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Space-Fractional Transport Equation

Arman Aghili[1]

[1] *University of Guilan, Faculty of Mathematical Sciences, Department of Applied Mathematics, Rasht - IRAN , P.O.Box 1841*

Abstract

In this article, the author consider certain space fractional equations using integral transforms and exponential operators. Transform method is a powerful tool for solving singular integral equations, evaluation of certain integrals and solution to partial fractional differential equations. The result reveals that the exponential operators method is very convenient and effective. Constructive examples occur throughout the paper.

Keywords: *Laplace transforms; Exponential operators; Transport equation; Parabolic cylinder function; Modified Bessel's function.*
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1. Introduction

In the last three decades, it has turned out that many phenomena in fluid mechanics, physics, biology, engineering and other areas of sciences can be successfully modelled by the use of fractional derivatives. Fractional differential equations arise in unification of diffusion and wave propagation phenomenon. The space fractional transport equation, which is a mathematical model of a wide range of important physical phenomena, is a partial differential equation obtained from the classical transport equation by replacing the first derivative by a fractional Riemann-Liouville derivative of order α. In this work, we considered methods and results for a partial fractional differential equations which arise in applications. Several methods have been introduced to solve partial fractional differential equations, the popular Laplace transform method [2], [3], [4], [10], the Fourier transform method [1], and operational method [5]. However, most of these methods are suitable for special types of fractional differential equations, mainly the linear with constant coefficients. More detailed information can be found in a survey paper by Kilbas and Trujillo [13], Atanakovic and Stankovic [7], [8], used the Laplace transform in a certain space of distributions to solve a system of partial differential equations with fractional derivatives, and indicated that such a system may serve as a certain model for a visco-elastic rod. Oldham and Spanier I [14], [15] respectively, by reducing a boundary value problem involving Fick's second law in electro-analytic chemistry to a formulation based on the partial Riemann-Liouville fractional with half derivative. Oldham and Spanier [15] gave other application of such equations for diffusion problems. Schneider [18], considered the time fractional diffusion and wave equations and obtained the solution in terms of Fox functions. Usta [20], used radial basis functions methods to solve the fractional Poisson equations via Kansa's collocation method. However, there are some physical problems that are modeled with both space and time fractional advection-dispersion equation like space-time fractional Fokker-Planck equation which is an effective tool for processes with both traps and flights, in which the time fractional term characterizes the traps and the space fractional term charaterizes the flights [10].

1.1. Definitions and Notations

Definition 1.1. *The left Riemann-Liouville fractional derivative of order α $(0<\alpha<1)$ of $\phi(t)$ is defined as follows [16].*

$$D_a^{R.L,\alpha}\phi(t)=\frac{1}{\Gamma(1-\alpha)}\frac{d}{dt}\int_a^t\frac{1}{(t-\xi)^\alpha}\phi(\xi)d\xi.$$

Definition 1.2. *The left Caputo fractional derivative of order α $(0<\alpha<1)$ of $\phi(t)$, is as follows*

$$D_a^{c,\alpha}\phi(t)=\frac{1}{\Gamma(1-\alpha)}\int_a^t\frac{1}{(t-\xi)^\alpha}\phi'(\xi)d\xi.$$

Note. *Recently, certain authors [21], [22], generalize the conformable fractional derivative and integral and obtain several results such as the product rule, quotient rule, chain rule and mean-value theorem for the conformable fractional derivative.*

The use of Laplace transforms in applications is quite extensive. Therefore, we will make no attempt to consider all the various applications involving this integral transform, but rather briefly discusses how the Laplace transform is used in several areas of applications.

Email addresses: **arman.aghili@gmail.com, ORCID: https://orcid.org/0000-0002-3758-2599 (A. Aghili)**

Definition 1.3. *The Laplace transform of function $f(t)$ is as follows*

$$\mathscr{L}\{f(t)\} = \int_0^{\infty} e^{-st} f(t)dt := F(s).$$

If $\mathscr{L}\{f(t)\} = F(s)$, then $\mathscr{L}^{-1}\{F(s)\}$ is given by

$$f(t) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{st} F(s)ds.$$

Where $F(s)$ is analytic in the region $\mathrm{Re}(s) > c$. The above complex integration is known as Bromwich integral [11].

Corollary 1.4. *Let $\mathscr{L}(\phi(t)) = \Phi(s)$ then the following identity holds true.*

$$\mathscr{L}\left[\int_0^{\infty} \frac{\sin(2\sqrt{t\xi})}{\sqrt{\pi\xi}}\phi(\xi)d\xi\right] = \frac{1}{s\sqrt{s}}\Phi\left(\frac{1}{s}\right).$$

Proof. See [6]. □

Singular integral equations arise in many problems of mathematical physics. The mathematical formulation of physical phenomena often involves singular integral equations. Applications in many important fields, like elastic contact problems, the theory of porous filtering contain integral and integro-differential equation with singular kernel.

Example 1.5. *Let us solve the following singular integral equation with trigonometric kernel.The Laplace transform provides a useful technique for the solution of such singular integral equations.*

$$\int_0^{+\infty} \frac{\sin(2\sqrt{t\xi})}{\sqrt{\pi\xi}}\phi(\xi)d\xi = \gamma + \ln t. \tag{1.1}$$

Solution. In view of the above corollary upon taking the Laplace transform of the given integral equation, yields

$$\frac{1}{s\sqrt{s}}\Phi\left(\frac{1}{s}\right) = -\frac{\ln s}{s}, \tag{1.2}$$

solving the above equation, leads to

$$\Phi\left(\frac{1}{s}\right) = -\sqrt{s}\ln s,$$

or

$$\Phi(s) = \frac{\ln s}{\sqrt{s}},$$

so that upon taking the inverse Laplace transform, we arrive at the solution

$$\phi(t) = \mathscr{L}^{-1}\frac{\ln s}{\sqrt{s}} = \frac{1}{\sqrt{\pi}}(\psi(0.5) - \ln t),$$

where $\psi(.)$ stands for the logarithmic derivative of the gamma function.
Lastly, the substitution of the obtained solution into the integral equation (1.1), yields the following integral identity

$$\int_0^{+\infty} \frac{\sin(2\sqrt{t\xi})}{\pi\sqrt{\xi}}(\psi(0.5) - \ln\xi)d\xi = \gamma + \ln t.$$

At this stage, by making the substitution $t = 1$, in the above relation we get the following

$$\gamma = \int_0^{+\infty} \frac{\sin(2\sqrt{\xi})}{\pi\sqrt{\xi}}(\psi(0.5) - \ln\xi)d\xi.$$

Many problems of physical interest lead to Laplace transform whose inverses are not readily expressed in terms of tabulated functions. Therefore, it is highly desirable to have methods for inversion. In this section an algorithm to invert the Laplace transform is presented.

Remark. In the next Lemmas, we need the following integral representation for the modified Bessel's function of the second kind [6], [11],

1. $K_\nu(ax) = \left(\frac{ax}{2}\right)^{\nu}\int_0^{+\infty} e^{-\xi - \frac{a^2x^2}{4\xi}}\frac{d\xi}{2\xi^{\nu+1}},$

and in special cases $\nu = 0, \nu = 1$, we have

2. $K_0(ax) = \int_0^{+\infty} e^{-\xi - \frac{a^2x^2}{4\xi}}\frac{d\xi}{2\xi},$

3. $K_1(ax) = \left(\frac{ax}{2}\right)\int_0^{+\infty} e^{-\xi - \frac{a^2x^2}{4\xi}}\frac{d\xi}{2\xi^2},$

Lemma 1.6. *By using an appropriate integral representation for the modified Bessel's functions of the second kind of order ν, $K_\nu(2\sqrt{s})$, show that*

$$\mathscr{L}^{-1}\{s^{\frac{\nu}{2}}K_\nu(2\sqrt{s})\} = \frac{e^{-\frac{1}{t}}}{2t^{\nu+1}}.$$

Solution. In view of the Definition 1.3, taking the inverse Laplace transform of the given $s^{\frac{\nu}{2}}K_\nu(2\sqrt{s})$, we obtain

$$f(t) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{st}(s^{\frac{\nu}{2}}K_\nu(2\sqrt{s}))ds, \tag{1.3}$$

at this stage, using the following integral representation for $K_\nu(2\sqrt{s})$.

$$s^{\frac{\nu}{2}}K_\nu(2\sqrt{s}) = \frac{1}{2}\int_0^\infty e^{-\xi-\frac{s}{\xi}}\xi^{\nu-1}d\xi. \tag{1.4}$$

By setting relation (1.4) in (1.3), we obtain

$$f(t) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{st}\left(\frac{1}{2}\int_0^\infty e^{-\xi-\frac{s}{\xi}}\xi^{\nu-1}d\xi\right)ds, \tag{1.5}$$

let us change the order of integration in relation (1.5), we arrive at

$$f(t) = \frac{1}{2}\int_0^\infty e^{-\xi}\xi^{\nu-1}\left(\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{s(t-\frac{1}{\xi})}ds\right)d\xi,$$

the value of the inner integral is $\delta(t-\frac{1}{\xi})$, therefore

$$f(t) = \int_0^\infty e^{-\xi}\xi^{\nu-1}\delta\left(t-\frac{1}{\xi}\right)d\xi,$$

after making a change of variable $t-\frac{1}{\xi}=\psi$, and considerable algebra, we obtain

$$f(t) = \frac{1}{2}\int_{-\infty}^{t}\delta(\psi)\frac{e^{\frac{1}{(t-\psi)}}}{(t-\psi)^{\nu-1}}\frac{d\psi}{(t-\psi)^2} = \frac{e^{-\frac{1}{t}}}{2t^{\nu+1}}.$$

Let us consider the special cases $\nu = 0.5$ and $\nu = 0$, we get the following relations

$$\mathscr{L}^{-1}\{s^{\frac{1}{4}}K_{\frac{1}{2}}(2\sqrt{s}\} = \frac{e^{-\frac{1}{t}}}{2t\sqrt{t}}. \tag{1.6}$$

$$\mathscr{L}^{-1}\{K_0(2\sqrt{s}\} = \frac{e^{-\frac{1}{t}}}{2t}. \tag{1.7}$$

Corollary 1.7. *We have the following identity*

$$\mathscr{L}^{-1}\{s^{\frac{1}{4}}K_{\frac{1}{2}}(2\sqrt{s})K_0(2\sqrt{s})\} = \int_0^t \frac{e^{-\frac{1}{\eta}-\frac{1}{t-\eta}}}{2\eta(t-\eta)\sqrt{\eta}}d\eta.$$

Proof. By using convolution theorem for the Laplace transforms and in view of the relations (1.6) and (1.7) we get the result. □

Bessel's function shows up in many problems of engineering and physics, in Fourier theory and harmonic analysis. The earliest systematic study of the Bessel's function was made in a problem connected with planetary motions.

Lemma 1.8. *The following integral identity holds true.*

$$\frac{a}{\sqrt{a^2b^2+c^2}}K_1\left(\frac{\sqrt{a^2b^2+c^2}}{b}\right) = \int_0^\infty e^{-a\sqrt{b^2x^2+1}}\cos(cx)dx.$$

Proof. Let us start with the following elementary integral

$$\frac{2\lambda}{\sqrt{\pi}}\int_0^\infty e^{-\lambda^2\xi^2-\frac{\eta^2}{\xi^2}}d\xi = e^{-2\lambda\eta}.$$

In view of the above integral, the right hand side of the identity can be written as follows

$$\int_0^\infty e^{-a\sqrt{b^2x^2+1}}\cos(cx)dx = \frac{a}{\sqrt{\pi}}\int_0^\infty \cos(cx)\left(\int_0^\infty e^{-\frac{a^2\xi^2}{4}-\frac{b^2x^2+1}{\xi^2}}d\xi\right)dx.$$

Changing the order of integration, leads to

$$\int_0^\infty e^{-a\sqrt{b^2x^2+1}}\cos(cx)dx = \frac{a}{\sqrt{\pi}}\int_0^\infty e^{-\frac{a^2\xi^2}{4}-\frac{1}{\xi^2}}\left(\int_0^\infty e^{-\frac{b^2x^2}{\xi^2}}\cos(cx)dx\right)d\xi.$$

After evaluating the inner integral and simplifying, we get the following

$$\int_0^\infty e^{-a\sqrt{b^2x^2+1}}\cos(cx)dx = \frac{a}{2b}\int_0^\infty e^{-\frac{(a^2b^2+c^2)\xi^2}{4b^2}-\frac{1}{\xi^2}}\xi d\xi.$$

At this point, making a change of variable $\xi^2 = \frac{1}{u}$, after simplifying and using integral representation for modified Bessel's function of second kind of first order, we arrive at

$$\int_0^\infty e^{-a\sqrt{b^2x^2+1}}\cos(cx)dx = \frac{a}{4b}\int_0^\infty e^{-u-\frac{(a^2b^2+c^2)u^{-1}}{4b^2}}\frac{du}{u^2} = \frac{a}{\sqrt{a^2b^2+\omega^2}}K_1\left(\frac{\sqrt{a^2b^2+\omega^2}}{b}\right),$$

We can write the above relation in terms of Fourier-cosine transform as below

$$\mathscr{F}_c(e^{-a\sqrt{b^2x^2+1}};x->\omega) = \frac{a}{\sqrt{a^2b^2+\omega^2}}K_1\left(\frac{\sqrt{a^2b^2+\omega^2}}{b}\right),$$

the inverse Fourier cosine transform of which leads to

$$e^{-a\sqrt{b^2x^2+1}} = \frac{2}{\pi}\int_0^\infty \cos(x\omega)\frac{a}{\sqrt{a^2b^2+\omega^2}}K_1\left(\frac{\sqrt{a^2b^2+\omega^2}}{b}\right)d\omega.$$

□

Note. If we set $x = 0$ in the above integral we obtain

$$e^{-a} = \frac{2}{\pi}\int_0^\infty \frac{a}{\sqrt{a^2b^2+\omega^2}}K_1\left(\frac{\sqrt{a^2b^2+\omega^2}}{b}\right)d\omega.$$

1.1.1. Generalized Abel's Integral Equation

The Abel integral equation is well studied, and there exists many sources to its applications in different fields. In [19], The authors established analytical solutions within local fractional Volterra and Abel's integral equations via the Yang–Laplace transform method.
In the next example we consider a singular integral equation.This type of singular integral equation arises in the theory of wave propagation over a flat surface. Such singular integral equations occur rather frequently in mathematical physics and possess very interesting properties.

Example 1.9. *Let us solve the following generalized Abel integral equation of the second kind.*

$$\phi(t) = \frac{e^{-\frac{\lambda^2}{t}}}{\sqrt{t}} + \frac{1}{\Gamma(\alpha)}\int_0^t \frac{\phi(\xi)}{(t-\phi)^{1-\alpha}}d\xi. \qquad 0<\alpha<1.$$

Note. The above singular integral equation can be written in terms of Riemann-Liouville fractional integral [14] as below

$$\phi(t) = \frac{e^{-\frac{\lambda^2}{t}}}{\sqrt{t}} + I_{0,x}^{R-L,\alpha}\phi(t). \qquad 0<\alpha<1.$$

Solution. Upon taking the Laplace transform of the given integral equation, after simplifying we arrive at

$$\Phi(s) = \mathscr{L}(\frac{e^{-\frac{\lambda^2}{t}}}{\sqrt{t}}) + \frac{\Phi(s)}{s^\alpha},$$

solving transformed equation leads to

$$\Phi(s) = \mathscr{L}\left(\frac{e^{-\frac{\lambda^2}{t}}}{\sqrt{t}}\right)\left(1-\frac{1}{s^\alpha}\right)^{-1} = \mathscr{L}\left(\frac{e^{-\frac{\lambda^2}{t}}}{\sqrt{t}}\right) + \mathscr{L}\left(\frac{e^{-\frac{\lambda^2}{t}}}{\sqrt{t}}\right)\left(\frac{1}{s^\alpha-1}\right).$$

At this stage, taking the inverse Laplace transform leads to

$$\phi(t) = \frac{e^{-\frac{\lambda^2}{t}}}{\sqrt{t}} + \int_0^t \frac{e^{-\frac{\lambda^2}{t-\eta}}}{\sqrt{t-\eta}}\mathscr{L}^{-1}\left(\frac{1}{s^\alpha-1}\right)d\eta.$$

Let us consider the special case $\alpha = 0.5$, we obtain the following

$$\phi(t) = \frac{e^{-\frac{\lambda^2}{t}}}{\sqrt{t}} + \int_0^t \frac{e^{-\frac{\lambda^2}{t-\eta}}}{\sqrt{t-\eta}} \left(\frac{1}{\sqrt{\pi\eta}} + e^{\eta} Erfc(-\sqrt{\eta}) \right) d\eta.$$

Definition 1.10. *The most important use of the Caputo fractional derivative is treated in initial value problems where initial conditions are expressed in terms of integer order derivatives. In this respect, it is interesting to know the Laplace transform of this kind of derivative.*

$$\mathscr{L}\{D_a^{c,\alpha} f(t)\} = sF(s) - f(0+), 0 < \alpha < 1.$$

and generally [16]

$$\mathscr{L}\{D_a^{c,\alpha} f(t)\} = s^{\alpha-1}F(s) - \sum_{k=0}^{k=m-1-k} s^{\alpha-1-k} f^k(0+), m-1 < \alpha < m.$$

The Laplace transform provides a useful technique for the solution of such fractional singular integro-differential equations.

Lemma 1.11. *Let $\mathscr{L}\{f(t)\} = F(s)$, then, the following identities hold true.*

1. $\mathscr{L}^{-1}[\exp(-k\sqrt{s})] = \frac{k}{(2\sqrt{\pi})} \int\limits_0^\infty \exp(-s\xi - \frac{k^2}{4\xi}) d\xi$
2. $\exp(-\omega s^\alpha) = \frac{1}{\pi} \int_0^\infty \exp(-\omega r^\alpha \cos\alpha\pi) \sin(\omega r^\alpha \sin\alpha\pi) (\int_0^\infty \exp(-(s+r)\tau) d\tau) dr.$

Proof. See [2]. □

Example 1.12. *Let us solve the following fractional Volterra integral equation of convolution type with Bessel's function as kernel.*

$$\lambda \int_0^t J_0(2\sqrt{\eta(t-\xi)}) D^{c,\alpha} \phi(\xi) d\xi = (\frac{t}{r})^{\frac{\beta}{2}} I_\beta(2\sqrt{rt}) \qquad \phi(0) = 0, \qquad r + \eta \geq 1.$$

Solution. Upon taking the Laplace transform of the given integral equation, we obtain

$$\lambda s^\alpha \Phi(s) \frac{e^{-\frac{\eta}{s}}}{s} = \frac{e^{\frac{r}{s}}}{s^{1+\beta}},$$

solving the above equation, leads to

$$\Phi(s) = \frac{e^{\frac{\eta+r}{s}}}{\lambda s^{\alpha+\beta}},$$

at this point, taking the inverse Laplace transform term wise, after simplifying we obtain

$$\phi(t) = \frac{1}{\lambda} \left(\frac{t}{r+\eta})^{\frac{\beta+\alpha-1}{2}} I_{\alpha+\beta-1}(2\sqrt{(r+\eta)t} \right).$$

Note: $I_\eta(.)$ stands for the modified Bessel's function of the first kind of order η.

Lemma 1.13. *Assume that $\mathscr{L}\{f(t)\} = F(s)$, then we have the following identity*

$$\mathscr{L}\{t^\nu f(t^2)\} = \frac{1}{4\sqrt{\pi}} \int_0^\infty \xi^{\nu-2} e^{-\frac{s^2\xi^2}{2}} D_\nu(s\xi) F\left(\frac{1}{2\xi^2}\right) d\xi.$$

Proof. See [15]. □

The above Lemma has some interesting applications as below

Lemma 1.14. *We have the following identities*

1. $\exp(-\sqrt{\lambda})\lambda^{\frac{\nu-1}{2}} = \frac{1}{2\sqrt{\pi}} \int_0^\infty \xi^{\nu-2} \exp\left(-\left(\frac{\xi^2}{2} + \frac{\lambda}{2\xi^2}\right)\right) D_\nu(\xi) d\xi.$

2. $\frac{2\sqrt{\pi}}{e} = \int_0^\infty \xi^{\nu-2} \exp\left(-\left(\frac{\xi^2}{2} + \frac{\lambda}{2\xi^2}\right)\right) D_\nu(\xi) d\xi.$

Proof. In Lemma 1.13, let us assume that $f(t)=\delta(t-\lambda)$, then we have $F(s)=e^{-\lambda s}$,

$$\mathscr{L}\{t^{\nu}f(t^2)\}=\int_0^{\infty}e^{-st}t^{\nu}\delta(t^{\nu}-\lambda)dt=\frac{1}{2}\lambda^{\frac{\nu-1}{2}}e^{-s\sqrt{\lambda}}.$$

After setting all the obtained results in the above identity and taking $s=1$, after simplifying we arrive at

$$e^{-\sqrt{\lambda}}\lambda^{\frac{\nu-1}{2}}=\frac{1}{2\sqrt{\pi}}\int_0^{\infty}\xi^{\nu-2}\exp\left(-\left(\frac{\xi^2}{2}+\frac{\lambda}{2\xi^2}\right)\right)D_{\nu}(\xi)d\xi.$$

At this stage, if we put $\lambda=1$ in the above identity, after simplifying we get the following

$$\int_0^{\infty}\xi^{\nu-2}\exp\left(-\left(\frac{\xi^2}{2}+\frac{\lambda}{2\xi^2}\right)\right)D_{\nu}(\xi)d\xi=\frac{2\sqrt{\pi}}{e}$$

□

Theorem 1.15. *The following integral identity holds true*

$$\frac{1}{4\sqrt{2}}\int_0^{\infty}\xi^{\nu-\frac{1}{2}}e^{-\frac{\xi^2}{2}}D_{\nu}(\xi)d\xi=\Gamma(\nu+2).$$

Proof. Let us take $f(t)=\sqrt{t}$, then we have

$$\mathscr{L}(f(t))=F(s)=\mathscr{L}(\sqrt{t})=\frac{\sqrt{\pi}}{2s\sqrt{s}}.$$

At this point, we can evaluate $\mathscr{L}(t^{\nu}\sqrt{t^2})$, in two different ways as follows first, by the definition of the Laplace transforms we have

$$\mathscr{L}(t^{\nu}\sqrt{t^2})=\int_0^{+\infty}e^{-st}t^{\nu+1}dt=\frac{\Gamma(\nu+2)}{s^{\nu+2}},$$

second, by using the Lemma 1.8, we get

$$\mathscr{L}(t^{\nu}t)=\frac{1}{4\sqrt{\pi}}\int_0^{\infty}\xi^{\nu-2}e^{-\frac{s^2\xi^2}{2}}D_{\nu}(s\xi)\sqrt{\frac{\pi}{2}}\xi^{\frac{3}{2}}d\xi,$$

consequently, we obtain the following result

$$\mathscr{L}(t^{\nu+1})=\frac{1}{4\sqrt{2}}\int_0^{\infty}\xi^{\nu-\frac{1}{2}}e^{-\frac{s^2\xi^2}{2}}D_{\nu}(s\xi)d\xi=\frac{\Gamma(\nu+2)}{s^{\nu+2}},$$

simplifying leads to the following relation

$$\frac{1}{4\sqrt{2}}\int_0^{\infty}\xi^{\nu-\frac{1}{2}}e^{-\frac{s^2\xi^2}{2}}D_{\nu}(s\xi)d\xi=\frac{\Gamma(\nu+2)}{s^{\nu+2}},$$

in the above integral relation, let us choose $s=1$, after simplifying,
we obtain

$$\frac{1}{4\sqrt{2}}\int_0^{\infty}\xi^{\nu-\frac{1}{2}}e^{-\frac{\xi^2}{2}}D_{\nu}(\xi)d\xi=\Gamma(\nu+2).$$

In the above relation $D_{\nu}(.)$ stands for the Weber Parabolic Cylinder function of order ν [6]. We have the following integral representation for the Weber Parabolic Cylinder function

$$D_{\nu}(t)=\sqrt{\frac{2}{\pi}}e^{\frac{t^2}{4}}\int_0^{\infty}\xi^{\nu}e^{-\frac{\xi^2}{2}}\cos\left(t\xi-\frac{\pi\nu}{2}\right)d\xi.$$

□

Definition 1.16. *The Stieltjes transform (second iterate of the Laplace transform) of a function $\phi(x):R_{+}->C$ is defined by means of*

$$\mathscr{S}[\phi(x)](s)=\mathscr{L}[\mathscr{L}[\phi(x)](s)=\int_0^{+\infty}\frac{\phi(x)}{x+s}dx=\Phi(s).$$

Provided that the integral exists.
Inversion formula for the Stieltjes transform.
To formally recover the function$\phi(x)$ from the transform function $\Phi(s)$, we consider the inversion formula [6], [12].

$$\mathscr{S}^{-1}[\Phi(s);x]=\frac{1}{\pi}Im[lim_{s->xe^{-i\pi}}\Phi(s)].$$

Lemma 1.17. *We have the following integral identity*

$$\int_0^{+\infty}\frac{J_{\nu}^2(\sqrt{\xi})}{\xi+\eta}d\xi=2I_{\nu}(\sqrt{\eta})K_{\nu}(\sqrt{\eta}).$$

Proof. In view of the definition (1.2), we need to evaluate

$$\mathscr{S}^{-1}[2I_\nu(\sqrt{\eta})K_\nu(\sqrt{\eta})] = \frac{1}{\pi}Im[lim_{\eta->\xi e^{-i\pi}}[2I_\nu(\sqrt{\eta})K_\nu(\sqrt{\eta})]].$$

Thus, we have

$$\mathscr{S}^{-1}[2I_\nu(\sqrt{\eta})K_\nu(\sqrt{\eta})] = \frac{1}{\pi}Im[2I_\nu(\sqrt{\xi e^{-i\pi}})K_\nu(\sqrt{\xi e^{-i\pi}})].$$

At this stage, let us recall the following well known identities for the modified Bessel's functions

$$K_\nu(z) = \frac{\pi}{2}\frac{I_{-\nu}(z) - I_\nu(z)}{\sin(\pi\nu)}, \quad I_\nu(z) = e^{-\frac{i\nu\pi}{2}}J_\nu(iz), \quad e^{i\pi\nu} = \cos(\pi\nu) + i\sin(\pi\nu).$$

Therefore, after simplifying we arrive at

$$\mathscr{S}^{-1}[2I_\nu(\sqrt{\eta})K_\nu(\sqrt{\eta})] == \frac{1}{\sin\pi\nu}Im[(J_\nu(\sqrt{\xi})J_{-\nu}(\sqrt{\xi}) - \cos\pi\nu) + i(\sin\pi\nu J_\nu^2(\xi))] = J_\nu^2(\sqrt{\xi}).$$

□

2. Solution to Non-Uniform Space Fractional Transport Equation with Decay via the Exponential Operators Method

In recent years a lot of attention has been devoted to the study of the exponential differential operators method to investigate various scientific models.

Lemma 2.1. *The following exponential operator identities hold true.*

1. $\exp(\pm\lambda\frac{d}{dt})\Phi(t) = \Phi(t\pm\lambda)$,
2. $\exp(\pm\lambda t\frac{d}{dt})\Phi(t) = \Phi(te^{\pm\lambda})$,
3. $\exp(\lambda q(t)\frac{d}{dt})\Phi(t) = \Phi(Q(F(t)+\lambda))$.

Where $F(t)$ is the primitive function of $\frac{1}{q(t)}$ and $Q(t)$ is the inverse function of $F(t)$.

Remark 2.2. *The most commonly known exponential operator is the operator of translation* $\exp\left(\lambda\frac{d}{dt}\right)$*, which acts on the function $\Phi(t)$ as follows:*

1. $$\exp\left((\lambda\frac{d}{dt}\right)\Phi(t) = \Phi(t+\lambda),$$

and

2. $$\exp\left(\lambda t\frac{d}{dt}\right)\Phi(t) = \Phi(e^\lambda t).$$

Proof. For detail proof of the above Lemma, see [9]. □

It is well known that the combined use of integral transform and operational exponent method provides a fast and universal mathematical tool for obtaining the exact solutions of partial differential equations. Let us consider the non-uniform space fractional transport equation with decay of the following form

$$u_t + v(x)D_x^{R.L,\alpha}u = \eta u, u(x,0) = \phi(x), \eta > 0, 0 < \alpha < 1,$$

where the wave speed $v(x)$ is now allowed to depend on the spatial position, and $\frac{dx}{dt} = v(x)$ is called characteristic curve for the transport equation with wave speed $v(x)$ [11], [17].

Problem 2.3. *Let us solve the following space fractional transport equation with decay.*

$$u_t + v(x)D_x^{R.L,\alpha}u = \eta u, \tag{2.1}$$

$$u(x,0) = \phi(x) \qquad 0 < \alpha \le 1.$$

Solution: In order to obtain a solution for equation (2.1), let us rewrite the above equation as follows

$$u_t = (\eta - v(x)D_x^{R.L,\alpha})u, \tag{2.2}$$

$$u(x,0) = \phi(x) \qquad 0 < \alpha \le 1, \eta > 0. \tag{2.3}$$

For solving the above space fractional equation (2.2), (2.3), in view of [2], first by solving the first order PDE with respect to t, and applying the initial condition (2.3), we get the following relation

$$u(x,t) = \exp(\eta t)\exp(-tv(x)D_x^{R.L,\alpha})\phi(x).$$

At this point, in order to find the result of the action of exponential operator, we may use part two of Lemma 1.6 by setting; $w = tv(x)$ and $s = \frac{\partial}{\partial x}$ to obtain

$$e^{-tv(x)(\frac{\partial}{\partial x})^{\alpha}}\phi(x) = \frac{1}{\pi}\int_0^{\infty} e^{-r^{\alpha}tv(x)\cos\alpha\pi}\sin(tv(x)r^{\alpha}\sin\alpha\pi)\left(\int_0^{\infty} e^{-r\tau-\tau\frac{\partial}{\partial x}}\phi(x)\right)d\tau dr,$$

thus, it follows that

$$u(x,t) = \frac{e^{\eta t}}{\pi}\int_0^{\infty} e^{-r^{\alpha}tv(x)\cos\alpha\pi} sin(tv(x)r^{\alpha}\sin\alpha\pi)\left(\int_0^{\infty} e^{-r\tau}\phi(x-\tau)d\tau\right)dr.$$

At this stage, let us consider the special case $\alpha = 0.5$, after simplifying we get

$$u(x,t) = \frac{e^{\eta t}}{\pi}\int_0^{\infty}\sin(tv(x)\sqrt{r})\left(\int_0^{\infty} e^{-r\tau}\phi(x-\tau)d\tau\right)dr,$$

changing the order of integration we obtain

$$u(x,t) = \frac{e^{\eta t}}{\pi}\int_0^{\infty}\phi(x-\tau)\left(\int_0^{\infty} e^{-\tau r}\sin(tv(x)\sqrt{r})dr)d\tau\right),$$

evaluation of the inner integral and simplifying leads to the following

$$u(x,t) = \frac{2e^{\eta t}}{\sqrt{\pi}}\int_0^{\infty}\frac{tv(x)}{4\tau\sqrt{\tau}}e^{-\frac{t^2v^2(x)}{4\tau}}\phi(x-\tau)d\tau). \tag{2.4}$$

Let us make a change of variable $\frac{t^2v^2(x)}{4\tau} = \xi$ in the above integral, after simplifying we get the solution to space fractional transport equation

$$u(x,t) = \frac{e^{\eta t}}{\sqrt{\pi}}\int_0^{\infty}\xi^{-\frac{1}{2}}e^{-\xi}\phi\left(x-\frac{t^2v^2(x)}{4\xi}\right)d\xi.$$

Note: It is easy to verify that $u(x,0) = \phi(x)$.

Example 2.4. *Let us solve the following space-fractional transport equation with decay.*

$$u_t + v(x)D_x^{R.L,\alpha}u = \eta u, \tag{2.5}$$

$$u(x,0) = \exp(x) \qquad v(x) = \frac{\beta^2}{2x^2}, \eta > 0, \alpha = 0.5. \tag{2.6}$$

Solution: In order to obtain a solution for equation (2.5)-(2.6), we may use relation (2.4) to obtain the solution as below

$$u(x,t) = \frac{2e^{x+\eta t}}{\sqrt{\pi}}\int_0^{\infty}\left(\frac{t\beta^2}{8x^2}\right)\xi^{-\frac{3}{2}}\exp\left(-\xi-\frac{t^2\beta^4}{16x^4\xi}\right)d\xi.$$

After simplifying, we have the following solution

$$u(x,t) = \frac{e^{x+\eta t}}{\sqrt{\pi}}\left(\frac{t\beta^2}{2x^2}\right)\int_0^{\infty} e^{-\xi-\frac{\left(\frac{t\beta^2}{4x^2}\right)^2}{\xi}}\frac{d\xi}{2\xi^{0.5+1}} = \frac{e^{x+\eta t}}{\sqrt{\pi}}\left[\frac{\beta\sqrt{t}}{x\sqrt{2}}K_{\frac{1}{2}}\left(\frac{t\beta^2}{4x^2}\right)\right]. \tag{2.7}$$

At this point, using the fact that $K_{\frac{1}{2}}(\phi) = \sqrt{\frac{\pi}{2\phi}}e^{-\phi}$, after simplifying we obtain

$$u(x,t) = e^{\eta t + x - \frac{\beta^2 t}{4x^2}}.$$

Note: For small ϕ we have the following asymptotic formula for the modified Bessel's functions of the second kind

$$lim_{\phi->0}[\phi^{\nu}K_{\nu}(\phi)] = 2^{\nu-1}\Gamma(\nu).\nu > 0. \tag{2.8}$$

Now, in (2.7), if we take limit as t tends to zero, in view of (2.8) we obtain $u(x,0) = e^x$.

3. Conclusion

The paper is devoted to study and application of the Laplace transform for solving certain space fractional partial differential equations, singular integral equation with trigonometric kernel and evaluation of certain integrals. The combined use of the integral transform and exponential operator method provides powerful method for analysing linear systems. The main purpose of this work is to develop a method for finding analytic solution of fractional PDEs.

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