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Dual Jacobsthal Quaternions

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Abstract
In this paper, dual Jacobsthal quaternions were defined. Also, the relations between dual Jacobsthal quaternions which connected with Jacobsthal and Jacobsthal-Lucas numbers were investigated. Furthermore, Binet's formula, Honsberger identity, D'ocagne's identity, Cassini's identity and Catalan's identity for these quaternions were given.

Keywords: Jacobsthal number, Jacobsthal-Lucas number, Jacobsthal quaternion, dual Jacobsthal quaternion.
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1. Introduction

In 1843, Hamilton [1] introduced the set of quaternions which can be represented as

$$H = \{ q = q_0 + i\,q_1 + j\,q_2 + k\,q_3 \mid q_0, q_1, q_2, q_3 \in \mathbb{R} \} \tag{1.1}$$

where

$$i^2 = j^2 = k^2 = -1, \; i\,j = -j\,i = k, \quad j\,k = -k\,j = i, \quad k\,i = -i\,k = j.$$

After the work of Hamilton, several authors worked on different quaternions and their generalizations. ([2]-[22]).

In 1973, Sloane [23] introduced the set of Jacobsthal numbers.
Further, in 1988, Horadam [24]-[25] defined the Jacobsthal and Jacobsthal-Lucas sequences $\{J_n\}$ and $\{j_n\}$ with the recurrence relations respectively, as follows

$$J_0 = 0, \; J_1 = 1, \; J_n = J_{n-1} + 2J_{n-2}, \; for\; n \geq 2, \tag{1.2}$$

and

$$j_0 = 2, \; j_1 = 1, \; j_n = j_{n-1} + 2j_{n-2}, \; for\; n \geq 2, \tag{1.3}$$

In 1996, Horadam studied on the Jacobsthal and Jacobsthal-Lucas sequences and he gave Cassini-like formula as follows [26]

$$J_{n+1}J_{n-1} - J_n^2 = (-1)^n . 2^{n-1} \tag{1.4}$$

$$j_{n+1}j_{n-1} - j_n^2 = 3^2.(-1)^{n+1}.2^{n-1} \tag{1.5}$$

The first eleven terms of Jacobsthal sequence $\{J_n\}$ are $0,1,1,3,5,11,21,43, 85,171$ and 341. This sequence is given by the formula

$$J_n = \frac{2^n - (-1)^n}{3} \tag{1.6}$$

The first eleven terms of Jacobsthal-Lucas sequence $\{j_n\}$ are $2,1,5,7,17,31$
$65,127,257,511$ and 1025. This sequence is given by the formula

$$j_n = 2^n + (-1)^n \tag{1.7}$$

Also, we can see the matrix representations of Jacobsthal and Jacobsthal-Lucas numbers in [27],[28]. The members of these integer sequences can also be obtained in different ways: Binet formulae or matrix method by Köken and Bozkurt [27]-[28]. Several authors worked on Jacobsthal numbers and polynomials in [29]-[32].

In 2015, Szynal-Liana and Włoch [33] defined the Jacobsthal quaternions and the Jacobsthal- Lucas quaternions respectively as follows

$$JQ_n = J_n + i\,J_{n+1} + j\,J_{n+2} + k\,J_{n+3}\,, \tag{1.8}$$

and

$$JLQ_n = j_n + i\,j_{n+1} + j\;j_{n+2} + k\,j_{n+3}\,. \tag{1.9}$$

where

$$i^2 = j^2 = k^2 = -1\,,\; i\,j = -j\,i = k\,, \quad j\,k = -k\,j = i\,, \quad k\,i = -i\,k = j.$$

In 2017, Torunbalcı Aydın and Yüce [34] given a new approach to Jacobsthal quaternions. Furthermore, some relations between Jacobsthal and Jacobsthal-Lucas quaternions are given in [34].

In 2017, Taşçı [35] defined k-Jacobsthal and k-Jacobsthal-Lucas quaternions as follows

$$QJ_{k,n} = J_{k,n} + i_1\,J_{k,n+1} + i_2\,J_{k,n+2} + i_3\,J_{k,n+3} \tag{1.10}$$

and

$$Q\,j_{k,n} = j_{k,n} + i_1\;j_{k,n+1} + i_2\;j_{k,n+2} + i_3\;j_{k,n+3} \tag{1.11}$$

where

$$i_1^2 = i_2^2 = i_3^2 = i_1\,i_2\,i_3 = -1\,.$$

In 2017, Cerda-Morales [36] worked on identities of third order Jacobsthal quaternions.
In 2018, Cerda-Morales [37] defined fourth-order Jacobsthal and Jacobsthal-Lucas quaternions as follows

$$QJ_n{}^{(4)} = J_n{}^{(4)} + i\,J_{n+1}{}^{(4)} + j\,J_{n+2}{}^{(4)} + k\,J_{n+3}{}^{(4)} \tag{1.12}$$

and

$$Q\,j_n{}^{(4)} = j_n{}^{(4)} + i\,j_{n+1}{}^{(4)} + j\,j_{n+2}{}^{(4)} + k\,j_{n+3}{}^{(4)} \tag{1.13}$$

In this paper, dual Jacobsthal and dual Jacobsthal-Lucas quaternions will be defined as follows

$$J_D = \{D_n^J = J_n + i\,J_{n+1} + j\,J_{n+2} + k\,J_{n+3} \mid J_n,\, n-th\,\text{Jacobsthal number}\}$$

and

$$j_D = \{D_n^j = j_n + i\,j_{n+1} + j\,j_{n+2} + k\,j_{n+3} \mid j_n, n-th \text{Jacobsthal-Lucas number}\}$$

where

$$i^2 = j^2 = k^2 = i\,j\,k = 0,\ \ i\,j = -j\,i = j\,k = -k\,j = k\,i = -i\,k = 0.$$

All the studies on Jacobsthal quaternions are summarized in Table 1.

Table 1. Types of Jacobsthal quaternions [33]-[35].

	Definition	Multiplication Rules
Jacobsthal quaternion	$JQ_n = (J_n, J_{n+1}, J_{n+2}, J_{n+3})$ $J_n = J_{n-1} + 2J_{n-2},\ J_1 = J_2 = 1$	$(1,i,j,k),\ i^2 = j^2 = k^2 = -1$ $i\,j = -j\,i = k,\ j\,k = -k\,j = i$ $k\,i = -i\,k = j$
k-Jacobsthal quaternion	$QJ_{k,n} = (J_{k,n}, J_{k,n+1}, J_{k,n+2}, J_{k,n+3})$ $QJ_{k,n+2} = k\,QJ_{k,n+1} + 2\,QJ_{k,n}$	$(1,i_1,i_2,i_3)$, $i_1^2 = i_2^2 = i_3^2 = i_1\,i_2\,i_3 = -1$
Dual Jacobsthal quaternion	$D_n^J = (J_n, J_{n+1}, J_{n+2}, J_{n+3})$ $J_n = J_{n-1} + 2J_{n-2},\ J_1 = J_2 = 1$	$(1,i,j,k)\ i^2 = j^2 = k^2 = i\,j\,k = 0$ $i\,j = -j\,i = j\,k = -kj = k\,i = -i\,k = 0$

2. Dual Jacobsthal Quaternions

In this section, the dual Jacobsthal quaternions will be defined. Also, the relations between dual Jacobsthal quaternions which connected with Jacobsthal and Jacobsthal-Lucas numbers were investigated.

Dual Jacobsthal quaternions is defined by relation recurrence (1.2) as follows

$$J_D = \{D_n^J = J_n + i\,J_{n+1} + j\,J_{n+2} + k\,J_{n+3} \,|\, J_n, n-th \text{Jacobsthal number}\} \tag{2.1}$$

where

$$i^2 = j^2 = k^2 = i\,j\,k = 0,\ \ i\,j = -j\,i = j\,k = -k\,j = k\,i = -i\,k = 0. \tag{2.2}$$

Also, the dual Jacobsthal-Lucas quaternion is defined by relation recurrence (1.3) as follows

$$j_D = \{D_n^j = j_n + i\,j_{n+1} + j\,j_{n+2} + k\,j_{n+3} \,|\, j_n, n-th \text{Jacobsthal-Lucas number}\}, \tag{2.3}$$

$$i^2 = j^2 = k^2 = i\,j\,k = 0,\ \ i\,j = -j\,i = j\,k = -k\,j = k\,i = -i\,k = 0.$$

Let $D_n^{J_1}$ and $D_n^{J_2}$ be n-th terms of the dual Jacobsthal quaternion sequence $(D_n^{J_1})$ and $(D_n^{J_2})$ such that

$$D_n^{J_1} = J_n + i\,J_{n+1} + j\,J_{n+2} + k\,J_{n+3} \tag{2.4}$$

and

$$D_n^{J_2} = K_n + i\,K_{n+1} + j\,K_{n+2} + k\,K_{n+3} \tag{2.5}$$

Then, the addition and subtraction of the dual Jacobsthal quaternions is defined by

$$\begin{aligned} D_n^{J_1} \pm D_n^{J_2} = &\quad (J_n + i\,J_{n+1} + j\,J_{n+2} + k\,J_{n+3}) \\ &\quad \pm (K_n + i\,K_{n+1} + j\,K_{n+2} + k\,K_{n+3}) \\ = &\quad (J_n \pm K_n) + i\,(J_{n+1} \pm K_{n+1}) + j\,(J_{n+2} \pm K_{n+2}) \\ &\quad + k\,(J_{n+3} \pm K_{n+3})\,. \end{aligned} \tag{2.6}$$

Multiplication of the dual Jacobsthal quaternions is defined by

$$\begin{aligned} D_n^{J_1}\, D_n^{J_2} = &\quad (J_n + i\,J_{n+1} + j\,J_{n+2} + k\,J_{n+3}) \\ &\quad (K_n + i\,K_{n+1} + j\,K_{n+2} + k\,K_{n+3}) \\ = &\quad (J_n\,K_n) + i(J_nK_{n+1} + J_{n+1}K_n) + j(J_nK_{n+2} + J_{n+2}K_n) \\ &\quad + k(J_nK_{n+3} + J_{n+3}K_n). \end{aligned} \tag{2.7}$$

The scalar and the vector part of D_n^J which is the n-th term of the dual Jacobsthal quaternion (D_n^J) are denoted by

$$S_{D_n^J} = J_n \ \text{ and } \ V_{D_n^J} = iJ_{n+1} + jJ_{n+2} + kJ_{n+3}. \tag{2.8}$$

Thus, the dual Jacobsthal quaternion D_n^J is given by $D_n^J = S_{D_n^J} + V_{D_n^J}$.
Then, relation (2.7) is defined by

$$D_n^{J_1}\, D_n^{J_2} = S_{D_n^{J_1}}\,.S_{D_n^{J_2}} + S_{D_n^{J_1}}\,.V_{D_n^{J_2}} + S_{D_n^{J_2}}\,.V_{D_n^{J_1}}\,. \tag{2.9}$$

The conjugate of the dual Jacobsthal quaternion D_n^J is denoted by $\overline{D_n^J}$ and it is

$$\overline{D_n^J} = J_n - i\,J_{n+1} - j\,J_{n+2} - k\,J_{n+3}. \tag{2.10}$$

The norm of D_n^J is defined as

$$N_{D_n^J} = \left\|D_n^J\right\|^2 = D_n^J\,\overline{D_n^J} = J_n^2. \tag{2.11}$$

Then, we give the following theorem using statements (2.1), (2.2) and

$$\begin{cases} J_nJ_{n+1} + 2J_{n-1}J_n = J_{2n}, \\ J_nJ_{m+1} + 2J_{n-1}J_m = J_{n+m}, \\ J_{n+1} + 2J_{n-1} = j_n, \\ J_n\,j_n = J_{2n}. \end{cases} \tag{2.12}$$

Theorem 2.1. *Let J_n and D_n^J be the n-th terms of the Jacobsthal sequence (J_n) and the dual Jacobsthal quaternion sequence (D_n^J), respectively. In this case, for $n \geq 1$ we can give the following relations:*

$$D_n^J + \overline{D_n^J} = 2J_n, \tag{2.13}$$

$$(D_n^J)^2 + D_n^J\,\overline{D_n^J} = 2J_n\,D_n^J, \tag{2.14}$$

$$D_{n+1}^J + 2\,D_n^J = D_{n+2}^J, \tag{2.15}$$

$$D_n^J - i\,D_{n+1}^J - j\,D_{n+2}^J - k\,D_{n+3}^J = J_n, \tag{2.16}$$

Proof. Proof of four equality can easily be done by the equations

$$D_n^J = J_n + i\,J_{n+1} + j\,J_{n+2} + k\,J_{n+3}, \tag{2.17}$$

$$D^J_{n+1} = J_{n+1} + iJ_{n+2} + jJ_{n+3} + kJ_{n+4} \tag{2.18}$$

(2.13):

$$\begin{aligned} D^J_n + \overline{D^J_n} = &\ (J_n + iJ_{n+1} + jJ_{n+2} + kJ_{n+3}) \\ &+(J_n - iJ_{n+1} - jJ_{n+2} - kJ_{n+3}) \\ = &\ (J_n + J_n) + i(J_{n+1} - J_{n+1}) + j(J_{n+2} - J_{n+2}) \\ &+k(J_{n+3} - J_{n+3}) \\ = &\ 2J_n. \end{aligned}$$

(2.14):

$$\begin{aligned} (D^J_n)^2 + D^J_n\overline{D^J_n} = &\ (J_n + iJ_{n+1} + jJ_{n+2} + kJ_{n+3}) \\ &(J_n + iJ_{n+1} + jJ_{n+2} + kJ_{n+3}) \\ &+(J_n + iJ_{n+1} + jJ_{n+2} + kJ_{n+3}) \\ &(J_n - iJ_{n+1} - jJ_{n+2} - kJ_{n+3}) \\ = &\ (J_nJ_n) + i(J_nJ_{n+1} + J_{n+1}J_n) + j(J_nJ_{n+2} + J_{n+2}J_n) \\ &+k(J_nJ_{n+3} + J_{n+3}J_n) \\ &+J_nJ_n + i(-J_nJ_{n+1} + J_{n+1}J_n) \\ &+j(-J_nJ_{n+2} + J_{n+2}J_n) \\ &+k(-J_nJ_{n+3} + J_{n+3}J_n) \\ = &\ 2J_nJ_n + 2iJ_nJ_{n+1} + 2jJ_nJ_{n+2} + 2kJ_nJ_{n+3} \\ = &\ 2J_n(J_n + iJ_{n+1} + jJ_{n+2} + kJ_{n+3}) \\ = &\ 2J_nD^J_n \end{aligned}$$

(2.15):

$$\begin{aligned} D^J_{n+1} + 2D^J_n = &\ (J_{n+1} + iJ_{n+2} + jJ_{n+3} + kJ_{n+4}) \\ &+2(J_n + iJ_{n+1} + jJ_{n+2} + kJ_{n+3}) \\ = &\ (J_{n+1} + 2J_n) + i(J_{n+2} + 2J_{n+1}) + j(J_{n+3} + 2J_{n+2}) \\ &+k(J_{n+4} + 2J_{n+3}) \\ = &\ J_{n+2} + iJ_{n+3} + jJ_{n+4} + kJ_{n+5} \\ = &\ D^J_{n+2}. \end{aligned}$$

(2.16):

$$\begin{aligned} D^J_n - iD^J_{n+1} - jD^J_{n+2} - kD^J_{n+3} = &\ (J_n + iJ_{n+1} + jJ_{n+2} + kJ_{n+3}) \\ - &\ i(J_{n+1} + iJ_{n+2} + jJ_{n+3} + kJ_{n+4}) \\ - &\ j(J_{n+2} + iJ_{n+3} + jJ_{n+4} + kJ_{n+5}) \\ - &\ k(J_{n+3} + iJ_{n+4} + jJ_{n+5} + kJ_{n+6}) \\ = &\ J_n. \end{aligned}$$

□

Theorem 2.2. *Let D^J_n and D^j_n be the n-th terms of the dual Jacobsthal quaternion sequence (D^J_n) and the dual Jacobsthal-Lucas quaternion sequence (D^j_n), respectively. The following relations are satisfied*

$$D^J_{n+1} + 2D^J_{n-1} = D^j_n, \tag{2.19}$$

$$2D^J_{n+1} - D^J_n = D^j_n. \tag{2.20}$$

Proof. (2.19): From equations (2.17), (2.18) and identity between Jacobsthal number and Jacobsthal-Lucas number $j_n = J_{n+1} + 2J_{n-1}$, it follows that

$$\begin{aligned} D^J_{n+1} + 2D^J_{n-1} = &\ (J_{n+1} + iJ_{n+2} + jJ_{n+3} + kJ_{n+4}) \\ &+2(J_{n-1} + iJ_n + jJ_{n+1} + kJ_{n+2}) \\ = &\ (J_{n+1} + 2J_{n-1}) + i(J_{n+2} + 2J_n) \\ &+j(J_{n+3} + 2J_{n+1}) + k(J_{n+4} + 2J_{n+2}) \\ = &\ j_n + ij_{n+1} + jj_{n+2} + kj_{n+3} \\ = &\ D^j_n. \end{aligned}$$

(2.20): Using the identity between Jacobsthal number and Jacobsthal -Lucas number $J_n + j_n = 2J_{n+1}$, we get

$$\begin{aligned} 2\,D_{n+1}^J - D_n^J = &\quad 2(J_{n+1} + i\,J_{n+2} + j\,J_{n+3} + k\,J_{n+4}) \\ &-(J_n + i\,J_{n+1} + j\,J_{n+2} + k\,J_{n+3}) \\ = &\quad (2J_{n+1} - J_n) + i\,(J_{n+2} - J_{n+1}) \\ &+ j\,(2J_{n+3} - J_{n+2}) + k\,(2J_{n+4} - J_{n+3}) \\ = &\quad j_n + i\,j_{n+1} + j\,j_{n+2} + k\,j_{n+3} \\ = &\quad D_n^j\,. \end{aligned}$$

□

Theorem 2.3. *Let D_n^J be the n-th term of the dual Jacobsthal quaternion sequence (D_n^J) and $\overline{D_n^J}$ be conjugate of D_n^J. Then, we can give the following relations between these quaternions:*

$$\begin{aligned} (D_n^J)^2 &= 2\,J_n\,D_n^J - J_n^2, \\ (D_n^J)^2 + 2\,(D_{n-1}^J)^2 &= 2\,D_{2n-1}^J - J_{2n-1}, \\ D_n^J\,\overline{D_n^J} + 2\,D_{n-1}^J\,\overline{D_{n-1}^J} &= J_n^2 + 2\,J_{n-1}^2 = J_{2n-1}, \\ D_{n+1}^J\,\overline{D_{n+1}^J} + 2\,D_n^J\,\overline{D_n^J} &= J_{n+1}^2 + 2\,J_n^2 = J_{2n+1}, \\ D_{n+1}^J\,\overline{D_{n+1}^J} - 2\,D_n^J\,\overline{D_n^J} &= J_{n+1}^2 - 2\,J_n^2 = J_{2n+1} - 4\,J_n^2 \end{aligned} \tag{2.21}$$

Proof. It can be proved easily by using (2.10). Now, we will prove first two equalities

$$\begin{aligned} (D_n^J)^2 = &\quad J_nJ_n + i\,(J_nJ_{n+1} + J_{n+1}J_n) + j\,(J_nJ_{n+2} + J_{n+2}J_n) \\ &+k\,(J_nJ_{n+3} + J_{n+3}J_n) \\ = &\quad 2J_n(J_n + i\,J_{n+1} + j\,J_{n+2} + k\,J_{n+3}) - J_nJ_n \\ = &\quad 2\,J_n\,D_n^J - J_n^2. \end{aligned}$$

$$\begin{aligned} (D_n^J)^2 + 2\,(D_{n-1}^J)^2 = &\quad J_n^2 + 2\,i\,(J_n\,J_{n+1}) + 2\,j\,(J_n\,J_{n+2}) + 2\,k\,(J_n\,J_{n+3}) \\ &+2(J_{n-1}^2) + 4\,i\,(J_{n-1}\,J_n) + 4\,j\,(J_{n-1}\,J_{n+1}) \\ &+4\,k\,(J_{n-1}\,J_{n+2}) \\ = &\quad (J_n^2 + 2J_{n-1}^2) + i\,(2J_nJ_{n+1} + 4J_{n-1}J_n) \\ &+ j\,(2J_nJ_{n+2} + 4J_{n-1}J_{n+1}) \\ &+k\,(2J_nJ_{n+3} + 4J_{n-1}J_{n+2}) \\ = &\quad J_{2n-1} + 2i\,J_{2n} + 2\,j\,J_{2n+1} + 2k\,J_{2n+2} \\ = &\quad 2\,D_{2n-1}^J - J_{2n-1}. \end{aligned}$$

We can prove last three equalities by using equation (2.12) as follows:

$$\begin{aligned} D_n^J\,\overline{D_n^J} + 2\,D_{n-1}^J.\overline{D_{n-1}^J} &= J_n^2 + 2\,J_{n-1}^2 = J_{2n-1}, \\ D_{n+1}^J\,\overline{D_{n+1}^J} + 2\,D_n^J\,\overline{D_n^J} &= J_{n+1}^2 + 2\,J_n^2 = J_{2n+1}, \\ D_{n+1}^J\,\overline{D_{n+1}^J} - 2\,D_n^J\,\overline{D_n^J} &= J_{n+1}^2 - 2J_n^2 = J_{2n+1} - 4\,J_n^2 \end{aligned}$$

where identities $J_mJ_{n+1} + 2J_{m-1}J_n = J_{m+n}$ and $J_n^2 + 2J_{n-1}^2 = J_{2n-1}$ were used. □

Theorem 2.4. *Let D_n^J be the n-th term of dual Jacobsthal quaternion sequence (D_n^J). Then, we have the following identities*

$$\sum_{s=1}^{n} D_s^J = \frac{1}{2}[D_{n+2}^J - D_2^J], \tag{2.22}$$

$$\sum_{s=0}^{p} D_{n+s}^J = \frac{1}{2}[D_{n+p+2}^J - D_{n+1}^J], \tag{2.23}$$

$$\sum_{s=1}^{n} D_{2s-1}^J = \frac{2\,D_{2n}^J}{3} + \frac{1}{3}\,[n(2\,D_2^J - D_3^J) - 2\,D_0^J], \tag{2.24}$$

$$\sum_{s=1}^{n} D^J_{2s} = \frac{2D^J_{2n+1}}{3} - \frac{1}{3}\left[n(2D^J_2 - D^J_3) + 2D^J_1\right]. \tag{2.25}$$

Proof. (2.22) Hence, we can write

$$\begin{aligned}
\sum_{s=1}^{n} D^J_s &= \sum_{s=1}^{n} J_s + i\sum_{s=1}^{n} J_{s+1} + j\sum_{s=1}^{n} J_{s+2} + k\sum_{s=1}^{n} J_{s+3} \\
&= \frac{1}{2}[(J_{n+2}-1) + i(J_{n+3}-3) + j(J_{n+4}-5) + k(J_{n+5}-11)] \\
&= \frac{1}{2}[(J_{n+2}-J_2) + i(J_{n+3}-J_3) + j(J_{n+4}-J_4) + k(J_{n+5}-J_5)] \\
&= \frac{1}{2}[J_{n+2} + iJ_{n+3} + jJ_{n+4} + kJ_{n+5} - (J_2 + iJ_3 + jJ_4 + kJ_5)] \\
&= \frac{1}{2}[D^J_{n+2} - D^J_2].
\end{aligned}$$

(2.23) Hence, we can write

$$\begin{aligned}
\sum_{s=0}^{p} D^J_{n+s} &= \sum_{s=0}^{p} J_{n+s} + i\sum_{s=0}^{p} J_{n+s+1} + j\sum_{s=0}^{p} J_{n+s+2} + k\sum_{s=0}^{p} J_{n+s+3} \\
&= \frac{1}{2}[(J_{n+p+2}-J_{n+1}) + i(J_{n+p+3}-J_{n+2}) + j(J_{n+p+4}-J_{n+3})] \\
&\quad + \frac{1}{2}[k(J_{n+p+5}-J_{n+4})] \\
&= \frac{1}{2}[J_{n+p+2} + iJ_{n+p+3} + jJ_{n+p+4} + kJ_{n+p+5} \\
&\quad - (J_{n+1} + iJ_{n+2} + jJ_{n+3} + kJ_{n+4})] \\
&= \frac{1}{2}[D^J_{n+p+2} - D^J_{n+1}].
\end{aligned}$$

(2.24): Using $\sum\limits_{i=0}^{n-1} J_{2i+1} = \frac{2J_{2n}+n}{3}$ and $\sum\limits_{i=0}^{n} J_{2i} = \frac{2J_{2n+1}-n-2}{3}$, we get

$$\begin{aligned}
\sum_{s=1}^{n} D^J_{2s-1} &= (J_1 + J_3 + \ldots + J_{2n-1}) + i(J_2 + J_4 + \ldots + J_{2n}) \\
&\quad + j(J_3 + J_5 + \ldots + J_{2n+1}) + k(J_4 + J_6 + \ldots + J_{2n+2}) \\
&= \frac{(2J_{2n}+n)}{3} + i\,\frac{(2J_{2n+1}-n-2)}{3} + j\,\frac{(2J_{2n+2}+n-2)}{3} \\
&\quad + k\,\frac{(2J_{2n+3}-n-6)}{3} \\
&= \frac{2}{3}[J_{2n} + iJ_{2n+1} + jJ_{2n+2} + kJ_{2n+3}] \\
&\quad + \frac{1}{3}[n(1-i+j-k) - 2(i+j+3k)] \\
&= \frac{2D^J_{2n}}{3} + \frac{1}{3}[n(2D^J_2 - D^J_3) - 2D^J_0].
\end{aligned}$$

(2.25): Using $\sum_{i=0}^{n} J_{2i} = \frac{2J_{2n+1}-n-2}{3}$ we obtain

$$\begin{aligned}
\sum_{s=1}^{n} D^J_{2s} &= (J_2+J_4+\ldots+J_{2n})+i(J_3+J_5+\ldots+J_{2n+1}) \\
&\quad + j(J_4+J_6+\ldots+J_{2n+2})+k(J_5+J_7+\ldots+J_{2n+3}) \\
&= \frac{(2J_{2n+1}-n-2)}{3}+i\,\frac{(2J_{2n+2}+n-2)}{3}+j\,\frac{(2J_{2n+3}-n-6)}{3} \\
&\quad +k\,\frac{(2J_{2n+4}+n-10)}{3} \\
&= \frac{2}{3}[J_{2n+1}+iJ_{2n+2}+jJ_{2n+3}+kJ_{2n+4}] \\
&\quad +\frac{1}{3}[-n(1-i+j-k)-2(1+i+3j+5k)] \\
&= \frac{2D^J_{2n+1}}{3}-\frac{1}{3}[n(2D^J_2-D^J_3)+2D^J_1].
\end{aligned}$$

□

Theorem 2.5. *Let D^J_n and D^j_n be the n-th terms of the dual Jacobsthal quaternion sequence (D^J_n) and the dual Jacobsthal-Lucas quaternion sequence (D^j_n) ,respectively. Then, we have*

$$D^j_n\overline{D^J_n}-\overline{D^j_n}D^J_n = 2[J_nD^j_n-j_nD^J_n], \tag{2.26}$$

$$D^j_n\overline{D^J_n}+\overline{D^j_n}D^J_n = 2j_nJ_n = 2J_{2n}, \tag{2.27}$$

$$D^j_nD^J_n-\overline{D^j_n}\,\overline{D^J_n} = 2[D^j_nJ_n+D^J_nj_n-2J_{2n}], \tag{2.28}$$

$$D^j_nD^J_n+\overline{D^j_n}\,\overline{D^J_n} = 2J_{2n}. \tag{2.29}$$

Proof. (2.26):

$$\begin{aligned}
D^j_n\overline{D^J_n}-\overline{D^j_n}D^J_n = & \;(j_n+ij_{n+1}+jj_{n+2}+kj_{n+3}) \\
& \;(J_n-iJ_{n+1}-jJ_{n+2}-kJ_{n+3}) \\
& -(j_n-ij_{n+1}-jj_{n+2}-kj_{n+3}) \\
& \;(J_n+iJ_{n+1}+jJ_{n+2}+kJ_{n+3}) \\
= & \;(j_nJ_n-j_nJ_n) \\
& +2i(j_{n+1}J_n-j_nJ_{n+1}) \\
& +2j(j_{n+2}J_n-j_nJ_{n+2}) \\
& +2k(j_{n+3}J_n-j_nJ_{n+3}) \\
= & \;2J_n(j_n+ij_{n+1}+jj_{n+2}+kj_{n+3}) \\
& -2j_n(J_n+iJ_{n+1}+jJ_{n+2}+kJ_{n+3}) \\
= & \;2[J_nD^j_n-j_nD^J_n].
\end{aligned}$$

(2.27):

$$\begin{aligned}
D^j_n\overline{D^J_n}+\overline{D^j_n}D^J_n = & \;(j_n+ij_{n+1}+jj_{n+2}+kj_{n+3}) \\
& \;(J_n-iJ_{n+1}-jJ_{n+2}-kJ_{n+3}) \\
& +(j_n-ij_{n+1}-jj_{n+2}-kj_{n+3}) \\
& \;(J_n+iJ_{n+1}+jJ_{n+2}+kJ_{n+3}) \\
= & \;j_n[J_n-iJ_{n+1}-jJ_{n+2}-kJ_{n+3}] \\
& +(ij_{n+1}+jj_{n+2}+kj_{n+3})J_n \\
& +j_n[J_n+iJ_{n+1}+jJ_{n+2}+kJ_{n+3}] \\
& +(-ij_{n+1}-jj_{n+2}-kj_{n+3})J_n \\
= & \;2j_nJ_n = 2J_{2n}.
\end{aligned}$$

(2.28):

$$\begin{aligned} D_n^j D_n^J - \overline{D_n^j}\,\overline{D_n^J} = & \ (j_n + i\, j_{n+1} + j\, j_{n+2} + k\, j_{n+3}) \\ & (J_n + i\, J_{n+1} + j\, J_{n+2} + k\, J_{n+3}) \\ & -(j_n - i\, j_{n+1} - j\, j_{n+2} - k\, j_{n+3}) \\ & (J_n - i\, J_{n+1} - j\, J_{n+2} - k\, J_{n+3}) \\ = & \ (j_n J_n - j_n J_n) + 2\,(i\, j_{n+1} + j\, j_{n+2} + k\, j_{n+3}) J_n \\ & +2\,(j_n + i\, j_{n+1} + j\, j_{n+2} + k\, j_{n+3}) J_n - 2\, j_n J_n \\ = & \ 2\,(D_n^j J_n + D_n^J j_n - 2\, j_n J_n) \\ = & \ 2\,(D_n^j J_n + D_n^J j_n - 2\, J_{2n}). \end{aligned}$$

(2.29):

$$\begin{aligned} D_n^j D_n^J + \overline{D_n^j}\,\overline{D_n^J} = & \ (j_n + i\, j_{n+1} + j\, j_{n+2} + k\, j_{n+3}) \\ & (J_n + i\, J_{n+1} + j\, J_{n+2} + k\, J_{n+3}) \\ & +(j_n - i\, j_{n+1} - j\, j_{n+2} - k\, j_{n+3}) \\ & (J_n - i\, J_{n+1} - j\, J_{n+2} - k\, J_{n+3}) \\ = & \ 2\, j_n J_n = 2 J_{2n}. \end{aligned}$$

In proofs, the identities of Jacobsthal and Jacobsthal-Lucas numbers given below were used, respectively,

$$J_m j_n - J_n j_m = (-1)^n 2^{n+1} J_{m-n},\ j_n J_n = J_{2n} \text{ and } j_{n+2} = j_{n+1} + 2\, j_n.$$

□

Theorem 2.6. ***(Binet's Formula)**. Let D_n^J and D_n^j be $n-th$ terms of dual Jacobsthal quaternion sequence (D_n^J) and the dual Jacobsthal-Lucas quaternion sequence (D_n^j), respectively. For $n \geq 1$, Binet's formula for these quaternions are as follows respectively,*

$$D_n^J = \frac{1}{\alpha - \beta}\left[\underline{\alpha}\, \alpha^n - \underline{\beta}\, \beta^n\right] \tag{2.30}$$

and

$$D_n^j = (\underline{\underline{\alpha}}\, \alpha^n + \underline{\underline{\beta}}\, \beta^n) \tag{2.31}$$

where

$$\underline{\alpha} = 1 + i\,(1-\beta) + j\,(3-\beta) + k\,(5-3\beta),\ \alpha = 2,$$

$$\underline{\beta} = 1 + i\,(1-\alpha) + j\,(3-\alpha) + k\,(5-3\alpha),\ \beta = -1,$$

$$\underline{\underline{\alpha}} = (1-2\beta) + i\,(5-\beta) + j\,(7-5\beta) + k\,(17-7\beta),\ \alpha = 2,$$

$$\underline{\underline{\beta}} = (2\alpha-1) + i\,(\alpha-5) + j\,(5\alpha-7) + k\,(7\alpha-17),\ \beta = -1.$$

Proof. The characteristic equation of recurrence relations $D_{n+2}^J = D_{n+1}^J + 2 D_n^J$ is

$$t^2 - t - 2 = 0.$$

The roots of this equation are $\alpha = 2$ and $\beta = -1$
where $\alpha + \beta = 1$, $\alpha - \beta = 3$, $\alpha\beta = -2$.
Using recurrence relation and initial values $D_0^J = (0, 1, 1, 3)$,
$D_1^J = (1, 1, 3, 5)$ the Binet's formula for D_n^J, we get

$$D_n^J = A\ \alpha^n + B\, \beta^n = \frac{1}{3}\left[\underline{\alpha}\, \alpha^n - \underline{\beta}\, \beta^n\right],$$

where $A = \frac{D_1^J - D_0^J \beta}{\alpha - \beta}$, $B = \frac{\alpha D_0^J - D_1^J}{\alpha - \beta}$ and

$\underline{\alpha} = 1 + i(1-\beta) + j(3-\beta) + k(5-3\beta)$, $\underline{\beta} = 1 + i(1-\alpha) + j(3-\alpha) + k(5-3\alpha)$. Similarly, using recurrence relations $D_{n+2}^j = D_{n+1}^j + 2D_n^j$, the Binet's formula for D_n^j is obtained as follows:

$$D_n^j = (\underline{\underline{\alpha}}\, \alpha^n + \underline{\underline{\beta}}\, \beta^n)$$

□

Theorem 2.7. ***(Honsberger Identity)*** *For* $n, m \geq 0$ *the Honsberger identity for the dual Jacobsthal quaternions* D_n^J *and* D_m^J *is given by*

$$D_n^J D_m^J + 2 D_{n-1}^J D_{m-1}^J = 2 D_{n+m-1}^J - J_{n+m-1}. \tag{2.32}$$

Proof. (2.32):

$$\begin{aligned} D_n^J D_m^J = \;& J_n J_m + i\,(J_n J_{m+1} + J_{n+1} J_m) + j\,(J_n J_{m+2} + J_{n+2} J_m) \\ & + k\,(J_n J_{m+3} + J_{n+3} J_m) \end{aligned} \tag{2.33}$$

and

$$\begin{aligned} 2 D_{n-1}^J D_{m-1}^J = \;& 2(J_{n-1} J_{m-1}) + 2i\,(J_{n-1} J_m + J_n J_{m-1}) \\ & + 2j\,(J_{n-1} J_{m+1} + J_{n+1} J_{m-1}) \\ & + 2k\,(J_{n-1} J_{m+2} + J_{n+2} J_{m-1}) \end{aligned} \tag{2.34}$$

Finally, adding equations (2.33) and (2.34) side by side, we obtain

$$\begin{aligned} D_n^J D_m^J + 2 D_{n-1}^J D_{m-1}^J = \;& J_{n+m-1} + i\,(2J_{n+m}) \\ & + j\,(2J_{n+m+1}) + k\,(2J_{n+m+2}) \\ = \;& 2 D_{n+m-1}^J - J_{n+m-1}. \end{aligned}$$

where the identity $J_{n+m} = J_m J_{n+1} + 2 J_{m-1} J_n$ was used [27] and [28]. □

Theorem 2.8. ***D'ocagne's Identity*** *For* $n, m \geq 0$ *the D'ocagne's identity for the dual-complex Jacobsthal quaternions* D_n^J *and* D_m^J *is given by*

$$D_m^J D_{n+1}^J - D_{m+1}^J D_n^J = (-1)^n\, 2^n\, J_{m-n}\,(1 + i + 5j + 7k). \tag{2.35}$$

Proof. (2.35):

$$\begin{aligned} D_m^J D_{n+1}^J - D_{m+1}^J D_n^J = \;& [(J_m J_{n+1} - J_{m+1} J_n)] \\ & + i\,[(J_m J_{n+2} - J_{m+1} J_{n+1}) + (J_{m+1} J_{n+1} - J_{m+2} J_n)] \\ & + j\,[(J_m J_{n+3} - J_{m+1} J_{n+2}) + (J_{m+2} J_{n+1} - J_{m+3} J_n)] \\ & + k\,[(J_m J_{n+4} - J_{m+1} J_{n+3}) + (J_{m+3} J_{n+1} - J_{m+4} J_n)] \\ = \;& (-1)^n\, 2^n\, J_{m-n}\,(1 + i + 5j + 7k). \end{aligned}$$

where the identity $J_m J_{n+1} - J_{m+1} J_n = (-1)^n\, 2^n\, J_{m-n}$ was used [27] and [28].

□

Theorem 2.9. ***(Cassini's Identity)****. Let* D_n^J *and* D_n^j *be* $n-th$ *terms of dual Jacobsthal quaternion sequence* (D_n^J) *and the dual Jacobsthal-Lucas quaternion sequence* (D_n^j)*, respectively. Then, we have*

$$D_{n-1}^J D_{n+1}^J - (D_n^J)^2 = (-1)^n\, 2^{n-1} (1 + i + 5j + 7k). \tag{2.36}$$

$$D_{n-1}^j D_{n+1}^j - (D_n^j)^2 = (-2)^{n-1}\, 3^2 (1 + i + 5j + 7k). \tag{2.37}$$

Proof. (2.36):

$$\begin{aligned} D^J_{n-1}D^J_{n+1}-(D^J_n)^2 = & \ (J_{n-1}J_{n+1}-J_n^2) \\ & +i\,(J_{n-1}J_{n+2}-J_nJ_{n+1}) \\ & +j\,(J_{n-1}J_{n+3}-2J_nJ_{n+2}+J_{n+1}^2) \\ & +k\,(J_{n-1}J_{n+4}+J_{n+1}J_{n+2}-2J_nJ_{n+3}) \\ = & \ (J_{n-1}J_{n+1}-J_n^2) \\ & +i\,(J_{n-1}J_{n+1}-J_n^2) \\ & +5j\,(J_{n-1}J_{n+1}-J_n^2) \\ & +7k\,(J_{n-1}J_{n+1}-J_n^2) \\ = & \ (-1)^n\,2^{n-1}(1+i+5j+7k). \end{aligned}$$

and (2.37):

$$\begin{aligned} D^j_{n-1}D^j_{n+1}-(D^j_n)^2 = & \ (j_{n-1}j_{n+1}-j_n^2) \\ & +i\,(j_{n-1}j_{n+2}-j_nj_{n+1}) \\ & +j\,(j_{n-1}j_{n+3}-2j_nj_{n+2}+j_{n+1}^2) \\ & +k\,(j_{n-1}j_{n+4}+j_{n+1}j_{n+2}-2j_nj_{n+3}) \\ = & \ (j_{n-1}j_{n+1}-j_n^2) \\ & +i\,(j_{n-1}j_{n+1}-j_n^2) \\ & +5j\,(j_{n-1}j_{n+1}-j_n^2) \\ & +7k\,(j_{n-1}j_{n+1}-j_n^2) \\ = & \ (-2)^{n-1}\,3^2(1+i+5j+7k). \end{aligned}$$

where identities of Jacobsthal numbers and Jacobsthal-Lucas numbers as follows:

$$\begin{aligned} J_m\,J_{n-1}-J_{m-1}\,J_n=(-1)^n\,2^{n-1}\,J_{m-n}, \quad J_{n+2}=J_{n+1}+2J_n \\ j_m\,j_{n-1}-j_{m-1}\,j_n=(-2)^{n-1}\,3^2\,j_{m-n}, \quad j_{n+2}=j_{n+1}+2j_n. \end{aligned}$$

were used [27] and [28]. □

Theorem 2.10. ***(Catalan's Identity)****. Let D^J_n and D^j_n be $n-th$ terms of dual Jacobsthal quaternion sequence (D^J_n) and the dual Jacobsthal-Lucas quaternion sequence (D^j_n), respectively. Then, we have*

$$D^J_{n+r}D^J_{n-r}-(D^J_n)^2=-(-2)^{n-r}\,J_r^2\,(1+i+5\,j+7\,k). \tag{2.38}$$

$$D^j_{n+r}D^j_{n-r}-(D^j_n)^2=-(-2)^{n-r}\,3^2\,j_r^2\,(1+i+5\,j+7\,k). \tag{2.39}$$

Proof. (2.38):

$$\begin{aligned} D^J_{n+r}D^J_{n-r}-(D^J_n)^2 = & \ (J_{n+r}J_{n-r}-J_n^2) \\ & +i\,[(J_{n+r}J_{n-r+1}-J_nJ_{n+1}) \\ & +(J_{n+r+1}J_{n-r}-J_{n+1}J_n) \\ & +j\,[(J_{n+r}J_{n-r+2}-J_nJ_{n+2}) \\ & +(J_{n+r+2}J_{n-r}-J_{n+2}J_n)] \\ & +k\,[(J_{n+r}J_{n-r+3}-J_nJ_{n+3}) \\ & +(J_{n+r+3}J_{n-r}-J_{n+3}J_n)] \\ = & \ -(-2)^{n-r}\,J_r^2\,(1+i+5\,j+7\,k). \end{aligned}$$

and (2.39):

$$\begin{aligned} D^j_{n+r}D^j_{n-r}-(D^j_n)^2 = & \ (j_{n+r}j_{n-r}-j_n^2) \\ & +i\,[(j_{n+r}j_{n-r+1}-j_nj_{n+1})+(j_{n+r+1}j_{n-r}-j_{n+1}j_n)] \\ & +j\,[(j_{n+r}j_{n-r+2}-j_nj_{n+2})+(j_{n+r+2}j_{n-r}-j_{n+2}j_n)] \\ & +k\,[(j_{n+r}j_{n-r+3}-j_{n+}j_{n+3})+(j_{n+r+3}j_{n-r}-j_{n+3}j_n)] \\ = & \ -(-2)^{n-r}\,3^2\,J_r^2\,(1+i+5\,j+7\,k). \end{aligned}$$

where identities of Jacobsthal numbers and Jacobsthal-Lucas numbers as follows:

$$J_{n+r}J_{n-r} - J_nJ_n = -(-2)^{n-r}J_r^2$$
$$j_{n+r}j_{n-r} - j_nj_n = (-2)^{n-r}3^2J_r^2.$$

were used [29]. □

3. Conclusion

The difference between the Jacobsthal and the dual Jacobsthal quaternions originated from the quaternionic units, i.e., the quaternionic units for the Jacobsthal quaternion are

$$i^2 = j^2 = k^2 = -1, \; i\,j = -j\,i = k, \;\; j\,k = -k\,j = i, \;\; k\,i = -i\,k = j$$

whereas for the dual Jacobsthal quaternions they are

$$i^2 = j^2 = k^2 = i\,j\,k = 0, \; i\,j = -j\,i = j\,k = -k\,j = k\,i = -i\,k = 0.$$

The set J_D forms a commutative ring under the dual Jacobsthal quaternion multiplication and also it is a vector space of dimensions four on R and its basis is the set $\{1,i,j,k\}$. The interesting property of dual Jacobsthal quaternions is that by their means one can express the Galilean transformation in one quaternion equation. Since the multiplication and ratio of two dual Jacobsthal quaternions $D_n^{J_1}$ and $D_n^{J_2}$ is again a dual Jacobsthal quaternion, the set of dual Jacobsthal quaternions form a division algebra under addition and multiplication. There have been several studies on curve theory and magnetism by using the isomorphism between dual quaternion space and Galilean space G^4. Similar applications for dual Jacobsthal and dual Jacobsthal-Lucas quaternions can be applied to these areas.
Galilean transformation expressed by the dual four-component numbers shows the linkage between the space and time exists in the Newtonian physics. Moreover, it may have a considerable heuristic value for the study of the underlying mathematical formalism of physical laws. This study fills the gap in the literature by providing dual Jacobsthal quaternions.

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