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On The Trigonometric Functions in $\mathbb{R}^2_{\pi 3}$

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Abstract

In this paper, we give trigonometric functions by using reference angle in the plane $\mathbb{R}^2_{\pi 3}$ associated with the induced metric $d_{\pi 3}$ that is a member of the family of $d_{\pi n}-$distances. Also we investigate the change of the length a line segment under the rotations in the plane $\mathbb{R}^2_{\pi 3}$.

Keywords: *non-Euclidean geometry, $d_{\pi 3}-$distances, trigonometric functions in the plane $\mathbb{R}^2_{\pi 3}$*
2010 Mathematics Subject Classification: *51F99*

1. Introduction

Trigonometric functions have an important place in mathematics, engineer, physics. Trigonometry is an area of mathematics that studies the relationships of angles and sides of triangles. Trigonometry provides deeper insights into both the properties of circles and the properties of triangles. Trigonometry was developed after geometry for the purposes of astronomy.
Professor Cayley afterwards gave an example of the non-Euclidean trigonometry in the plane, obtaining the formula for the special case in which the fundamental conic or " absolute " is a circle, and the constants of measurement have particular values, [11].
Iso-taxicab geometry is a non-Euclidean geometry defined by K. O. Sowell.in 1989. In this geometry presented by Sowell three distance functions arise depending upon the relative positions of the points A and B. There are three axes at the origin; the x-axis, the y-axis and the y'-axis. The iso-taxicab trigonometric functions in iso-taxicab plane with three axes were given in [5,6,7].
A family of distances, $d_{\pi n}$, that includes Taxicab, Chinese-Checker and Iso-taxi distances, [2,3,11,12], as special cases introduced and the group of isometries of the plane with $d_{\pi n}-$metric is the semi-direct product of D_{2n} and $T(2)$ was shown in [1].
The definition of $d_{\pi n}-$distances family is given as follows;

Definition 1.1. *Let $A=(x_1,y_1)$ and $B=(x_2,y_2)$ be any two points in $\mathbb{R}^2$, a family of $d_{\pi n}-$distances is defined by;*

$$d_{\pi n}(A,B)=\frac{1}{\sin\frac{\pi}{n}}\left(\left|\sin\frac{k\pi}{n}-\sin\frac{(k-1)\pi}{n}\right||x_1-x_2|+\left|\cos\frac{(k-1)\pi}{n}-\cos\frac{k\pi}{n}\right||y_1-y_2|\right)$$

$$\begin{cases} 1\leq k\leq\left[\frac{n-1}{2}\right], k\in\mathbb{Z} & , \quad if \tan\frac{(k-1)\pi}{n}\leq\left|\frac{y_2-y_1}{x_2-x_1}\right|\leq\tan\frac{k\pi}{n} \\ k=\left[\frac{n+1}{2}\right] & , \quad if \tan\frac{\left[\frac{n-1}{2}\right]\pi}{n}\leq\left|\frac{y_2-y_1}{x_2-x_1}\right|<\infty \text{ or } x_1=x_2 \end{cases}$$

The plane $\mathbb{R}^2$ with the $d_{\pi n}-$distance is denoted by $\mathbb{R}^2_{\pi n}$.
For $n=3$ and accordingly $k=1$, $k=2$, we obtain the formula of $d_{\pi 3}-$distance between the points A and B according to the inclination in the plane $\mathbb{R}^2_{\pi 3}$:

$$d_{\pi 3}(A,B)=\begin{cases} |x_1-x_2|+\frac{1}{\sqrt{3}}|y_1-y_2| & , \quad 0\leq\left|\frac{y_2-y_1}{x_2-x_1}\right|\leq\sqrt{3} \\ \frac{2}{\sqrt{3}}|y_1-y_2| & , \quad \sqrt{3}\leq\left|\frac{y_2-y_1}{x_2-x_1}\right|<\infty \text{ or } x_1=x_2 \end{cases} \tag{1.1}$$

In this paper, firstly we will explain the relationship between the coordinates of any point in the iso-taxicab plane and the coordinates in the plane $\mathbb{R}^2_{\pi 3}$. Then we will define unit circle in the plane $\mathbb{R}^2_{\pi 3}$ and obtain $\sin_{\pi 3}\theta$, $\cos_{\pi 3}\theta$, $\tan_{\pi 3}\theta$, $\cot_{\pi 3}\theta$ trigonometric functions in the same

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plane and present a trigonometric table for some θ values of these trigonometric functions. We will compare these θ values obtained with the iso-taxicab trigonometric functions with some of the θ values presented by I. Kocayusufoglu and we will explain the reasons of the differences between those values.

In the following parts, we will calculate the measure of an angle on the unit circle in terms of the length of arc that is the piece of the unit circle and define the concept of the reference angle in the plane $\mathbb{R}^2_{\pi 3}$. Finally, we will find the change of the length of a line segment after rotations in the plane $\mathbb{R}^2_{\pi 3}$.

2. The Relationship Between The Coordinates of a Point in the Planes $\mathbb{R}^2_{\pi 3}$ and $\mathbb{R}^2$

Let's show the coordinates of any point with $A_{\pi 3} = (a,b)$ in the plane $\mathbb{R}^2_{\pi 3}$ and $A = (x,y)$ in the plane $\mathbb{R}^2$. Also note that, the ordinate of this point of $\mathbb{R}^2_{\pi 3}$ coordinate find by drawing line which have $\dfrac{\pi}{3}$ radian between $x-$axis. We obtain these equations with the help of the following figure.

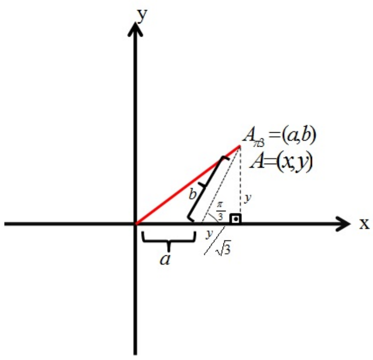


Figure 2.1

$$a = x - \frac{y}{\sqrt{3}} \quad , \quad b = \frac{2}{\sqrt{3}}y \tag{2.1}$$

These equations give us the relationship between the coordinates of a point in the planes $\mathbb{R}^2_{\pi 3}$ and $\mathbb{R}^2$.

3. Trigonometric Functions in the plane $\mathbb{R}^2_{\pi 3}$

$\mathbb{R}^2_{\pi 3}$ unit circle is defined as the set of points that are 1, $d_{\pi 3}-$distance away from the origin in the plane $\mathbb{R}^2_{\pi 3}$. If we take $A = (0,0)$ and $B = (x,y)$ in equation (1.1), we obtain the equation of $\mathbb{R}^2_{\pi 3}$ unit circle in the plane $\mathbb{R}^2_{\pi 3}$.

$$d_{\pi 3}((0,0),(x,y)) = \begin{cases} |x| + \dfrac{1}{\sqrt{3}}|y| = 1 & , \quad 0 \leq \left|\dfrac{y}{x}\right| \leq \sqrt{3} \\ \dfrac{2}{\sqrt{3}}|y| = 1 & , \quad \sqrt{3} \leq \left|\dfrac{y}{x}\right| < \infty \end{cases} \tag{3.1}$$

When the last equation is analyzed in terms of absolute values, the unit circle splits into branches according to regions. These branches are

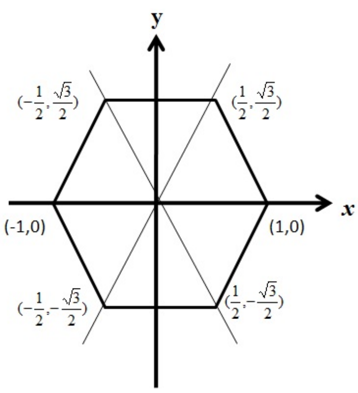


Figure 3.1: Unit circle in the plane $\mathbb{R}^2_{\pi 3}$

$$
\begin{array}{lll}
I & \to & y=-\sqrt{3}x+\sqrt{3} \\
II & \to & y=\dfrac{\sqrt{3}}{2} \\
III & \to & y=\sqrt{3}x+\sqrt{3} \\
IV & \to & y=-\sqrt{3}x-\sqrt{3} \\
V & \to & y=-\dfrac{\sqrt{3}}{2} \\
VI & \to & y=\sqrt{3}x-\sqrt{3}.
\end{array}
$$

Now, let's define sine and cosine functions in the plane $\mathbb{R}^2_{\pi3}$ depending on the point (x,y) on $\mathbb{R}^2_{\pi3}$ unit circle. We must first indicate the point (x,y) on the $\mathbb{R}^2_{\pi3}$ unit circle with the help of Euclidean trigonometric functions;

$$
\begin{cases}
|x|+\dfrac{1}{\sqrt{3}}|y|=1 & , \quad 0\leq\left|\dfrac{y}{x}\right|\leq\sqrt{3} \\
\dfrac{2}{\sqrt{3}}|y|=1 & , \quad \sqrt{3}\leq\left|\dfrac{y}{x}\right|<\infty \\
y=(\tan\theta)x. &
\end{cases}
$$

Here value of x and y are found as follow;

$$
\begin{cases}
|x|=\dfrac{\sqrt{3}|\cos\theta|}{\sqrt{3}|\cos\theta|+|\sin\theta|} \\
|y|=\dfrac{\sqrt{3}}{2}.
\end{cases}
$$

Now if we analyzed in terms of absolute values and according to exist situations of Euclidean angle θ;

Region I: If $0\leq\theta\leq\dfrac{\pi}{3}$

$$
\begin{cases}
y=-\sqrt{3}x+\sqrt{3} \\
y=(\tan\theta)x
\end{cases}
$$

and

$$
\begin{aligned}
x&=\frac{\sqrt{3}\cos\theta}{\sin\theta+\sqrt{3}\cos\theta} \\
y&=\frac{\sqrt{3}\sin\theta}{\sin\theta+\sqrt{3}\cos\theta}
\end{aligned} \tag{3.2}
$$

This is also the equation in the kind of Euclidean trigonometric value of a point on $\mathbb{R}^2_{\pi3}$ unit circle. Similar equations can be calculated for other regions.

Let's denote $\mathbb{R}^2_{\pi3}$ sine function as $\sin_{\pi3}$, $\mathbb{R}^2_{\pi3}$ cosine function as $\cos_{\pi3}$ and $\mathbb{R}^2_{\pi3}$ tangent function as $\tan_{\pi3}$. The $(\cos_{\pi3}\theta,\sin_{\pi3}\theta)$ point on $\mathbb{R}^2_{\pi3}$ unit circle shows the $\mathbb{R}^2_{\pi3}$ coordinates of the intersection point of $\mathbb{R}^2_{\pi3}$ unit circle and the line that passes through the origin and has positive.θ with the x-axis.

Using equation (2.1) we get;

$$
\tan_{\pi3}\theta=\frac{b}{a}=\frac{\dfrac{2y}{\sqrt{3}}}{x-\dfrac{y}{\sqrt{3}}} \tag{3.3}
$$

and

$$
\begin{cases}
\sin_{\pi3}\theta=\dfrac{2}{\sqrt{3}}y \\
\cos_{\pi3}\theta=x-\dfrac{y}{\sqrt{3}}.
\end{cases} \tag{3.4}
$$

When the values of x and y that are obtained in equation (3.2) are used in (3.3) and (3.4) equations, we get;

$$
\begin{aligned}
\sin_{\pi3}\theta&=\frac{2\sin\theta}{\sin\theta+\sqrt{3}\cos\theta} \quad , \quad \cos_{\pi3}\theta=\frac{\sqrt{3}\cos\theta-\sin\theta}{\sin\theta+\sqrt{3}\cos\theta} \\
\tan_{\pi3}\theta&=\frac{2\sin\theta}{\sqrt{3}\cos\theta-\sin\theta}.
\end{aligned}
$$

These values of $\sin_{\pi3}\theta$, $\cos_{\pi3}\theta$, $\tan_{\pi3}\theta$ can be calculated in similar ways for other regions. The calculated $\sin_{\pi3}\theta$, $\cos_{\pi3}\theta$ values for all regions are shown as;

$$
\sin_{\pi3}\theta=\begin{cases}
\dfrac{2\sin\theta}{|\sin\theta|+\sqrt{3}|\cos\theta|} & , \quad I-III-IV-VI \\
1 & , \quad II \\
-1 & , \quad V
\end{cases} \tag{3.5}
$$

$$\cos_{\pi 3}\theta = \begin{cases} \dfrac{\sqrt{3}\cos\theta - \sin\theta}{|\sin\theta| + \sqrt{3}|\cos\theta|} & , \quad I-III-IV-VI \\ \dfrac{\sqrt{3}\cos\theta - \sin\theta}{2|\sin\theta|} & , \quad II-V \end{cases} \tag{3.6}$$

Hence, we obtain trigonometric functions in the plane $\mathbb{R}^2_{\pi 3}$ with the help of Euclidean trigonometric functions. We can draw a table for some θ values;

θ	0	$\frac{\pi}{12}$	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	π	$\frac{4\pi}{3}$	$\frac{3\pi}{2}$	$\frac{5\pi}{3}$	2π
$\sin_{\pi 3}\theta$	0	$\frac{\sqrt{3}-1}{\sqrt{3}+1}$	$\frac{1}{2}$	$\frac{2}{\sqrt{3}+1}$	1	1	1	0	-1	-1	-1	0
$\cos_{\pi 3}\theta$	1	$\frac{2}{\sqrt{3}+1}$	$\frac{1}{2}$	$\frac{\sqrt{3}-1}{\sqrt{3}+1}$	0	$-\frac{1}{2}$	-1	-1	0	$\frac{1}{2}$	1	1

Similarly, some $\tan_{\pi 3}\theta$ and $\cot_{\pi 3}\theta$ values are found as follows;

θ	0	$\frac{\pi}{12}$	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	π	$\frac{4\pi}{3}$	$\frac{3\pi}{2}$	$\frac{5\pi}{3}$	2π
$\tan_{\pi 3}\theta$	0	$\frac{\sqrt{3}-1}{2}$	1	$\frac{2}{\sqrt{3}-1}$	∞	-2	-1	0	∞	-2	-1	0
$\cot_{\pi 3}\theta$	∞	$\frac{2}{\sqrt{3}-1}$	1	$\frac{\sqrt{3}-1}{2}$	0	$-\frac{1}{2}$	-1	∞	0	$-\frac{1}{2}$	-1	∞

Corollary 3.1. *We know that a family of $d_{\pi n}-$distances in $\mathbb{R}^2$ analytical plane where $n=3$ gives the iso-taxicab.distance. The trigonometric values we obtained are not completely the same compared to the $\sin_I\theta$, $\cos_I\theta$, $\tan_I\theta$, $\cot_I\theta$ values obtained by I. Kocayusufoglu [5]. This is because θ he has in iso-taxicab trigonometric equals the value of lenght of the arc. However the values we obtained in this paper are the trigonometric values of θ in the $\mathbb{R}^2$ analytic plane.*

4. The Angle Measurement in the plane $\mathbb{R}^2_{\pi 3}$ and The Reference Angle

4.1. The Angle Measurement in the plane $\mathbb{R}^2_{\pi 3}$

There are at least two common ways of defining angle measurement: in terms of an inner product and in terms of unit circle. For Euclidean space, these definitions agree. However, $d_{\pi 3}$ metric is not an inner product since the natural norm derived form the metric does not satisfy the parallelogram law. Thus, we will define angle measurement on the $\mathbb{R}^2_{\pi 3}$ unit circle.

Definition 4.1. *A $\mathbb{R}^2_{\pi 3}-$radian is an angle whose vertex is the center of $\mathbb{R}^2_{\pi 3}$ unit circle and intercepts an arc of length 1. The $\theta_{\pi 3}-$angle measure of an angle θ is the number of $\mathbb{R}^2_{\pi 3}-$radians subtended by the angle on the $\mathbb{R}^2_{\pi 3}$ unit circle about vertex.*

It follows immediately that a $\mathbb{R}^2_{\pi 3}$ unit circle has 6 $\mathbb{R}^2_{\pi 3}-$radians since the $\mathbb{R}^2_{\pi 3}$ unit circle has a circumference of 6. For reference purposes the Euclidean angles $\frac{\pi}{3}$, $\frac{\pi}{2}$ and π in standart position now have measure 1, $\frac{3}{2}$ and 3, respectively.
The following theorem gives the formula for determining the $\theta_{\pi 3}-$angle measures of some other Euclidean angles.

Theorem 4.2. *An acute Euclidean angle θ in standart position has a $\theta_{\pi 3}-$angle measure;*
For $0 \le \theta \le \frac{\pi}{3}$

$$\theta_{\pi 3} = 2 - \frac{2\sqrt{3}}{\sqrt{3}+\tan\theta} = \frac{2\sin\theta}{\sin\theta+\sqrt{3}\cos\theta} \tag{4.1}$$

For $\frac{\pi}{3} \le \theta \le \frac{\pi}{2}$

$$\theta_{\pi 3} = \frac{3}{2} - \frac{\sqrt{3}}{2}\cot\theta = \frac{3\sin\theta - \sqrt{3}\cos\theta}{2\sin\theta} \tag{4.2}$$

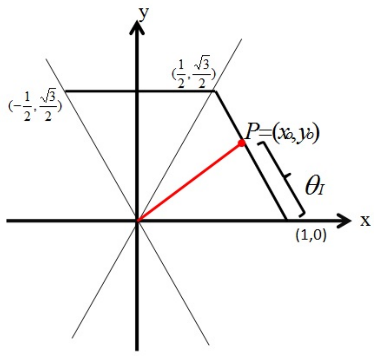
Proof. The $\theta_{\pi 3}-$angle measure of Euclidean angle θ between the angles of $0 \le \theta \le \frac{\pi}{3}$ (I. region) equals to the $d_{\pi 3}-$distance from $(1,0)$ to the intersection of the lines $y=-\sqrt{3}x+\sqrt{3}$ and $y=x.\tan\theta$. Let's take this intersection point as $P=(x_0,y_0)$. The $x-$coordinate of this intersection is:

$$x_0 = \frac{\sqrt{3}}{\sqrt{3}+\tan\theta}$$

and thus $y-$coordinate of P is;

$$y_0 = -\sqrt{3}xo+\sqrt{3}.$$

Hence, $\theta_{\pi 3}-$angle measure from $(1,0)$ to P is;



$$\begin{aligned}\theta_{\pi3} &= \frac{2}{\sqrt{3}}(-\sqrt{3}x_0+\sqrt{3}) = 2-2x_0\\ &= 2-\frac{2\sqrt{3}}{\sqrt{3}+\tan\theta}\\ &= \frac{2\sin\theta}{\sin\theta+\sqrt{3}\cos\theta}\end{aligned}$$

Similarly, the $\theta_{\pi3}-$angle measure of Euclidean angle θ between the angles of $\frac{\pi}{3}\leq\theta\leq\frac{\pi}{2}$ (II. region) equals to the $d_{\pi3}-$distance from $(\frac{1}{2},\frac{\sqrt{3}}{2})$ to the intersection of the $y=\frac{\sqrt{3}}{2}$ and $y=x.\tan\theta$ lines and the sum of 1 unit distance. The coordinates of $P(x_0,y_0)$ obtain as

$$y_0=\frac{\sqrt{3}}{2} \text{ and } y=x_0.\tan\theta\rightarrow\frac{\sqrt{3}}{2}=x_0.\tan\theta$$

$$x_0=\frac{\sqrt{3}\cos\theta}{2\sin\theta} \text{ and } y_0=\frac{\sqrt{3}}{2}.$$

Thus, $\theta_{\pi3}-$angle measure from $(1,0)$ to P is;

$$\begin{aligned}\theta_{\pi3} &= 1+(\frac{1}{2}-\frac{\sqrt{3}\cos\theta}{2\sin\theta})\\ &= \frac{3}{2}-\frac{\sqrt{3}}{2}\cot\theta.\end{aligned}$$

These formulas can be calculated similarly for obtuse angles. □

Definition 4.3. *The reference angle of an angle θ is the smallest angle between θ and $x-$axis.*

Now, let's find the $\theta_{\pi3}-$angle measure of an acute angle θ according to the existing positions.

Corollary 4.4. *Case 1 $\rightarrow$ If an acute Euclidean angle θ with Euclidean reference angle ψ is contained entirely in I. region, then the angle has a $\theta_{\pi3}-$angle measure of*

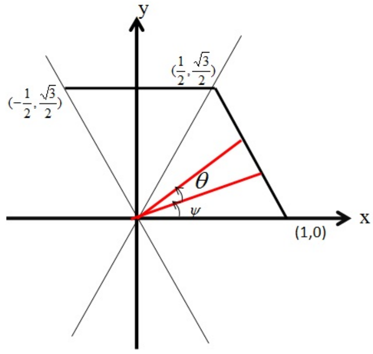


Figure 4.1

$$\begin{aligned}\theta_{\pi3} &= 2-\frac{2\sqrt{3}}{\sqrt{3}+\tan(\theta+\psi)}-2+\frac{2\sqrt{3}}{\sqrt{3}+\tan\psi}\\ &= \frac{2\sqrt{3}\sin\theta}{\left[\sqrt{3}\cos(\theta+\psi)+\sin(\theta+\psi)\right]\left[\sqrt{3}\cos\psi+\sin\psi\right]}.\end{aligned}$$

Case $2 \rightarrow$ *If an acute Euclidean angle* θ *with Euclidean reference angle* ψ *is contained entirely in* II*. region, then the angle has a* $\theta_{\pi 3}-$*angle measure of*

$$\begin{aligned} \theta_{\pi 3} &= \frac{3}{2}-\frac{\sqrt{3}}{2}\cot(\theta+\psi)-\frac{3}{2}+\frac{\sqrt{3}}{2}\cot\psi \\ &= \frac{\sqrt{3}\sin\theta}{2\sin\psi.\sin(\theta+\psi)}. \end{aligned}$$

Case $3 \rightarrow$ *If an acute Euclidean angle* θ *with Euclidean reference angle* ψ *is contained in* I*. region and* ψ *in* I*. region, then the angle has a* $\theta_{\pi 3}-$*angle measure of*

$$\begin{aligned} \theta_{\pi 3} &= \frac{3}{2}-\frac{\sqrt{3}}{2}\cot(\theta+\psi)-2+\frac{2\sqrt{3}}{\sqrt{3}+\tan\psi} \\ &= -\frac{1}{2}+\frac{2\sqrt{3}\cos\psi}{\sqrt{3}\cos\psi+\sin\psi}-\frac{\sqrt{3}\cos(\theta+\psi)}{2\sin(\theta+\psi)}. \end{aligned}$$

This corrolary implies the $\theta_{\pi 3}-$angle measure of an Euclidean angle in non-standart position is not necessarily equal to the $\theta_{\pi 3}-$angle measure of the same Euclidean angle in standart position. Thus we can easily say the rotations of angles are not invariant.
Now we determine the $\theta_{\pi 3}-$angle measure of angle θ that is in standard position for each hextant containing θ.

$$\theta_{\pi 3}=\begin{cases} 2-\dfrac{2\sqrt{3}}{\sqrt{3}+\tan\theta} & , \quad 0\leq\theta\leq\dfrac{\pi}{3} \\ \dfrac{3}{2}-\dfrac{\sqrt{3}}{2}\cot\theta & , \quad \dfrac{\pi}{3}\leq\theta\leq\dfrac{2\pi}{3} \\ 3-\dfrac{2\tan\theta}{\tan\theta-\sqrt{3}} & , \quad \dfrac{2\pi}{3}\leq\theta\leq\pi \\ 3+\dfrac{2\tan\theta}{\tan\theta+\sqrt{3}} & , \quad \pi\leq\theta\leq\dfrac{4\pi}{3} \\ \dfrac{9}{2}+\dfrac{2\tan\theta}{\tan\theta+\sqrt{3}} & , \quad \dfrac{4\pi}{3}\leq\theta\leq\dfrac{5\pi}{3} \\ 6-\dfrac{2\tan\theta}{\tan\theta-\sqrt{3}} & , \quad \dfrac{5\pi}{3}\leq\theta\leq 2\pi \end{cases} \tag{4.3}$$

Considering the known definitions of the trigonometric functions $\sin_{\pi 3}\theta$, $\cos_{\pi 3}\theta$ and using the equations of (3.5) and (3.6) we get;

$$\sin_{\pi 3}\theta_{\pi 3}=\begin{cases} \theta_{\pi 3} & , \quad 0\leq\theta_{\pi 3}\leq 1 \\ 1 & , \quad 1\leq\theta_{\pi 3}\leq 2 \\ 3-\theta_{\pi 3} & , \quad 2\leq\theta_{\pi 3}\leq 4 \\ -1 & , \quad 4\leq\theta_{\pi 3}\leq 5 \\ \theta_{\pi 3}-6 & , \quad 5\leq\theta_{\pi 3}\leq 6 \end{cases} \tag{4.4}$$

$$\cos_{\pi 3}\theta_{\pi 3}=\begin{cases} 1-\theta_{\pi 3} & , \quad 0\leq\theta_{\pi 3}\leq 2 \\ -1 & , \quad 2\leq\theta_{\pi 3}\leq 3 \\ \theta_{\pi 3}-4 & , \quad 3\leq\theta_{\pi 3}\leq 5 \\ 1 & , \quad 5\leq\theta_{\pi 3}\leq 6 \end{cases} \tag{4.5}$$

Using the equation (4.1) for the I. hextant we get that;

$$\begin{aligned} \sin_{\pi 3}\theta_{\pi 3} &= \frac{2\sqrt{3}\sin\theta}{\left[\sqrt{3}\cos(\theta+\psi)+\sin(\theta+\psi)\right]\left[\sqrt{3}\cos\psi+\sin\psi\right]} \\ &= \frac{2\sqrt{3}\sin\theta}{\left[\cos\theta+\sqrt{3}\sin(\theta+2\psi)+2\cos(\theta+\psi)\cos\psi\right]} \end{aligned} \tag{4.6}$$

$$\begin{aligned} \cos_{\pi 3}\theta_{\pi 3} &= 1-\frac{2\sqrt{3}\sin\theta}{\left[\cos\theta+\sqrt{3}\sin(\theta+2\psi)+2\cos(\theta+\psi)\cos\psi\right]} \\ &= \frac{\cos\theta+\sqrt{3}\sin(\theta+2\psi)+2\cos(\theta+\psi)\cos\psi-2\sqrt{3}\sin\theta}{\cos\theta+\sqrt{3}\sin(\theta+2\psi)+2\cos(\theta+\psi)\cos\psi}. \end{aligned} \tag{4.7}$$

It can be computed with equations (4.6) and (4.7)

$$\sin_{\pi 3}\psi_{\pi 3}=\frac{2\sqrt{3}\sin\psi}{3\cos\psi+\sqrt{3}\sin\psi} \tag{4.8}$$

$$\cos_{\pi 3}\psi_{\pi 3}=\frac{3\cos\psi-\sqrt{3}\sin\psi}{3\cos\psi+\sqrt{3}\sin\psi} \tag{4.9}$$

setting ψ to zero and θ to ψ.

Conversely, using equations (4.8) and (4.9), we get the $\sin\psi$ and $\cos\psi$ Euclidean trigonometric functions in terms of $\sin_{\pi 3}$ and $\cos_{\pi 3}$ for the angle ψ that is in standart position ;

$$\sin\psi = \frac{\sqrt{3}\sin_{\pi 3}\psi_{\pi 3}}{2\sqrt{1-\sin_{\pi 3}\psi_{\pi 3}+\sin^2_{\pi 3}\psi_{\pi 3}}} \tag{4.10}$$

and

$$\cos\psi = \sqrt{\frac{4-4\sin_{\pi 3}\psi_{\pi 3}+\sin^2_{\pi 3}\psi_{\pi 3}}{4(1-\sin_{\pi 3}\psi_{\pi 3}+\sin^2_{\pi 3}\psi_{\pi 3})}}. \tag{4.11}$$

Now, let's find the sine and cosine Euclidean trigonometric functions of the angle θ in terms of $\sin_{\pi 3}$ ve $\cos_{\pi 3}$ with the angle ψ being the reference.
Using equations (4.6) and (4.7) we get the Euclidean trigonometric functions $\sin\theta$ and $\cos\theta$ in terms of $\sin_{\pi 3}$ and $\cos_{\pi 3}$ as follows;

$$\sin\theta = \frac{1}{\sqrt{\left[\frac{2\sqrt{3}+2\sin_{\pi 3}\theta_{\pi 3}\sin\psi\cos\psi-\sqrt{3}\sin_{\pi 3}\theta_{\pi 3}\cos 2\psi}{\sin_{\pi 3}\theta_{\pi 3}+\sqrt{3}\sin 2\psi\sin_{\pi 3}\theta_{\pi 3}+2\cos^2\psi\sin_{\pi 3}\theta_{\pi 3}}\right]^2+1}} \tag{4.12}$$

$$\cos\theta = \sqrt{1-\left[\frac{1}{\sqrt{\left[\frac{2\sqrt{3}+2\sin_{\pi 3}\theta_{\pi 3}\sin\psi\cos\psi-\sqrt{3}\sin_{\pi 3}\theta_{\pi 3}\cos 2\psi}{\sin_{\pi 3}\theta_{\pi 3}+\sqrt{3}\sin 2\psi\sin_{\pi 3}\theta_{\pi 3}+2\cos^2\psi\sin_{\pi 3}\theta_{\pi 3}}\right]^2+1}}\right]^2}. \tag{4.13}$$

Note: It should be taken into consideration that we found the values of $\sin\psi$ and $\cos\psi$ that are in (4.12) and (4.13) equations in terms of $\mathbb{R}^2_{\pi 3}$ trigonometric functions of the (4.10) and (4.11) equations. $\sin\theta$, $\cos\theta$, $\sin_{\pi 3}\theta_{\pi 3}$ and $\cos_{\pi 3}\theta_{\pi 3}$ can be calculated in a similar way for the other hextants.

5. Change of The Lengths under Rotations in the plane $\mathbb{R}^2_{\pi 3}$

The change of the lengths under some non-Euclidean planes are studied in [4,8,9,10]. Now let's find the change of the $d_{\pi 3}-$length in the $\mathbb{R}^2_{\pi 3}$ plane of a line segment after rotation.

Theorem 5.1. *Let OA be a line segment, not on the $x-$axis with reference angle ψ and $d_{\pi 3}(O,A)=k$. If OA' is the image of OA under the rotation with an angle $\theta_{\pi 3}$(or θ) then*

$$\begin{aligned} d_{\pi 3}(O,A') &= k\sqrt{\frac{\cos^2_{\pi 3}\psi_{\pi 3}+\sin^2_{\pi 3}\psi_{\pi 3}+\cos_{\pi 3}\psi_{\pi 3}\sin_{\pi 3}\psi_{\pi 3}}{\cos^2_{\pi 3}(\theta+\psi)_{\pi 3}+\sin^2_{\pi 3}(\theta+\psi)_{\pi 3}+\cos_{\pi 3}(\theta+\psi)_{\pi 3}\sin_{\pi 3}(\theta+\psi)_{\pi 3}}} \\ &= k\frac{3\left|\cos(\theta+\psi)\right|+\sqrt{3}\left|\sin(\theta+\psi)\right|}{\left|3\cos\psi\right|+\left|\sqrt{3}\sin\psi\right|}. \end{aligned}$$

Proof. Let $d_{\pi 3}(O,A)=k$ be the $d_{\pi 3}-$length of the line segment OA. Rotating OA through an angle θ we get the line segment OA'. If ψ is the reference angle of θ then

$$A=(k\cos_{\pi 3}\psi_{\pi 3},k\sin_{\pi 3}\psi_{\pi 3}).$$

Let's calculate $d_{\pi 3}(O,A')=k'$, where $A'=(k'\cos_{\pi 3}(\theta+\psi)_{\pi 3},k'\sin_{\pi 3}(\theta+\psi)_{\pi 3})$. Because of the equality of Euclidean lengths of the line segments OA and OA' we get;

$$d_E(O,A)=d_E(O,A')$$

and as shown in the figure below;

$$\begin{aligned} (k\cos_{\pi 3}\psi_{\pi 3}+\frac{k}{2}\sin_{\pi 3}\psi_{\pi 3})^2+(\frac{\sqrt{3}}{2}k\sin_{\pi 3}\psi_{\pi 3})^2 &= (k'\cos_{\pi 3}(\theta+\psi)_{\pi 3}+\frac{k'}{2}\sin_{\pi 3}(\theta+\psi)_{\pi 3})^2+(\frac{\sqrt{3}}{2}k'\sin_{\pi 3}(\theta+\psi)_{\pi 3})^2 \\ k^2(\cos^2_{\pi 3}\psi_{\pi 3}+\sin^2_{\pi 3}\psi_{\pi 3}+\cos_{\pi 3}\psi_{\pi 3}\sin_{\pi 3}\psi_{\pi 3}) &= (k')^2(\cos^2_{\pi 3}(\theta+\psi)_{\pi 3}+\sin^2_{\pi 3}(\theta+\psi)_{\pi 3}+\cos_{\pi 3}(\theta+\psi)_{\pi 3}\sin_{\pi 3}(\theta+\psi)_{\pi 3}) \end{aligned}$$

$$k'=k\sqrt{\frac{\cos^2_{\pi 3}\psi_3+\sin^2_{\pi 3}\psi_{\pi 3}+\cos_{\pi 3}\psi_{\pi 3}\sin_{\pi 3}\psi_{\pi 3}}{\cos^2_{\pi 3}(\theta+\psi)_{\pi 3}+\sin^2_{\pi 3}(\theta+\psi)_{\pi 3}+\cos_{\pi 3}(\theta+\psi)_{\pi 3}\sin_{\pi 3}(\theta+\psi)_{\pi 3}}}.$$

Similarly, using the equation below and as shown in the following figure we obtain;

$$d_E(O,A)=d_E(O,A')$$

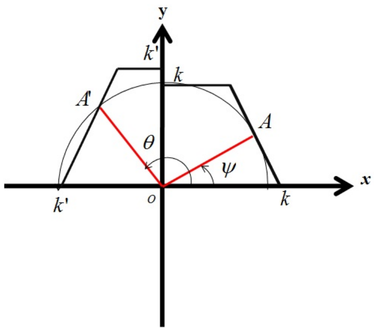


Figure 5.1: Rotation of the line segment *OA* with angle θ

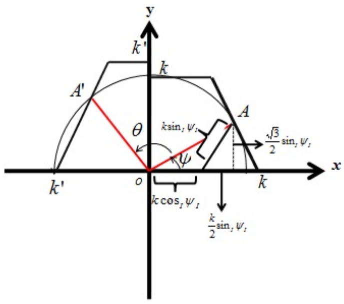


Figure 5.2

and

$$
\begin{aligned}
d_{\pi 3}(O,A) \quad = \quad k \quad &= \left| k_e \cos\psi - \frac{1}{\sqrt{3}} k_e \sin\psi \right| + \left| \frac{2}{\sqrt{3}} k_e \sin\psi \right| \\
&= k_e \left(|\cos\psi| + \left| \frac{1}{\sqrt{3}} \sin\psi \right| \right).
\end{aligned}
$$

Here, k_e referring the Euclidean length of k;

$$
k_e = \frac{k}{|\cos\psi| + \left| \frac{1}{\sqrt{3}} \sin\psi \right|}.
$$

Also;

$$
\begin{aligned}
d_{\pi 3}(O,A') \quad = \quad k' \quad &= \left| k_e \cos(\theta+\psi) - \frac{1}{\sqrt{3}} k_e \sin(\theta+\psi) \right| + \left| \frac{2}{\sqrt{3}} k_e \sin(\theta+\psi) \right| \\
&= |k_e \cos(\theta+\psi)| + \frac{1}{\sqrt{3}} |k_e \sin(\theta+\psi)| \\
&= k_e \left(|\cos(\theta+\psi)| + \frac{1}{\sqrt{3}} |\sin(\theta+\psi)| \right)
\end{aligned}
$$

and

$$
k_e = \frac{k^{'}}{|\cos(\theta+\psi)| + \frac{1}{\sqrt{3}} |\sin(\theta+\psi)|}.
$$

Because of the equality of Euclidean lengths of these two line segments we obtain;

$$
\frac{k}{|\cos\psi| + \frac{1}{\sqrt{3}} |\sin\psi|} = \frac{k^{'}}{|\cos(\theta+\psi)| + \frac{1}{\sqrt{3}} |\sin(\theta+\psi)|}
$$

and

$$
d_{\pi 3}(O,A^{'}) \quad = \quad k^{'} \quad = \quad k \frac{3|\cos(\theta+\psi)| + \sqrt{3}|\sin(\theta+\psi)|}{3|\cos\psi| + \sqrt{3}|\sin\psi|}. \tag{5.1}
$$

□

The following corrolary shows how we can find the $d_{\pi 3}-$length, after rotation of a line segment with an angle θ in standart form.

Corollary 5.2. *Let OA be a line segment on the x−axis. If $OA^{'}$ is the image of OA under the rotation with an angle $\theta_{\pi 3}$(or θ in the standart form) then*

$$d_{\pi 3}(O,A^{'}) = k' = k\frac{3\left|\cos\theta\right| + \sqrt{3}\left|\sin\theta\right|}{3}$$

Using the value $\psi = 0$ in equation (5.1), we get the equation in the corrolary.

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