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Collocation Method for the KdV-Burgers-Kuramoto Equation with Caputo Fractional Derivative

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Abstract: The present article focuses on obtaining numerical solutions of time fractional KdV-Burger-Kuramoto equation (KBK) with the finite element collocation method. The finite element collocation methods are common and effective tool for solving nonlinear problems because of their reasonable computational costs. The idea underlying the method is seeking the numerical solutions in a form of a linear combination of unknown functions with basis at nodal points by avoid of integration. Thus, in this article, we achieve more accurate numerical results are obtained with the application of the method to KBK equation. Additionally, we show the efficiency and effectiveness of the method using comparisons of numerical results with exact solutions via error norms and their simulations.

Key words: KdV-Burger-Kuramoto equation , Collocation finite element method, Quintic B-spline.

1. Introduction

The date of scientific modelling is as old as humanity history and it is a necessary human way to copy nature with the aim of representation of a particular nature system, how something works or prediction and explanation of the life. The role of modelling in science is so important. Let us try to explain this importance with some examples; if you want to investigate the behaviour of an atom or a cell, you can't become so small, if you want to define some features a dinosaur, they are no longer exist, so you can not jump in a time machine and experience it yourself. Therefore, modelling and simulations have a vital role in science. There are three types of models, these are physical modelling, conceptual modelling, and mathematical modelling. The physical modelling is a way to simulate of an object and relation of parts with each other. The conceptual modelling is dealing with ideas, it is a type of a mental modelling. Lastly, mathematical modelling is a translation of a particular phenomenon into mathematical language.

Engineers, researchers and people who work on natural, economics or scientific subjects

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use variables, constants and the relationship between them in order to investigate systems and characterize the problems which are born from real life. These problems change over time and the rates of change are expressed via derivative in mathematics. So, the evolution equations become an important tool and the final step of mathematical modelling. Changing model and researches help the development the theory of evolution equations. With all that, some materials or system which are dependent on memory, some damping systems and some physical problems including anomalous diffusion are modelled correctly by fractional differential equations which are the best way of modelling [3]. Therefore, fractional calculus is getting become a new growing field in mathematical modelling with its rich theories and physical background. For this reason many books, surveys, journals and papers address problems involving fractional calculus and investigate the solutions of the problems. In order to contribute to literature, we are consider a well known time fractional model with initial and boundary conditions read as

$$D_t^{\alpha} u = -\frac{1}{2}\left(u\right)_x^2 + \lambda_1 u_{xx} - \lambda_2 u_{xxx} - \lambda_3 u_{xxxx} + f\left(x,t\right), \qquad (x,t) \in [0,1] \times [0,1] \tag{1}$$

$$\begin{array}{ll} u(x,0) = u_0\left(x\right), & \\ u(x_l,t) = g_1\left(t\right), & u\left(x_r,t\right) = g_2\left(t\right), \\ u'(x_l,t) = h_1\left(t\right), & u'\left(x_r,t\right) = h_2\left(t\right) \end{array} \tag{2}$$

where λ_1, λ_2 and λ_3 are any constants, f, u_0, g, h are smooth functions and α $(0 < \alpha \leq 1)$ is a parameter describing the order of fractional derivative. The model describes long waves on a viscous fluid flowing down along an inclined plane, unstable drift waves in plasma and turbulent cascade model in a barotropic atmosphere [4–6].

In this article, we are going to seek numerical solutions of time fractional KdV-Burgers-Kuramoto (KBK) equation given in Eq. (1) .For this motivation, collocation method based on finite element approximation are used. At the first step of the article, we achived utilize spatial discretization $L1$ algorithm and space discretization of the KBK equation are done with collocation finite element technique using quintic b-spline basis. In fact, many different types of splines can be used in order to get smoother approximations for solution of the partial differential equations. Here, we choose the degree of at most 5 th order spline in order to protect continuity between elements and this part of the article ends with the equation into algebraic equation system. Finally, the article is finished with “conclusion” part.

2. Collocation Method for KdV-Burger-Kuramoto Equation

Now, we are going to obtain a numerical scheme for splitting equation system given in Eq. (1) using collocation finite element method. At the beginning of the process, the interval $I = [a, b]$

will be divide into N equal elements such as

$$a = x_0 < x_1 < \cdots < x_N = b$$

where $h = x_{m+1} - x_m$ is length of every sub-interval will be refer as step size. Then, a collocation solution u_h are going to be constructed on an interval $I = [a, b]$ as following form

$$u_h = \sum_{m=-2}^{N+2} \delta_m(t)\,\phi_m(x) \tag{3}$$

where $\{x_m\}_{m=0}^{N}$ are nodal points of each element called as collocation points,

$$\delta_m(t) \ \ (m = -2(1)N+2)$$

are time dependent unknown parameters, $\phi_m(x)$ are quintic B-spline basis. Each basis are fourth times continuously differentiable on the entire interval. Now, let us take a break and present define of the quintic B-spline basis $\phi_m(x)$

$$\phi_m(x) = \frac{1}{h^5}\begin{cases} (x - x_{m-3})^5, & [x_{m-3}, x_{m-2}] \\ (x - x_{m-3})^5 - 6(x - x_{m-2})^5, & [x_{m-2}, x_{m-1}] \\ (x - x_{m-3})^5 - 6(x - x_{m-2})^5 - 15(x - x_{m-1})^5, & [x_{m-1}, x_m] \\ (x_{m+3} - x)^5 - 6(x_{m+2} - x)^5 - 15(x_{m+1} - x)^5, & [x_m, x_{m+1}] \\ (x_{m+3} - x)^5 - 6(x_{m+2} - x)^5, & [x_{m+1}, x_{m+2}] \\ (x_{m+3} - x)^5, & [x_{m+2}, x_{m+3}] \\ 0, & otherwise \end{cases} \tag{4}$$

Also, let us give the values of $\phi_m(x), \phi'_m(x), \phi''_m(x), \phi'''_m(x)$ and $\phi^{iv}_m(x)$ order derivatives at nodal points via a table

	x_{m-3}	x_{m-2}	x_{m-1}	x_m	x_{m+1}	x_{m+2}	x_{m+3}
$\phi_m(x)$	0	1	26	66	26	1	0
$\phi'_m(x)$	0	5/h	50/h	0	-50/h	-5/h	0
$\phi''_m(x)$	0	$20/h^2$	$40/h^2$	-$120/h^2$	$40/h^2$	$20/h^2$	0
$\phi'''_m(x)$	0	$60/h^3$	-$120/h^3$	0	$120/h^3$	-$60/h^3$	0

(5)

Before substituting collocation solution given in (3) into equation system, we need to discrete the system using Crack Nicolson formula for ordinary order derivatives and $L1$ algorithm for fractional order time derivative given as follows

$$u = \frac{u^{n+1} + u^n}{2},$$

$$D_t^{\alpha} u = \frac{(\Delta t)^{-\alpha}}{\Gamma(2-\alpha)} \sum_{k=0}^{n-1} b_k^{\alpha}\left[u^{n-k} - u^{n-1-k}\right].$$

Thus, the system given in Eqs. $(1)-(2)$ are rewritten as

$$\begin{aligned}\frac{(\Delta t)^{-\alpha}}{\Gamma(2-\alpha)}\sum_{k=0}^{n-1} b_k^{\alpha}\left[u^{n-k}-u^{n-1-k}\right] &= -u\left[\frac{(u_x)^{n+1}+(u_x)^n}{2}\right]+\lambda_1\left[\frac{(u_{xx})^{n+1}+(u_{xx})^n}{2}\right] \\ &-\lambda_2\left[\frac{(u_{xxx})^{n+1}+(u_{xxx})^n}{2}\right]-\lambda_3\left[\frac{(u_{xxxx})^{n+1}+(u_{xxxx})^n}{2}\right]+f(x,t).\end{aligned}$$

In order to obtain numerical scheme for KBK equation, when we use the idea that approximate solution satisfies partial differential equation at collocation points x_m, we get following difference equation

$$\begin{aligned}&\delta_{m-2}^{n+1}\left(1-S\left(e_1 z_m+e_2+e_3-e_4\right)\right)+\delta_{m-1}^{n+1}\left(26-S\left(10 e_1 z_m+2 e_2-2 e_3+4 e_4\right)\right)\\ &+\delta_m^{n+1}\left(66+S\left(6 e_2+6 e_4\right)\right)+\delta_{m+1}^{n+1}\left(26+S\left(10 e_1 z_m-2 e_2-2 e_3-4 e_4\right)\right)\\ &+\delta_{m+2}^{n+1}\left(1+S\left(e_1 z_m-e_2+e_3+e_4\right)\right)\\ &=\\ &\delta_{m-2}^{n}\left(1+S\left(e_1 z_m+e_2+e_3-e_4\right)\right)+\delta_{m-1}^{n}\left(26+S\left(10 e_1 z_m+2 e_2-2 e_3+4 e_4\right)\right)\\ &\delta_m^{n}\left(66-S\left(6 e_2+6 e_4\right)\right)+\delta_{m+1}^{n}\left(26-S\left(10 e_1 z_m-2 e_2-2 e_3-4 e_4\right)\right)\\ &+\delta_{m+2}^{n}\left(1-S\left(e_1 z_m-e_2+e_3+e_4\right)\right)\\ &+f(x,t)-\sum_{k=1}^{n} b_k^{\alpha}\left\{\left(\delta_{m-2}^{n+1-k}+26\delta_{m-1}^{n+1-k}+66\delta_m^{n+1-k}+26\delta_{m+1}^{n+1-k}+\delta_{m+2}^{n+1-k}\right)\right.\\ &\left.-\left(\delta_{m-2}^{n-k}+26\delta_{m-1}^{n-k}+66\delta_m^{n-k}+26\delta_{m+1}^{n-k}+\delta_{m+2}^{n-k}\right)\right\}\end{aligned} \tag{6}$$

where $S=\Gamma(2-\alpha)(\Delta t)^{\alpha}, e_1=5/2h, e_2=10\lambda_1/h^2, e_3=30\lambda_2/h^3$ and $e_4=60\lambda_3/h^4$. Our problem has a nonlinearity, so we will convert the problem to linear form by performing transformations on the nonlinear term as $z_m=u_m$. The values of parameter z_m at the given x_m nodal points can be present as following

$$z_m=\delta_{m-2}+26\delta_{m-1}+66\delta_m+26\delta_{m+1}+\delta_{m+2}.$$

According to collocation method, there should be a match between the number of collocation points and basis functions [7]. The system given Eq (6) is consisting $(N+5)$ basis function points and $(N+1)$ selected collocation points, therefore, the numerical schemes are redefine as matching collocation points and basis function. Therefore, four unknown time parameters δ_{-2}, $\delta_{-1}, \delta_{N+1}$ and δ_{N+2} which is related with basis functions should be removed from each system and system redefine with $(N+1)$ collocation points and $(N+1)$ basis functions.

This refining procedure is done by using boundary conditions of the problem for u and u'

at x_0 and x_N nodal points such as

$$\begin{aligned}
&u(x_0,t) = \delta_{-2} + 26\delta_{-1} + 66\delta_0 + 26\delta_1 + \delta_2\\
&u'(x_0,t) = \tfrac{5}{h}\left(\delta_{-2} + 10\delta_{-1} - 10\delta_1 - \delta_2\right)\\
&u''(x_0,t) = \tfrac{20}{h^2}\left(\delta_{-2} + 2\delta_{-1} - 6\delta_0 + 2\delta_1 + \delta_2\right)\\
\\
&u(x_N,t) = \delta_{N-2} + 26\delta_{N-1} + 66\delta_N + 26\delta_{N+1} + \delta_{N+2}\\
&u'(x_N,t) = \tfrac{5}{h}\left(\delta_{N-2} + 10\delta_{N-1} - 106\delta_{N+1} - \delta_{N+2}\right)\\
&u''(x_N,t) = \tfrac{20}{h^2}\left(\delta_{N-2} + 2\delta_{N-1} - 6\delta_N + 2\delta_{N+1} + \delta_{N+2}\right).
\end{aligned} \tag{7}$$

Now, we have an algebraic equation system in the form $(N+1)\times(\ N+1)$. When boundaries and terms of the system Eq. (6) is rearranged in matrix norm, we get;

$$A\delta^{n+1} = B\delta^n. \tag{8}$$

The working procedure of the iteration given in (8) is to use the old value of δ^n to obtain new and presumably more accurate values of δ^{n+1} and update values of δ^n at each step.

2.1. Initial State

Iterative methods for the equation system given in (8) begin with an approximation to the solution, $\delta^{(0)}$ to seek provide a series as $\left\{\delta^{(1)},\delta^{(2)},\delta^{(3)},\ldots\right\}$ for better approximation. In this section, we are going to obtain an initial vector in order to begin iteration. For this purpose, we are going to use known values given in initial condition $u(x,t_0) = f(x)$ of the problem with approximate solution $u_h(x,t_0)$ following way

$$\begin{aligned}
&u_h(x_0,t_0) = \delta_{-2} + 26\delta_{-1} + 66\delta_0 + 26\delta_1 + \delta_2 = u_0(x_0)\\
&u_h(x_1,t_0) = \delta_{-1} + 26\delta_0 + 66\delta_1 + 26\delta_2 + \delta_3 = u_0(x_1)\\
&u_h(x_2,t_0) = \delta_0 + 26\delta_1 + 66\delta_2 + 26\delta_3 + \delta_3 = u_0(x_2)\\
&\qquad\vdots\\
&u_h(x_{N-1},t_0) = \delta_{N-3} + 26\delta_{N-2} + 66\delta_{N-1} + 26\delta_N + \delta_{N+1} = u_0(x_{N-1})\\
&u_h(x_N,t_0) = \delta_{N-2} + 26\delta_{N-1} + 66\delta_N + 26\delta_{N+1} + \delta_{N+2} = u_0(x_N).
\end{aligned} \tag{9}$$

As it is seen from the above equation system, there are fewer equations than unknowns. To obtain a unique solution of the system of equations, we need to eliminate four unknown variables from the system. Again, the equations given in (7) will help us to elimination. After some simple calculations, if we rewritten Eq. (9) in matrix form, we get

$$\begin{bmatrix}
54 & 60 & 6 & 0 & 0 & 0\\
25.25 & 67.5 & 26.25 & 1 & 0 & 0\\
1 & 26 & 66 & 26 & 1 & 0\\
\vdots & \ddots & \ddots & \ddots & \ddots & \vdots\\
0 & 1 & 26 & 66 & 26 & 1\\
0 & 0 & 1 & 26.25 & 67.5 & 25.25\\
0 & 0 & 1 & 6 & 60 & 54
\end{bmatrix}
\begin{bmatrix}
\delta_0^{(0)}\\ \delta_1^{(0)}\\ \delta_2^{(0)}\\ \vdots\\ \delta_{N-1}^{(0)}\\ \delta_N^{(0)}
\end{bmatrix}
=
\begin{bmatrix}
u_0(x_0)\\ u_0(x_1)\\ u_0(x_2)\\ \vdots\\ u_0(x_{N-1})\\ u_0(x_N)
\end{bmatrix}.$$

Now, number of unknown variables equal to the number equations. Thus, solving this system yields us obtain desired initial vector.

3. Numerical Results and Discussion

In this section, we are going to consider three examples of time fractional KBK equation for interval $I = [0.1]$ and different values of parameters $\lambda_1, \lambda_2, \lambda_3$ and fractional order α . The theoretical solution of this problem is available in literature, In order to validate the proposed method and measure the accuracy of the numerical results the error norms L_2 and L_∞ are calculated by giving formulas as follow

$$\begin{aligned} L_2 &= \|u - U_h\|_2 = \sqrt{h \sum_{j=0}^{N} \left|u_j - (u_h)_j\right|^2}, \\ L_\infty &= \|u - U_h\|_\infty = \max_{0 \le j \le N} \left|u_j - (u_h)_j\right| \end{aligned} \tag{10}$$

where u and u_h are theoretical solution and numerical solution of the equation, respectively.

3.1. Example 1

Consider a time fractional KBK equation with initial and boundary conditions given as

$$\begin{aligned} &D_t^\alpha u = -\tfrac{1}{2}(u_x)^2 + \lambda_1 u_{xx} - \lambda_2 u_{xxx} - \lambda_3 u_{xxxx} + f(x,t) \\ &u(x,0) = 0 \end{aligned}$$

where

$f(x,t) = \frac{2t^{2-\alpha}}{\Gamma(3-\alpha)} \cos(2\pi x) - \pi t^4 \sin(4\pi x) + 4\lambda_1 \pi^2 t^2 \cos(2\pi x) + 8\lambda_2 \pi^3 t^2 \sin(2\pi x) + 16\lambda_3 \pi^4 t^2 \cos(2\pi x)$

and the exact solution of time fractional KBK equation is $u(x,t) = t^2 \cos(2\pi x)$ is expressed in [1]. In this numerical experiment, we take $\lambda_1 = 1, \lambda_2 = 2$ and $\lambda_3 = 2$. In order to discuss what is the effects of the number of step and time partition on the accuracy, the error norms L_2 and L_∞ are presented for different values of time (Δt) and step size (h) and fractional order parameter α and $t_{final} = 1$ in Tables 1-4. It is observed that from obtained results,when time step decrease half-and-half, error norms decrease almost amount $\%4-5$. Thus the decreasing values of time size have a slightly effect on accuracy. Additionally, the effect of step size is related with order parameter of the equation α. For small α (see in Tables 1-2), decreasing values of step size has a high increasing effects on the error norm. However, when α parameter is getting higher, decreasing values of step size effects the error norms with slightly increase. But, when we choose higher values of α, one can see in Table 3 and 4, decreasing step size yields decreasing error norms as expected from collocation method.

Since the analytical solution and numerical solution overlap for many values of chosen variables, the plots of the numerical solution are shown in Figures. 1 for different values of time

Table 1: Example 1: The error norms L_2 and L_∞ for $\alpha = 0.1$ and $t_{final} = 1$

Δt	0.01		0.005		0.0025		0.00125	
h	$L_2 \times 10^4$	$L_\infty \times 10^4$	$L_2 \times 10^4$	$L_\infty \times 10^4$	$L_2 \times 10^4$	$L_\infty \times 10^4$	$L_2 \times 10^4$	$L_\infty \times 10^4$
0.01	0.501590	0.819300	0.481670	0.786761	0.471699	0.770474	0.466711	0.762326
0.005	2.455074	4.010208	2.474995	4.042753	2.484978	4.059062	2.489976	4.067225
0.0025	3.194241	5.217852	3.214273	5.250577	3.224043	5.266540	3.229241	5.275033
0.00125	3.378098	5.518191	3.399813	5.553667	3.411439	5.572622	3.414204	5.577186

Table 2: Example 1: The error norms L_2 and L_∞ for $\alpha = 0.5$ and $t_{final} = 1$

Δt	0.01		0.005		0.0025		0.00125	
h	$L_2 \times 10^4$	$L_\infty \times 10^4$	$L_2 \times 10^4$	$L_\infty \times 10^4$	$L_2 \times 10^4$	$L_\infty \times 10^4$	$L_2 \times 10^4$	$L_\infty \times 10^4$
0.01	2.077723	3.393803	2.057199	3.360276	2.046771	3.343242	2.041499	3.334631
0.00625	0.323262	0.528042	0.343799	0.561593	0.354238	0.578646	0.359514	0.587265
0.005	0.877381	1.433143	0.897912	1.466686	0.908363	1.483758	0.913635	1.492369
0.0025	1.616197	2.640104	1.636722	2.673632	1.647155	2.690677	1.652406	2.699259
0.00125	1.802030	2.943670	1.822648	2.977347	1.833003	2.994273	3.414204	5.577186

order parameter α. We chose the interval of graphics as $[0, 5]$, step size as $h = 0.01$ and time step size as $\Delta t = 0.001$, order as $\alpha = 0.2, 0.75$ and 0.9, respectively. We notice from Figures 1.

3.2. Example 2

In this numerical experiment, we are going to consider time fractional KBK equation with given initial and boundary conditions

$$\begin{aligned} u(x_l,t) &= -\frac{t^{2\alpha}\cos(x_l)}{\Gamma(1+2\alpha)}, \qquad u(x_r,t) = -\frac{t^{2\alpha}\cos(x_r)}{\Gamma(1+2\alpha)} \\ u'(x_l,t) &= -\frac{t^{2\alpha}\sin(x_l)}{\Gamma(1+2\alpha)}, \qquad u'(x_r,t) = -\frac{t^{2\alpha}\sin(x_r)}{\Gamma(1+2\alpha)} \\ u(x,0) &= 0 \end{aligned}$$

the forced term of the problem is $f(x,t) = \frac{t^{\alpha}\cos(x)}{\Gamma(1+\alpha)} - \frac{t^{4\alpha}\cos(x)\sin(x)}{(\Gamma(1+2\alpha))^2} + \lambda_1\frac{t^{2\alpha}\cos(x)}{\Gamma(1+2\alpha)} + \lambda_2\frac{t^{2\alpha}\sin(x)}{\Gamma(1+2\alpha)} + \lambda_3\frac{t^{2\alpha}\cos(x)}{\Gamma(1+2\alpha)}$ and analytical solution is $u(x,t) = \frac{t^{2\alpha}\cos(x)}{\Gamma(1+2\alpha)}$ [2]. In Example 2, the problem is discussed in interval $[x_l, x_r] = [0, 1]$ and different final times from 0.1 to 0.6. In order to obtain several comparison tables, according to absolute error is given in Table 5-6, other variables are chosen as $\lambda_1 = \lambda_2 = \lambda_3 = 1$ and $\alpha = 0.75, 1$, respectively. Also, for our numerical results space step is

Table 3: Example 1: The error norms L_2 and L_∞ for $\alpha = 0.75$ and $t_{final} = 1$

Δt	0.01		0.005		0.0025		0.00125	
h	$L_2 \times 10^4$	$L_\infty \times 10^4$	$L_2 \times 10^4$	$L_\infty \times 10^4$	$L_2 \times 10^4$	$L_\infty \times 10^4$	$L_2 \times 10^4$	$L_\infty \times 10^4$
0.01	3.070501	5.015458	3.054283	4.988967	3.045780	4.975077	3.041362	4.967861
0.00625	0.670316	1.095016	0.654085	1.068499	0.645574	1.054595	0.641152	1.047371
0.005	0.116385	0.190137	0.100139	0.163594	0.091631	0.149695	0.087216	0.142483
0.0025	0.622289	1.016518	0.638489	1.042986	0.646939	1.056796	0.651403	1.064078
0.00125	0.806107	1.316794	0.821751	1.342337	0.831972	1.359063	0.835825	1.365319

Table 4: Example 1: The error norms L_2 and L_∞ for $\alpha = 0.9$ and $t_{final} = 1$

Δt	0.01		0.005		0.0025		0.00125	
h	$L_2 \times 10^4$	$L_\infty \times 10^4$	$L_2 \times 10^4$	$L_\infty \times 10^4$	$L_2 \times 10^4$	$L_\infty \times 10^4$	$L_2 \times 10^4$	$L_\infty \times 10^4$
0.01	3.615892	5.90633	3.607046	5.891878	3.602242	5.884031	3.599663	5.879819
0.00625	1.216145	1.986654	1.207291	1.972187	1.202484	1.964333	1.199901	1.960115
0.005	0.662317	1.081895	0.653459	1.067420	0.648644	1.059554	0.646062	1.055336
0.0025	0.076282	0.124594	0.084937	0.138734	0.089913	0.146871	0.092330	0.150816
0.00125	0.260287	0.425191	0.269151	0.439681	0.272848	0.445669	0.275889	0.450654

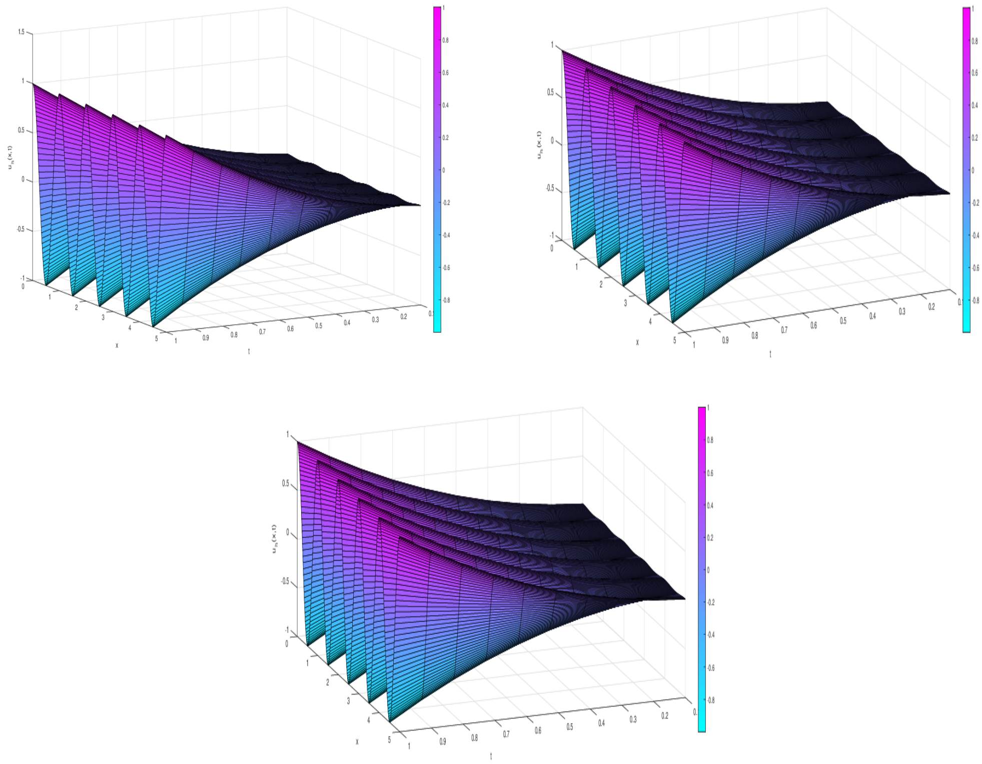

Figure 1: Physical behavior of numerical solutions of Example 1 for $h = 0.01$,$\Delta t = 0.01$ and $\alpha = 0.2, 0.75, 0.9$, respectively

$h = 0.1$ and time step size is $\Delta t = 0.001$. From Tables 5 and 6, we observe that error-norms has a slow rise according to increasing final time and numerical solutions are obtained via collocation method has better approximation for the different values of α according to results given in [2].

3D graph of problem 2 is exhibited in Figure 2 for the interval $[0, 5]$, $h = 0.01$, $\Delta t = 0.001$. and $\alpha = 0.2, 0.75, 0.9$

3.3. Example 3:

As a last example, we consider time fractional KBK equation with conditions as follow

Table 5: Example 2: A comparison absolute error norms between present method with [2] for $\alpha = 0.75$

$\alpha = 0.75$	t=0.1		t=0.2		t=0.3	
x	[2]	Collocation	[2]	Collocation	[2]	Collocation
0.1	2.12737E-4	1.425555827E-6	2.28422E-4	2.117782172E-6	7.56638E-3	2.636578023E-6
0.2	2.41145E-4	5.001672638E-6	2.65580E-3	7.401239780E-6	9.06168E-3	9.199881243E-6
0.3	2.86820E-4	9.754706651E-6	3.10412E-3	1.4359901838E-5	1.07513E-2	1.7813737870E-5
0.4	3.51375E-4	1.4750674667E-5	3.63645E-3	2.1591250347E-5	1.26528E-2	2.6732887819E-5
0.5	3.19966E-4	1.9083687300E-5	2.78217E-3	2.7763486907E-5	8.89023E-3	3.4314027437E-5
0.6	3.22493E-4	2.1882537269E-5	2.93705E-3	3.1623892290E-5	9.67846E-3	3.9007915198E-5
0.7	3.35989E-4	2.2274092353E-5	3.12840E-3	3.1981620122E-5	1.05325E-2	3.9359355101E-5
0.8	3.61167E-4	1.9321052204E-5	3.36005E-3	2.7616713636E-5	1.14623E-2	3.3928557400E-5
0.9	3.98257E-4	1.1906187843E-5	3.63348E-3	1.6993266607E-5	1.24718E-2	2.0873955458E-5
1	4.47018E-4	0.0000000	3.94789E-3	0.0000000	1.35589E-2	0.0000000000
	t=0.4		t=0.5		t=0.6	
	[2]	Collocation	[2]	Collocation	[2]	Collocation
0.1	1.76708E-2	3.062339073E-6	9.57606E-2	3.4260E-6	2.20683E-2	3.743322999E-6
0.2	2.18543E-2	1.0681868614E-5	5.85324E-2	1.19547E-5	2.74249E-2	1.3072045992E-5
0.3	2.64650E-2	2.0667842446E-5	1.30486E-2	2.31288E-5	3.19023E-2	2.5299349033E-5
0.4	3.15340E-2	3.0991736731E-5	4.06402E-2	3.46745E-5	3.56728E-2	3.7933801499E-5
0.5	2.04192E-2	3.9754064361E-5	3.80953E-2	4.44699E-5	7.45291E-3	4.8654986503E-5
0.6	2.29045E-2	4.5159269141E-5	6.76929E-2	5.05064E-5	8.91490E-3	5.5265016233E-5
0.7	2.54926E-2	4.5521775607E-5	1.00724E-1	5.08931E-5	9.53959E-3	5.5686801517E-5
0.8	2.82012E-2	3.9209323018E-5	1.36832E-1	4.38218E-5	9.45691E-3	4.7948594769E-5
0.9	3.10362E-2	2.4128363192E-5	1.75597E-1	2.69787E-5	8.77358E-3	2.9537364804E-5
1	3.39907E-2	0.0000000	2.16518E-1	0.0000000	7.57357E-3	0.00000000

$$\begin{array}{l} u(x_l,t) = -\frac{t^{2\alpha}\sin(x_l)}{\Gamma(1+2\alpha)}, \qquad u(x_r,t) = -\frac{t^{2\alpha}\sin(x_r)}{\Gamma(1+2\alpha)} \\ u'(x_l,t) = -\frac{t^{2\alpha}\cos(x_l)}{\Gamma(1+2\alpha)}, \qquad u'(x_r,t) = -\frac{t^{2\alpha}\cos(x_r)}{\Gamma(1+2\alpha)} \\ u(x,0) = 0 \\ f(x,t) = \frac{t^{\alpha}\sin(x)}{\Gamma(1+\alpha)} + \frac{t^{4\alpha}\cos(x)\sin(x)}{(\Gamma(1+2\alpha))^2} + \lambda_1\frac{t^{2\alpha}\sin(x)}{\Gamma(1+2\alpha)} \\ \qquad -\lambda_2\frac{t^{2\alpha}\cos(x)}{\Gamma(1+2\alpha)} + \lambda_3\frac{t^{2\alpha}\sin(x)}{\Gamma(1+2\alpha)} \end{array}$$

exact solution of the problem is given as $u(x,t) = \frac{t^{2\alpha}\sin(x)}{\Gamma(1+2\alpha)}$. As it is in the previous example, we choose the $\Delta t = 0.001$, $\lambda_1 = \lambda_2 = \lambda_3 = 1$ and collocation points as $N = 10$ on the interval $I = [0,1]$. Tables 7-8 are prepared for the giving the numerical comparisons for different values of x. Notice from Tables, the numerical solutions at different nodal points are closely agree with the theoretical solution of the problem and, they have better approximation. Moreover, the error is low according to using few collocation points. The plots of the solutions are presented for the interval $[0,5]$ and same values given for tables in Figure 3. However, in order to change of solution according to fractional order, α values are chosen as $0.2, 0.75$ and 0.9.

Table 6: Example 2: A comparison of absolute error norms between present method with [2] for $\alpha = 1$

$\alpha = 1$	t=0.1		t=0.2		t=0.3	
x	[2]	Collocation	[2]	Collocation	[2]	Collocation
0.1	1.84959E-4	4.44355464E-7	2.90830E-4	9.01012708E-7	9.67277E-5	1.354531357E-6
0.2	2.97674E-4	1.541729248E-6	4.23930E-4	3.123843477E-6	2.76159E-4	4.696476134E-6
0.3	4.25867E-4	2.967349766E-6	5.67587E-4	6.006607148E-6	4.61419E-4	9.030127274E-6
0.4	5.68389E-4	4.428565539E-6	7.19135E-4	8.953614116E-6	6.46241E-4	1.3458731643E-5
0.5	7.24113E-4	5.657686623E-6	8.76214E-4	1.1422144064E-5	8.25379E-4	1.7165512936E-5
0.6	8.91885E-4	6.405705017E-6	1.03672E-3	1.2910934156E-5	9.94495E-4	1.9397068698E-5
0.7	1.07048E-3	6.437027332E-6	1.19879E-3	1.2950401464E-5	1.15008E-3	1.9449231453E-5
0.8	1.25857E-3	5.525346771E-6	1.36074E-3	1.1094600671E-5	1.28944E-3	1.6655266726E-5
0.9	1.45468E-3	3.398148858E-6	1.52113E-3	6.809864703E-6	1.41067E-3	1.0218726996E-5
1	1.65715E-3	0.0000000	1.67872E-3	0.0000000	1.5127E-3	0.0000000

x	t=0.4		t=0.5	
	[2]	Collocation	[2]	Collocation
0.1	7.19605E-4	1.804412500E-6	2.50855E-3	2.249981928E-6
0.2	5.99905E-4	6.258237331E-6	2.64995E-3	7.807250628E-6
0.3	4.94075E-4	1.2035659966E-5	2.83828E-3	1.5020156239E-5
0.4	4.14924E-4	1.7940993125E-5	3.09695E-3	2.2396408915E-5
0.5	3.72850E-4	2.2884474856E-5	3.44474E-3	2.8574469993E-5
0.6	3.75989E-4	2.5860745298E-5	3.89591E-3	3.2297304739E-5
0.7	4.30298E-4	2.5930504938E-5	4.46022E-3	3.2390015051E-5
0.8	5.39558E-4	2.2205117775E-5	5.14289E-3	2.7741028965E-5
0.9	7.05318E-4	1.3623766681E-5	5.94439E-3	1.7023623612E-5
1	9.26776E-4	0.0000000	6.86019E-3	0.0000000

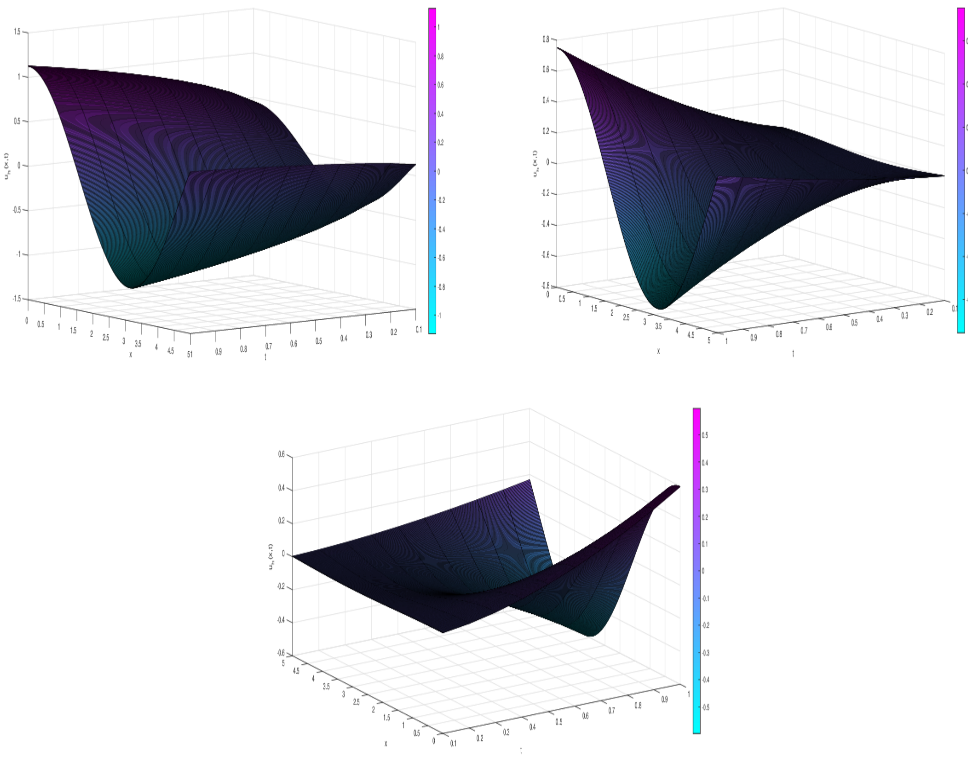

Figure 2: Physical behavior of numerical solutions of Example 2 for $h = 0.01$,$\Delta t = 0.01$ and $\alpha = 0.2, 0.75, 0.9$,respectively

Table 7: Example 3: A comparison of error norms L_2 and L_∞ between present method with [2] for $\alpha = 0.75$

$\alpha = 0.75$	t=0.05		t=0.1		t=0.15		t=0.2	
x	[2]	Collocation	[2]	Collocation	[2]	Collocation	[2]	Collocation
0.1	2.27075E-5	1.0199106688E-7	2.46815E-4	0.9151217315	7.82738E-4	0.8634200623	1.72484E-3	0.8377844024
0.2	2.61128E-5	3.2986384012E-7	2.14264E-4	2.9543475464	6.52493E-4	2.7907775679	1.40683E-3	2.7085592083
0.3	2.80289E-5	5.8028333723E-7	1.72666E-4	5.1829071135	4.94899E-4	4.9010987657	1.02750E-3	4.7583496137
0.4	2.78829E-5	7.7362185093E-7	1.19612E-4	6.8844581103	3.04250E-4	6.5112627940	5.76276E-4	6.3219508406
0.5	3.15224E-5	8.5736887105E-7	1.14364E-4	7.6027844858	3.08178E-4	7.1815593500	6.41627E-4	6.9660351142
0.6	3.54238E-5	8.0622215322E-7	8.08914E-5	7.1433136020	1.61045E-4	6.7340566055	2.66399E-4	6.5176961435
0.7	3.64308E-5	6.2527979888E-7	3.35835E-5	5.5673455606	2.33063E-5	5.2457341908	1.85579E-4	5.0652490343
0.8	3.38764E-5	3.6082416268E-7	3.04752E-5	3.2492842068	2.51819E-4	3.070467162	7.27157E-4	2.9614393981
0.9	2.71227E-5	1.0966740355E-7	1.14065E-4	1.0014537630	5.31069E-4	0.9511763780	1.37042E-3	0.9168263713
1	1.55712E-5	0.0000	2.19782E-4	0.0000	8.67153E-4	0.0000	2.12648E-3	0.0000
L_2		5.452250E-7		4.8498380E-7		4.579561 E-7		4.437573 E-7
L_∞		8.573689E-7		7.6027840E-7		7181559E-7		6.966035E-7

	t=0.25		t=0.3		t=0.4	
x	[2]	Collocation	[2]	Collocation	[2]	Collocation
0.1	3.10534E-3	0.8257542647	4.90591E-3	0.8207665259	9.42941E-3	0.8176118113
0.2	2.46587E-3	2.6686380135	3.76612E-3	2.6509976859	6.56591E-3	2.6379935367
0.3	1.71007E-3	4.6864354216	2.42812E-3	4.6515305627	3.23418E-3	4.6162744993
0.4	8.21037E-4	6.2231576346	8.67143E-4	6.1698002315	6.10355E-4	6.0970237738
0.5	1.04190E-3	6.8483029251	1.40877E-3	6.7764233864	1.55604E-3	6.6519550650
0.6	2.74380E-4	6.3903925955	3.61691E-4	6.3008818865	1.84669E-3	6.1165081634
0.7	6.28651E-4	4.9469387332	1.55294E-3	4.8498627656	5.70490E-3	4.6206941846
0.8	1.68784E-3	2.8788404717	3.38895E-3	2.7985508565	1.00748E-2	2.5827196035
0.9	2.92247E-3	0.8854851841	5.49997E-3	0.8492953489	1.50075E-2	0.7402677529
1	4.35005E-3	0.0000	7.91130E-3	0.0000	2.05479E-2	0.0000
L_2		4.356309 E-7		4.302325E-7		4.199293E-7
L_∞		6.848303 E-7		6.776423E-7		6.651955E-7

Table 8: Example 2: A comparison of error norms L_2 and L_∞ between present method with [2] for $\alpha = 1$

$\alpha = 1$	t=0.1		t=0.2		t=0.3	
x	[2]	Collocation	[2]	Collocation	[2]	Collocation
0.1	5.68684E-5	5.74636E-10	1.28221E-4	2.290067E-9	4.93990E-4	4.970277E-8
0.2	5.39855E-5	1.854416E-9	1.19191E-4	7.387768E-9	4.39997E-4	1.6028950E-7
0.3	4.85989E-5	3.258108E-9	1.04660E-4	1.2960332E-8	3.69101E-4	2.8057013E-7
0.4	4.05274E-5	4.335789E-9	8.31235E-5	1.7200593E-8	2.76311E-4	3.7081903E-7
0.5	2.95642E-5	4.788543E-9	5.30284E-5	1.8918339E-8	1.56464E-4	4.0519390E-7
0.6	1.54809E-5	4.486534E-9	1.27887E-5	1.7615798E-8	4.30468E-6	3.7348811E-7
0.7	1.96621E-6	3.486531E-9	3.91878E-5	1.3558206E-8	1.85428E-4	2.8273891E-7
0.8	2.30271E-5	2.049010E-9	1.04464E-4	7.839995E-9	4.17885E-4	1.5871350E-7
0.9	4.79469E-5	6.53934E-10	1.84526E-4	2.435719E-9	6.97968E-4	4.674238E-8
1	7.69510E-5	0.00000	2.80724E-4	0.00000	1.03017E-3	0.00000
L_2		3.0500 E-9		1.20129 E-8		2.56004 E-8
L_∞		4.7885E-9		1.89183 E-8		4.05194 E-8

	t=0.4		t=0.5		t=0.6	
x	[2]	Collocation	[2]	Collocation	[2]	Collocation
0.1	1.48989E-3	8.318817E-9	3.49306E-3	1.1919442E-8	6.95761E-3	1.5236319E-8
0.2	1.31699E-3	2.6816112E-8	3.07391E-3	38398689E-8	6.12894E-3	4.9038037E-8
0.3	1.10257E-3	4.6781142E-8	2.56670E-3	6.6651807E-8	5.13534E-3	8.4475253E-8
0.4	8.35246E-4	6.1435507E-8	1.95025E-3	8.6689906E-8	3.94182E-3	1.08247598E-7
0.5	5.03251E-4	6.6449302E-8	1.20250E-3	9.2297917E-8	2.51167E-3	1.12388559E-7
0.6	9.45733E-5	6.0263230E-8	3.00882E-4	8.1557400E-8	8.06986E-4	9.5051338E-8
0.7	4.02820E-4	4.4379625E-8	7.77358E-4	5.7311506E-8	1.21073E-3	6.1196285E-8
0.8	1.00074E-3	2.3629115E-8	2.05467E-3	2.7588516E-8	3.57988E-3	2.3214190E-8
0.9	1.71050E-3	6.246999E-9	3.55273E-3	5.580577E-9	6.33795E-3	6.50645E-10
1	2.54254E-3	0.00000	5.29182E-3	0.00000	9.52067E-3	0.00000
L_2		4.16716 E-8		5.72901 E-8		6.88772 E-8
L_∞		6.64493 E-8		9.22979E-8		1.123886E-7

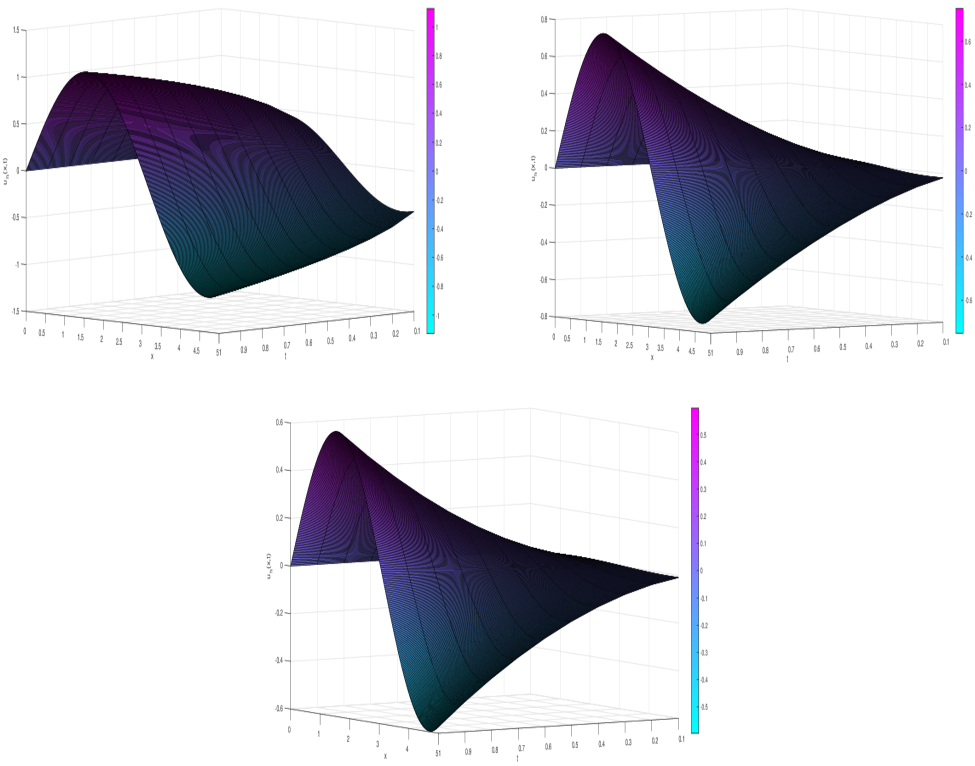

Figure 3: Physical behavior of numerical solutions of Example 3 for $h = 0.01$,$\Delta t = 0.01$ and $\alpha = 0.2, 0.75, 0.9$, respectively

4. Conclusion

In this article, we have applied a finite element collocation method combined with quintic B-spline basis is used to obtain a numerical scheme for time fractional KdV-Burgers-Kuramoto Equation in Caputo sense. Numerical solutions of the equation are discussed for three examples involving different forced terms. Newly obtained numerical results are presented via tables and graphics. The accuracy, compatibility and easy adaptability of the method is an interesting part of such methods and show that the method is reliable, practical and efficient tool for solving many practical problems which are observed in many physical phenomena.

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