha)\left[(1+2m\delta)^n g_{2m+1}-(1+2m\delta)^t h_{2m+1}\right] \\
&-2(1+2m\alpha+m^2\alpha)\left[(1+m\delta)^{t+n}h_{m+1}g_{m+1}-(1+m\delta)^{2t}h_{m+1}^2\right]\Big\}a_{m+1}^2=\tau\left(\lambda\left(p_{2m}+q_{2m}\right)+\tfrac{\lambda(\lambda-1)}{2}(p_m^2+q_m^2)\right).
\end{aligned}
$$

Therefore, from (14), we get

$$a_{m+1}^2=\frac{\tau^2\lambda^2\left(p_{2m}+q_{2m}\right)}{A}, \tag{15}$$

where

$$
\begin{aligned}
&A=\tau\lambda(1+m)(1+2m\alpha)\left[(1+2m\delta)^n g_{2m+1}-(1+2m\delta)^t h_{2m+1}\right] \\
&-2\tau\lambda(1+2m\alpha+m^2\alpha)\left[(1+m\delta)^{t+n}h_{m+1}g_{m+1}-(1+m\delta)^{2t}h_{m+1}^2\right]-(\lambda-1)(1+m\alpha)^2\left[(1+m\delta)^n g_{m+1}-(1+m\delta)^t h_{m+1}\right]^2.
\end{aligned}
$$

Now taking the absolute value of (15) and appying Lemma 1 for the coefficients p_{2m} and q_{2m}, we have the following inequality

$$|a_{m+1}|\leq\frac{2\,|\tau|\,\lambda}{\sqrt{|A|}}.$$

Next, so as to obtain solution of the coefficient bound on $|a_{2m+1}|$, we subtract (12) from (10). We thus have

$$(1+2m\alpha)\left[(1+2m\delta)^n g_{2m+1}-(1+2m\delta)^t h_{2m+1}\right]\left[2a_{2m+1}-(1+m)a_{m+1}^2\right]=\tau\left(\lambda\left(p_{2m}-q_{2m}\right)+\frac{\lambda(\lambda-1)}{2}(p_m^2-q_m^2)\right). \tag{16}$$

Also using (15) in (16) we obtain that

$$a_{2m+1}=\frac{\tau\lambda\left(p_{2m}-q_{2m}\right)}{2(1+2m\alpha)\left[(1+2m\delta)^n g_{2m+1}-(1+2m\delta)^t h_{2m+1}\right]}+\frac{(m+1)\tau^2\lambda^2\left(p_{2m}+q_{2m}\right)}{2A}. \tag{17}$$

Taking the absolute value of (17) and applying Lemma 1.1 again for coefficients p_{2m}, p_m and q_{2m}, q_m we get the desired result. This completes the proof of Theorem 1. □

3 Concluding remark

Various choices of the functions h, g as mentioned above and by specializing on the parameters m, τ, t, n, δ we state some interesting results analogous to Theorem 1. The details involved may be left as an exercise for the interested reader.

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