Research Article GU J Sci 34(1): 196-209 (2021) DOI: 10.35378/gujs.691419

Gazi University

Journal of Science

http://dergipark.gov.tr/gujs

On Stancu type Szász-Mirakyan-Durrmeyer Operators Preserving e^{2ax},$a > 0$

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Highlights

- This paper focuses on linear positive operators preserving exponential functions.
- A Voronovskaya-type theorem is examined.
- Exponential modulus of continuity is investigated.

Article Info

Received: 19/02/2020
Accepted: 28/08/2020

Keywords

Exponential functions
Voronovskaya-type theorem,
Modulus of Continuity

Abstract

The present paper deals with the Szász-Mirakyan-Durrmeyer-Stancu operators preserving e^{2ax} for a>0. The uniform convergence of the constructed operators is mentioned in this paper. The rate of convergence is examined by employing two different modulus of continuities. After that, a Voronovskaya-type theorem is investigated for quantitative asymptotic estimation. Finally, a comparison is made theoretically to show that the new constructed operators perform well.

1. INTRODUCTION

In 1985, Mazhar and Totik [1] defined Durrmeyer-type generalization of the Szász-Mirakyan operators. In 2017, Acar et al. [2] introduced a modification of the Szász-Mirakyan operators preserving constants and e^{2ax}, $a > 0$. Then Deniz et al. [3] investigated the Szász-Mirakyan-Durrmeyer operators reproducing e^{2ax} for $a > 0$. For $0 \leq \alpha \leq \beta$ and $m > 0$ Stancu type Szász-Mirakyan-Durrmeyer operators are given by Gupta et al. [4]

$$S_{m,r}^{(\alpha,\beta)}(f;x) = m\sum_{k=0}^{\infty} e^{-mx}\frac{(mx)^k}{k!}\int_0^{\infty} e^{-mt}\frac{(mt)^{k+r}}{(k+r)!} f\left(\frac{mt+\alpha}{m+\beta}\right)dt. \tag{1}$$

We consider the generalized form of the Szász-Mirakyan-Durrmeyer-Stancu operators

$$S_{m,r}^{\alpha,\beta,\theta}(f;x) = m\sum_{k=0}^{\infty} e^{-m\theta(x)}\frac{(m\theta(x))^k}{k!}\int_0^{\infty} e^{-mt}\frac{(mt)^{k+r}}{(k+r)!} f\left(\frac{mt+\alpha}{m+\beta}\right)dt, \tag{2}$$

where $0 \leq \alpha \leq \beta$, $x \geq 0$ and $m > 0$. For notational convenience, we briefly denote the operators $S_{m,r}^{\alpha,\beta,\theta}$ as $S_{m,r}^{\theta}$. In this paper, we study the Szász-Mirakyan-Durrmeyer-Stancu operators preserving e^{2ax} for $a > 0$. In this situation, the function $\theta(x)$ which satisfies $S_{m,r}^{\theta}(e^{2at};x) = e^{2ax}$ is obtained as follows:

$$\begin{aligned} e^{2ax} &= m\sum_{k=0}^{\infty} e^{-m\theta(x)}\frac{(m\theta(x))^k}{k!}\int_0^{\infty} e^{-mt}\frac{(mt)^{k+r}}{(k+r)!} e^{\frac{2a(mt+\alpha)}{m+\beta}}dt \\ &= \left(\frac{m+\beta}{m+\beta-2a}\right)^{r+1} e^{\frac{2a\alpha}{m+\beta}+\theta(x)\left(\frac{m(m+\beta)}{m+\beta-2a}-m\right)}, \quad m+\beta > 2a. \end{aligned}$$

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By simple computations, we have

$$\theta(x)=\frac{m+\beta-2a}{2am}\left\{\frac{2a((m+\beta)x-\alpha)}{m+\beta}+(r+1)\ln\left(\frac{m+\beta-2a}{m+\beta}\right)\right\},\ \ m+\beta>2a. \tag{3}$$

The aim of the current paper is to investigate the approximation properties of the Stancu type Szász-Mirakyan-Durrmeyer operators preserving $e^{2ax}, a>0$ defined by (2), with $\theta(x)$ given in (3). By taking $\theta(x)=x$ and $\alpha=\beta=r=0$, we obtain the Szász-Mirakyan-Durrmeyer operators [1]. Some recent papers are Szász-Mirakyan type operators which fix exponentials [5], Szász-Mirakyan operators which preserve exponential functions [6], Baskakov-Szász-Stancu operators which preserve exponential functions [7], Baskakov-Szász-Mirakyan-type operators preserving exponential type functions [8] and Szász-Mirakyan-Kantorovich operators which preserve e^{-x} [9].

2. SOME AUXILIARY RESULTS

Here, for $0\le\alpha\le\beta$ and $m+\beta>2a$, we present three lemmas which are necessarily used in the proof of the theorems.

Lemma 1. Let $f(t)=e^{-At}$. Then for the Szász-Mirakyan-Durrmeyer-Stancu operators we have

$$S_{m,r}^{\theta}(e^{-At};x)=\left(1-\frac{A}{m+\beta+A}\right)^{r+1}e^{-A\left(\frac{m\theta(x)}{m+\beta+A}+\frac{\alpha}{m+\beta}\right)}. \tag{4}$$

Here, $\theta(x)$ is given by (3).

Lemma 2. Let $e_k(t)=t^k, k=0,1,2,3,4$. Then we have the next equalities:

$$\begin{aligned}
S_{m,r}^{\theta}(e_0;x)&=1,\\
S_{m,r}^{\theta}(e_1;x)&=\frac{m}{m+\beta}\theta(x)+\frac{r+\alpha+1}{m+\beta},\\
S_{m,r}^{\theta}(e_2;x)&=\frac{m^2}{(m+\beta)^2}\theta^2(x)+\frac{m(2r+2\alpha+4)}{(m+\beta)^2}\theta(x)+\frac{r^2+(3+2\alpha)r+\alpha^2+2\alpha+2}{(m+\beta)^2},\\
S_{m,r}^{\theta}(e_3;x)&=\frac{m^3}{(m+\beta)^3}\theta^3(x)+\frac{(3r+9+3\alpha)m^2}{(m+\beta)^3}\theta^2(x)+\frac{(3r^2+15r+18+6\alpha r+12\alpha+3\alpha^2)m}{(m+\beta)^3}\theta(x)\\
&\quad+\frac{r^3+6r^2+11r+6+3\alpha(r^2+3r+2)+3\alpha^2(r+1)+\alpha^3}{(m+\beta)^3},\\
S_{m,r}^{\theta}(e_4;x)&=\frac{m^4}{(m+\beta)^4}\theta^4(x)+\frac{(4r+16+4\alpha)m^3}{(m+\beta)^4}\theta^3(x)+\frac{(6r^2+(42+12\alpha)r+72+36\alpha+6\alpha^2)m^2}{(m+\beta)^4}\theta^2(x)\\
&\quad+\frac{(4r^3+(36+12\alpha)r^2+(104+60\alpha+12\alpha^2)r+96+72\alpha+24\alpha^2+4\alpha^3)m}{(m+\beta)^4}\theta(x)\\
&\quad+\frac{r^4+(10+4\alpha)r^3+(35+24\alpha+6\alpha^2)r^2+(50+44\alpha+18\alpha^2+4\alpha^3)r+24+24\alpha+12\alpha^2+4\alpha^3+\alpha^4}{(m+\beta)^4}.
\end{aligned}$$

Lemma 3. For $k=0,1,2,4.$ we briefly denote $\phi_x^k(t)=(t-x)^k$. Then for the central moments we get the equalities as follows:

$$\begin{aligned}
S_{m,r}^{\theta}(\phi_x^0;x)&=1,\\
S_{m,r}^{\theta}(\phi_x^1;x)&=\frac{m}{m+\beta}\theta(x)+\frac{r+\alpha+1}{m+\beta}-x,\\
S_{m,r}^{\theta}(\phi_x^2;x)&=\left(\frac{m}{m+\beta}\theta(x)-x\right)^2-\frac{2x(r+\alpha+1)}{m+\beta}+\frac{m(2r+2\alpha+4)\theta(x)+r^2+(3+2\alpha)r+\alpha^2+2\alpha+2}{(m+\beta)^2},\\
S_{m,r}^{\theta}(\phi_x^4;x)&=\frac{m^4}{(m+\beta)^4}\theta^4(x)+\frac{(4r+16+4\alpha)m^3}{(m+\beta)^4}\theta^3(x)+\frac{(6r^2+(42+12\alpha)r+72+36\alpha+6\alpha^2)m^2}{(m+\beta)^4}\theta^2(x)\\
&\quad+\frac{(4r^3+(36+12\alpha)r^2+(104+60\alpha+12\alpha^2)r+96+72\alpha+24\alpha^2+4\alpha^3)m}{(m+\beta)^4}\theta(x)\\
&\quad+\frac{r^4+(10+4\alpha)r^3+(35+24\alpha+6\alpha^2)r^2+(50+44\alpha+18\alpha^2+4\alpha^3)r+24+24\alpha+12\alpha^2+4\alpha^3+\alpha^4}{(m+\beta)^4}
\end{aligned}$$

$$-4x\left(\frac{m^3}{(m+\beta)^3}\theta^3(x)+\frac{(3r+9+3\alpha)m^2}{(m+\beta)^3}\theta^2(x)+\frac{(3r^2+15r+18+6\alpha r+12\alpha+3\alpha^2)m}{(m+\beta)^3}\theta(x)\right.$$
$$\left.+\frac{r^3+6r^2+11r+6+3\alpha(r^2+3r+2)+3\alpha^2(r+1)+\alpha^3}{(m+\beta)^3}\right)$$
$$+6x^2\left(\frac{m^2}{(m+\beta)^2}\theta^2(x)+\frac{m(2r+2\alpha+4)}{(m+\beta)^2}\theta(x)+\frac{r^2+(3+2\alpha)r+\alpha^2+2\alpha+2}{(m+\beta)^2}\right)$$
$$-4x^3\left(\frac{m}{m+\beta}\theta(x)+\frac{r+\alpha+1}{m+\beta}\right)+x^4.$$

Proof. By using the linearity of the $S_{m,r}^{\theta}$ operators and Lemma 2, we obtain

$$S_{m,r}^{\theta}(\phi_x^0;x)=S_{m,r}^{\theta}(e_0;x),$$
$$S_{m,r}^{\theta}(\phi_x^1;x)=S_{m,r}^{\theta}(e_1;x)-xS_{m,r}^{\theta}(e_0;x),$$
$$S_{m,r}^{\theta}(\phi_x^2;x)=S_{m,r}^{\theta}(e_2;x)-2xS_{m,r}^{\theta}(e_1;x)+x^2S_{m,r}^{\theta}(e_0;x),$$
$$S_{m,r}^{\theta}(\phi_x^4;x)=S_{m,r}^{\theta}(e_4;x)-4xS_{m,r}^{\theta}(e_3;x)+6x^2S_{m,r}^{\theta}(e_2;x)-4x^3S_{m,r}^{\theta}(e_1;x)+x^4S_{m,r}^{\theta}(e_0;x).$$

Remark 4. Taking into consideration the definition of $\theta(x)$, we get the following limit results for each $x\in[0,\infty)$, $m+\beta>2a$ and $0\le\alpha\le\beta$

$$\lim_{m\to\infty} mS_{m,r}^{\theta}(\phi_x^1;x)=-2ax \tag{5}$$

and

$$\lim_{m\to\infty} mS_{m,r}^{\theta}(\phi_x^2;x)=2x. \tag{6}$$

3. RESULTS

Let the subspace of all continuous and real-valued functions on the interval $[0,\infty)$ is denoted by $C^*[0,\infty)$ with the condition that $\lim_{x\to\infty} f(x)$ exists and also is finite, equipped with the uniform norm. In 1970, Boyanov and Veselinov [10] demonstrated the uniform convergence of a sequence of linear positive operators. For the new constructed operators (2) with $\theta(x)$ as shown in (3), we present the next theorem according to [10].

Theorem 5. If the Stancu type Szász-Mirakyan-Durrmeyer operators (2) satisfy

$$\lim_{m\to\infty} S_{m,r}^{\theta}(e^{-kt};x)=e^{-kx}, k=0,1,2. \tag{7}$$

uniformly in $[0,\infty)$, then for each $f\in C^*[0,\infty)$

$$\lim_{m\to\infty} S_{m,r}^{\theta}(f;x)=f(x) \tag{8}$$

uniformly in $[0,\infty)$.

Proof. As is already known that $\lim_{m\to\infty} S_{m,r}^{\theta}(1;x)=1$. Taking into consideration the equality (4) with $\theta(x)$ given in (3), we write

$$S_{m,r}^{\theta}(e^{-t},x)=e^{-x}+\frac{(1+2a)xe^{-x}}{m}+\mathcal{O}(m^{-2}) \tag{9}$$

and

$$S_{m,r}^{\theta}(e^{-2t},x)=e^{-2x}+\frac{4(1+a)xe^{-2x}}{m}+\mathcal{O}(m^{-2}). \tag{10}$$

Thus, we prove that

$$\lim_{m\to\infty} S^{\theta}_{m,r}(e^{-kt};x) = e^{-kx}, k = 0,1,2.$$

uniformly in the interval $[0,\infty)$. This proof guarantees that $\lim_{m\to\infty} S^{\theta}_{m,r}(f;x) = f(x)$ uniformly in the interval $[0,\infty)$ for any $f \in C^*[0,\infty)$.

After Boyanov and Veselinov [10], in 2010 Holhoş, [11] examined the uniform convergence of a sequence of linear positive operators. For a beneficial estimation of the positive and linear operators, the following theorem is presented.

Theorem 6. [11] For a sequence of positive and linear operators $A_m: C^*[0,\infty) \longrightarrow C^*[0,\infty)$, we get

$$\|A_m(f;x) - f(x)\|_{[0,\infty)} \le \|f\|_{[0,\infty)}\delta_m + (2+\delta_m)\omega^*(f, \sqrt{\delta_m + 2\sigma_m + \rho_m})$$

for each function $f \in C^*[0,\infty)$, where

$$\|A_m(e_0,x) - 1\|_{[0,\infty)} = \delta_m,$$

$$\|A_m(e^{-t},x) - e^{-x}\|_{[0,\infty)} = \sigma_m,$$

$$\|A_m(e^{-2t},x) - e^{-2x}\|_{[0,\infty)} = \rho_m$$

and the modulus of continuity is denoted by $\omega^*(f,\eta) = \sup_{\substack{|e^{-x}-e^{-t}|\le\eta \\ x,t>0}} |f(t) - f(x)|$. In these equalities, δ_m, σ_m and ρ_m tend to zero as $m \to \infty$.

Accordingly, we provide a quantitive estimation of the Szasz-Mirakyan-Durrmeyer-Stancu operators reproducing e^{2ax} for $a > 0$ as can be seen:

Theorem 7. For $f \in C^*[0,\infty)$, we get the following inequality

$$\left\|S^{\theta}_{m,r}f - f\right\|_{[0,\infty)} \le 2\omega^*(f, \sqrt{2\sigma_m + \rho_m}), \tag{11}$$

where

$$\left\|S^{\theta}_{m,r}(e^{-t},x) - e^{-x}\right\|_{[0,\infty)} = \sigma_m,$$

$$\left\|S^{\theta}_{m,r}(e^{-2t},x) - e^{-2x}\right\|_{[0,\infty)} = \rho_m.$$

In these equalities, σ_m and ρ_m tend to zero as $m \to \infty$. So, $S^{\theta}_{m,r}f$ converges f uniformly.

Proof. The Szasz-Mirakyan-Durrmeyer-Stancu operators $S^{\theta}_{m,r}$ preserve constants. So, $\delta_m = 0$. One can write as

$$\frac{k-n}{\ln k - \ln n} < \frac{k+n}{2} \tag{12}$$

for $0 < n < k$. By choosing $k = e^{-k_m x}$ and $n = e^{-x}$, we get

$$e^{-k_m x} - e^{-x} < \frac{1-k_m}{2}\left(xe^{-xk_m} + xe^{-x}\right).$$

Then let us notice that

$$\max_{x>0} \; xe^{-sx} = \frac{1}{es} \tag{13}$$

for each $s > 0$. Therefore, we have

$$e^{-k_m x} - e^{-x} < \frac{1-k_m}{2}\left(\frac{1}{ek_m} + \frac{1}{e}\right) < \frac{1-k_m^2}{2ek_m}.$$

In addition, by simple computations, we acquire

$$\begin{aligned} S_{m,r}^{\theta}(e^{-t}, x) &= \left(1 - \frac{1}{m+\beta+1}\right)^{r+1} e^{-\frac{m\theta(x)}{m+\beta+1} - \frac{\alpha}{m+\beta}} \\ &= e^{\frac{-\alpha}{m+\beta}\left(1-\frac{m+\beta-2a}{m+\beta+1}\right)}\left(1 - \frac{1}{m+\beta+1}\right)^{r+1}\left(1 + \frac{2a}{m+\beta-2a}\right)^{\frac{(r+1)(m+\beta-2a)}{2a(m+\beta+1)}} e^{-\frac{m+\beta-2a}{m+\beta+1}x} \\ &:= K_m e^{-k_m x}. \end{aligned}$$

Thus, we arrive at

$$\begin{aligned} \sigma_m &= \left\|S_{m,r}^{\theta}(e^{-t}, x) - e^{-x}\right\|_{[0,\infty)} = \left\|K_m e^{-k_m x} - e^{-x}\right\|_{[0,\infty)} \\ &= \left\|K_m(e^{-k_m x} - e^{-x}) + e^{-x}(K_m - 1)\right\|_{[0,\infty)} \\ &< K_m\left(\frac{1-k_m^2}{2ek_m}\right) + K_m - 1 \to 0 \end{aligned}$$

as $m \to \infty$. Here $k_m = \frac{m+\beta-2a}{m+\beta+1}$ and

$$K_m = e^{\frac{-\alpha}{m+\beta}\left(1-\frac{m+\beta-2a}{m+\beta+1}\right)}\left(1 - \frac{1}{m+\beta+1}\right)^{r+1}\left(1 + \frac{2a}{m+\beta-2a}\right)^{\frac{(r+1)(m+\beta-2a)}{2a(m+\beta+1)}}.$$

In the same manner, if we choose $k = e^{-n_m x}$, $n = e^{-2x}$ in (12) and use (13), we obtain

$$e^{-n_m x} - e^{-2x} < \frac{2-n_m}{2}(xe^{-xn_m} + xe^{-2x}) < \frac{2-n_m}{2}\left(\frac{1}{en_m} + \frac{1}{2e}\right) < \frac{4-n_m^2}{4en_m}.$$

On the other hand,

$$\begin{aligned} S_{m,r}^{\theta}(e^{-2t}, x) &= \left(1 - \frac{2}{m+\beta+2}\right)^{r+1} e^{-2\frac{m\theta(x)}{m+\beta+2} - \frac{\alpha}{m+\beta}} \\ &= e^{\frac{-2\alpha}{m+\beta}\left(1-\frac{m+\beta-2a}{m+\beta+2}\right)}\left(1 - \frac{2}{m+\beta+2}\right)^{r+1}\left(1 + \frac{2a}{m+\beta-2a}\right)^{\frac{(r+1)(m+\beta-2a)}{a(m+\beta+2)}} e^{-\frac{2(m+\beta-2a)}{m+\beta+2}x} \\ &:= M_m e^{-n_m x}. \end{aligned}$$

Thus, we find

$$\begin{aligned} \rho_m &= \left\|S_{m,r}^{\theta}(e^{-2t}, x) - e^{-2x}\right\|_{[0,\infty)} = \|M_m e^{-n_m x} - e^{-x}\|_{[0,\infty)} \\ &= \|M_m(e^{-n_m x} - e^{-x}) + e^{-x}(M_m - 1)\|_{[0,\infty)} \\ &< M_m\left(\frac{4-4n_m^2}{4en_m}\right) + M_m - 1 \to 0, \end{aligned}$$

as $m \to \infty$. Here $n_m = \frac{2(m+\beta-2a)}{m+\beta+2}$ and

$$M_m = e^{\frac{-2\alpha}{m+\beta}\left(1-\frac{m+\beta-2a}{m+\beta+2}\right)}\left(1-\frac{2}{m+\beta+2}\right)^{r+1}\left(1+\frac{2a}{m+\beta-2a}\right)^{\frac{(r+1)(m+\beta-2a)}{a(m+\beta+2)}}.$$

As a consequence, σ_m and ρ_m tend to zero as $m \to \infty$.

Section 4 investigates the rate of convergence with the help of the modulus of continuity.

4. THE MODULUS OF CONTINUITY

With the norm $\|f\|_{C_B} = \sup_{x\geq 0}|f(x)|$, $C_B[0,\infty)$ denotes the class of all uniform continuous and bounded functions f on $[0,\infty)$. For $f \in C_B[0,\infty)$,

$$\omega(f,\delta) := \sup_{0<h\leq\delta}\ \sup_{x,x+h\in[0,\infty)}|f(x+h)-f(x)|$$

presents the modulus of continuity.

$$\omega_2(f,\delta) := \sup_{0<h\leq\delta}\ \sup_{x,x+h,x+2h\in[0,\infty)}|f(x+2h)-2f(x+h)+f(x)|,$$

defines the second order modulus of continuity of the function $f \in C_B[0,\infty)$ for $\delta > 0$. Peetre's K-functionals are given by

$$K_2(f,\delta) := \inf_{g\in C_B^2[0,\infty)}\{\|f-g\|_{C_B[0,\infty)} + \delta\|g\|_{C_B^2[0,\infty)}\}.$$

Here, $C_B^2[0,\infty)$ describes the space of the functions, where f, f' and f'' belong to $C_B[0,\infty)$. The relationship between Peetre's K-functional and second order modulus of continuity is defined by [12],

$$K_2(f,\delta) \leq M\omega_2(f,\sqrt{\delta})$$

for $M > 0$.

Lemma 8. For $f \in C_B[0,\infty)$, we obtain $|S_{m,r}^{\theta}(f;x)| \leq \|f\|$.

Theorem 9. For $f \in C_B[0,\infty)$ and for all $x \in [0,\infty)$, there exists a constant $M > 0$, such that

$$|S_{m,r}^{\theta}(f;x) - f(x)| \leq M\omega_2(f,\sqrt{\mu_m}) + \omega\left(f,\left|\frac{m\theta(x)+r+\alpha+1}{m+\beta} - x\right|\right), \tag{14}$$

where

$$\mu_m = \frac{2m^2}{(m+\beta)^2}\theta^2(x) + 2m\left(\frac{2r+2\alpha+3}{(m+\beta)^2} - \frac{2x}{m+\beta}\right)\theta(x) + 2x^2 - \frac{4x(r+\alpha+1)}{m+\beta} + \frac{2r^2+(4\alpha+5)r+2\alpha^2+4\alpha+3}{(m+\beta)^2}. \tag{15}$$

Here, $\theta(x)$ is as shown in (3).

Proof. We define $\tilde{S}_{m,r}^{\theta}: C_B[0,\infty) \to C_B[0,\infty)$ auxiliary operators as follows

$$\tilde{S}_{m,r}^{\theta}(g;x) = S_{m,r}^{\theta}(g;x) + g(x) - g\left(\frac{m\theta(x)+r+\alpha+1}{m+\beta}\right), \tag{16}$$

where Eqn. (3) gives $\theta(x)$. It is important to notice that the operators given by (16) are linear and positive. From the Taylor expansion, we have for $g \in C_B^2[0,\infty)$

$$g(t)=g(x)+(t-x)g'(x)+\int_x^t (t-u)g''(u)du, \quad x,t\in[0,\infty). \tag{17}$$

When $\tilde{S}_{m,r}^{\theta}$ operators are applied to the equation (17) and then Lemma 3 is used, we get

$$\left|\tilde{S}_{m,r}^{\theta}(g;x)-g(x)\right|=\left|\tilde{S}_{m,r}^{\theta}\left(\int_x^t (t-u)g''(u)du;x\right)\right|.$$

$$\left|\tilde{S}_{m,r}^{\theta}(g;x)-g(x)\right|\le\left|S_{m,r}^{\theta}\left(\int_x^t (t-u)g''(u)du;x\right)\right|+\left|\int_x^{\frac{m\theta(x)+r+\alpha+1}{m+\beta}}\left(\frac{m\theta(x)+r+\alpha+1}{m+\beta}-u\right)g''(u)du\right|. \tag{18}$$

Further,

$$\left|S_{m,r}^{\theta}\left(\int_x^t (t-u)g''(u)du;x\right)\right|\le S_{m,r}^{\theta}\left(\int_x^t |t-u||g''(u)|du;x\right)\le ||g''||S_{m,r}^{\theta}(\phi_x^2;x) \tag{19}$$

and

$$\left|\int_x^{\frac{m\theta(x)+r+\alpha+1}{m+\beta}}\left(\frac{m\theta(x)+r+\alpha+1}{m+\beta}-u\right)g''(u)du\right|\le||g''||\left(\frac{m\theta(x)+r+\alpha+1}{m+\beta}-x\right)^2. \tag{20}$$

Rewrite (19) and (20) in (18), then we have

$$\begin{aligned}\left|\tilde{S}_{m,r}^{\theta}(g;x)-g(x)\right|&\le||g''||\left(S_{m,r}^{\theta}(\phi_x^2;x)+\left(\frac{m\theta(x)+r+\alpha+1}{m+\beta}-x\right)^2\right)\\&=||g''||\left(\left(\frac{m}{m+\beta}\theta(x)-x\right)^2-\frac{2x(r+\alpha+1)}{m+\beta}+\frac{m(2r+2\alpha+4)\theta(x)+r^2+(3+2\alpha)r+\alpha^2+2\alpha+2}{(m+\beta)^2}\right.\\&\quad\left.+\left(\frac{m\theta(x)+r+\alpha+1}{m+\beta}-x\right)^2\right)\\&:=||g''||\,\mu_m,\end{aligned} \tag{21}$$

where

$$\mu_m=\frac{2m^2}{(m+\beta)^2}\theta^2(x)+2m\left(\frac{2r+2\alpha+3}{(m+\beta)^2}-\frac{2x}{m+\beta}\right)\theta(x)+2x^2-\frac{4x(r+\alpha+1)}{m+\beta}+\frac{2r^2+(4\alpha+5)r+2\alpha^2+4\alpha+3}{(m+\beta)^2}. \tag{22}$$

By using the auxilary operators (16) and Lemma 8, we get

$$||\tilde{S}_{m,r}^{\theta}(f;x)||\le||S_{m,r}^{\theta}(f;x)||+2||f||\le3||f||. \tag{23}$$

With the help of (16), (21) and (23), for each $g\in C_B^2[0,\infty)$ we obtain

$$\begin{aligned}|S_{m,r}^{\theta}(f;x)-f(x)|&=\left|\tilde{S}_{m,r}^{\theta}(f;x)-f(x)+f\left(\frac{m\theta(x)+r+\alpha+1}{m+\beta}\right)-f(x)\right.\\&\quad\left.+\tilde{S}_{m,r}^{\theta}(g;x)-\tilde{S}_{m,r}^{\theta}(g;x)+g(x)-g(x)\right|\\&\le\left|\tilde{S}_{m,r}^{\theta}(f-g;x)-(f-g)(x)\right|+\left|f\left(\frac{m\theta(x)+r+\alpha+1}{m+\beta}\right)-f(x)\right|\\&\quad+\left|\tilde{S}_{m,r}^{\theta}(g;x)-g(x)\right|\\&\le4||f-g||+||g''||\mu_m+\left|f\left(\frac{m\theta(x)+r+\alpha+1}{m+\beta}\right)-f(x)\right|\\&\le K_2(f,\mu_m)+\omega\left(f,\left|\frac{m\theta(x)+r+\alpha+1}{m+\beta}-x\right|\right)\\&\le M\omega_2(f,\sqrt{\mu_m})+\omega\left(f,\left|\frac{m\theta(x)+r+\alpha+1}{m+\beta}-x\right|\right).\end{aligned} \tag{24}$$

Remark 10. We see that $\mu_m=\frac{2x}{m}+\mathcal{O}(m^{-2})\to0$, when $m\to\infty$. This result guarantees the convergence of the Theorem 9.

Section 5 investigates the rate of convergence with the help of exponential modulus of continuity.

5. THE EXPONENTIAL MODULUS OF CONTINUITY

The exponential growth of order $B > 0$ is given by

$$||f||_B := \sup_{x\in[0,\infty)} \left|f(x)e^{-Bx}\right| < \infty \tag{25}$$

for $f \in C[0,\infty)$. Also,

$$\omega_1(f,\delta,B) = \sup_{\substack{x\in[0,\infty)\\ h\le\delta}} |f(x)-f(x+h)|e^{-Bx} \tag{26}$$

gives the first order modulus of continuity of functions f with the exponential growth. Let K be a subspace of continuous functions space on $[0,\infty)$, which includes functions f with exponential growth with $||f||_B < \infty$.

Assume that the function f belong to Lipschitz class. So, for every $\delta < 1$ and $0 < c \le 1$

$$\omega_1(f,\delta,B) \le M\delta^c. \tag{27}$$

Theorem 11. Let $S_{m,r}^{\theta}: K \to C[0,\infty)$ be the sequence of positive and linear operators reproducing e^{2ax} for $a > 0$. It is assumed that $S_{m,r}^{\theta}$ give

$$S_{m,r}^{\theta}((t-x)^2e^{Bt};x) \le C_a(B,x)S_{m,r}^{\theta}(\phi_x^2;x), \tag{28}$$

for $0 < B < x < \frac{m}{B^2}$. Additionally, if $f \in C^2[0,\infty) \cap K$, $0 < c \le 1$ and $f'' \in Lip(c,B)$, then for $0 < B < x < \frac{m}{B^2}$, we obtain

$$\begin{aligned}&\left|S_{m,r}^{\theta}(f;x) - f(x) - f'(x)S_{m,r}^{\theta}(\phi_x^1;x) - \frac{1}{2}f''(x)S_{m,r}^{\theta}(\phi_x^2;x)\right|\\ &\qquad\le S_{m,r}^{\theta}(\phi_x^2;x)\left(\frac{\sqrt{C_a(2B,x)}}{2} + \frac{C_a(B,x)}{2} + e^{2Bx}\right)\omega_1\left(f'', \sqrt{\frac{S_{m,r}^{\theta}(\phi_x^4;x)}{S_{m,r}^{\theta}(\phi_x^2;x)}}, B\right),\end{aligned}$$

where $C_a(B,x) = Me^{Bx+1}$.

Proof. By considering Taylor expansion of the function $f \in C^2[0,\infty)$ at $x \in (0,\infty)$, we obtain

$$f(t) = f(x) + f'(x)(t-x) + f''(x)\frac{(t-x)^2}{2!} + H_2(f;t,x). \tag{29}$$

Here the remainder term is $H_2(f;t,x) = \frac{(t-x)^2}{2}(f''(\eta) - f''(x))$, and η is between t and x. Applying the operators $S_{m,r}^{\theta}$ to the equality (29), we get

$$\begin{aligned}\left|S_{m,r}^{\theta}(f;x) - f(x) - f'(x)S_{m,r}^{\theta}(\phi_x^1;x) - \frac{1}{2}f''(x)S_{m,r}^{\theta}(\phi_x^2;x)\right| &= \left|S_{m,r}^{\theta}(H_2(f;t,x);x)\right|\\ &\le S_{m,r}^{\theta}(|H_2(f;t,x)|;x).\end{aligned} \tag{30}$$

Additionally,

$$H_2(f;t,x) = \frac{(t-x)^2}{2}\left(f''(\eta) - f''(x)\right) \le \frac{(t-x)^2}{2}\begin{cases} e^{Bx}\omega_1(f'',h,B), & |t-x| \le h\\ e^{Bx}\omega_1(f'',kh,B), & h \le |t-x| \le kh\end{cases}$$

It was proved by Tachev et al. [13] that

$$\omega_1(f, kh, B) \leq k e^{B(k-1)h} \omega_1(f, h, B) \tag{31}$$

for each $h > 0$ and $k \in \mathbb{N}$. With the help of the inequality (31), we obtain

$$\begin{aligned}\frac{(t-x)^2 e^{Bx}}{2}\omega_1(f'', kh, B) &\leq \frac{(t-x)^2 e^{Bx}}{2} k e^{B(k-1)h}\omega_1(f'', h, B)\\ &\leq \frac{(t-x)^2}{2}\left(\frac{|t-x|}{h}+1\right) e^{Bx} e^{B|t-x|}\omega_1(f'', h, B)\\ &\leq \frac{(t-x)^2}{2}\left(\frac{|t-x|}{h}+1\right)\left(e^{Bt}+e^{2Bx}\right)\omega_1(f'', h, B).\end{aligned}$$

Thusly,

$$|H_2(f;t,x)| \leq \frac{(t-x)^2}{2}\left(\frac{|t-x|}{h}+1\right)\left(e^{Bt}+e^{2Bx}\right)\omega_1(f'', h, B). \tag{32}$$

Applying the operators $S_{m,r}^{\theta}$ to the inequality (32), we write

$$\begin{aligned}S_{m,r}^{\theta}(|H_2(f;t,x)|;x) &\leq \frac{1}{2} S_{m,r}^{\theta}\left(\left(\frac{|t-x|^3}{h}+|t-x|^2\right)\left(e^{Bt}+e^{2Bx}\right);x\right)\omega_1(f'', h, B)\\ &= \left(\frac{1}{2h} S_{m,r}^{\theta}\left(|t-x|^3 e^{Bt};x\right)+\frac{1}{2} S_{m,r}^{\theta}\left(|t-x|^2 e^{Bt};x\right)\right.\\ &\quad\left.+\frac{e^{2Bx}}{2h} S_{m,r}^{\theta}(|t-x|^3;x)+\frac{e^{2Bx}}{2} S_{m,r}^{\theta}(|t-x|^2;x)\right)\omega_1(f'', h, B).\end{aligned}$$

With some calculations we get

$$\begin{aligned}S_{m,r}^{\theta}\left(|t-x|^2 e^{Bt};x\right) &= S_{m,r}^{\theta}\left(t^2 e^{Bt};x\right)-2x S_{m,r}^{\theta}\left(t e^{Bt};x\right)+x^2 S_{m,r}^{\theta}\left(e^{Bt};x\right)\\ &= e^{B\left(\frac{\alpha}{m+\beta}+\frac{m\theta(x)}{m+\beta-B}\right)}\left\{\frac{m^2(m+\beta)^{r+3}}{(m+\beta-B)^{r+5}}\theta^2(x)\right.\\ &\quad+\left(\frac{(2r+4)m(m+\beta)^{r+2}}{(m+\beta-B)^{r+4}}+\frac{2\alpha m(m+\beta)^{r+1}}{(m+\beta-B)^{r+3}}-\frac{2xm(m+\beta)^{r+2}}{(m+\beta-B)^{r+3}}\right)\theta(x)\\ &\quad+\frac{(r^2+3r+2)(m+\beta)^{r+1}}{(m+\beta-B)^{r+3}}+\frac{2\alpha(r+1)(m+\beta)^{r}}{(m+\beta-B)^{r+2}}+\frac{\alpha^2(m+\beta)^{r-1}}{(m+\beta-B)^{r+1}}\\ &\quad\left.-\frac{2x(r+1)(m+\beta)^{r+1}}{(m+\beta-B)^{r+2}}-\frac{2x\alpha(m+\beta)^{r}}{(m+\beta-B)^{r+1}}+\frac{x^2(m+\beta)^{r+1}}{(m+\beta-B)^{r+1}}\right\}\\ &= e^{Bx}\left(1+\frac{3B(1-2ax+Bx)}{m}\right.\\ &\quad+\frac{B\left(-1-6\beta x+9Bx-6\beta Bx^2+20B^2x^2+16a^3x^3+5B^3x^3+r(-1+6ax-3Bx)\right)}{2xm^2}\\ &\quad+\frac{B\left(12a^2x^2(-1+Bx)-6Bx\alpha+2ax(3+3\beta x-18Bx-10B^2x^2+6\alpha)\right)}{2xm^2}\\ &\quad\left.+\mathcal{O}(m^{-3})\right)S_{m,r}^{\theta}(\phi_x^2;x).\end{aligned}$$

$$\begin{aligned}S_{m,r}^{\theta}\left(|t-x|^2 e^{Bt};x\right) &= e^{Bx}\left\{1+\frac{B^2x}{m}\left(\frac{3}{Bx}-\frac{6a}{B}+3\right)\right.\\ &\quad+\frac{1}{2!}\left(\frac{B^2x}{m}\right)^2\left(\frac{-1-r}{B^3x^3}+\frac{-6\beta+9B+6ar-3Br-6B\alpha+6a+12a\alpha}{B^3x^2}\right.\\ &\quad\left.+\frac{-6\beta B+20B^2-12a^2+12a\beta-36aB}{B^3x}+\frac{16a^3+5B^3+12a^2B-20aB^2}{B^3}\right)\\ &\quad+\frac{1}{3!}\left(\frac{B^2x}{m}\right)^3\left(\frac{-\left(3/2+3/2r^2+3\alpha+3r+3\alpha r+6a\right)}{B^5x^5}+\frac{6\beta-9B-9Br+6\beta r-6ar}{B^5x^4}\right.\\ &\quad+\frac{18\beta^2-54\beta B+20B^2+36\beta B\alpha-60B^2\alpha-40B^2r+36a^2r+18\beta Br-36a\beta r+66arB}{B^5x^3}\\ &\quad+\frac{36a^2(1+\alpha)-36a\beta-72a\alpha\beta+66aB+108aB\alpha}{B^5x^3}\\ &\quad+\frac{18\beta^2B-120\beta B+120B^3-30B^3\alpha-48a^3r-15B^3r-36a^2rB+60aB^2r+72a^2\beta}{B^5x^2}\\ &\quad+\frac{-108a^2B-72a^3B\alpha-36a\beta^2-180aB^2+120aB^2\alpha+216a\beta B-8(15+12\alpha)}{B^5x^2}\end{aligned}$$

$$+\frac{-30\beta B^3+63\beta^4+168a^4+120a^2B^2-72a^2\beta B-240aB^3+120a\beta B-96a^3\beta+144a^3\beta}{B^5x}$$
$$\left.\left.+\frac{7B^5-96a^5-48a^4B+60a^2B^3+40a^3B^2-42aB^4}{B^5}\right)+\mathcal{O}(m^{-4})\right\}S^{\theta}_{m,r}(\phi^2_x;x)$$
$$S^{\theta}_{m,r}\left(|t-x|^2e^{Bt};x\right)=e^{Bx}\left\{1M_0+\frac{B^2x}{m}M_1+\frac{1}{2!}\left(\frac{B^2x}{m}\right)^2M_2+\frac{1}{3!}\left(\frac{B^2x}{m}\right)^3M_3+\mathcal{O}(m^{-4})\right\}S^{\theta}_{m,r}(\phi^2_x;x)$$
$$S^{\theta}_{m,r}\left(|t-x|^2e^{Bt};x\right)=e^{Bx}\left\{\sum_{k=0}^{\infty}\frac{1}{k!}\left(\frac{B^2x}{m}\right)^kM_k\right\}S^{\theta}_{m,r}(\phi^2_x;x).$$

Let us choose $M_0=1$ and $M=\max\{M_0,M_1,M_2,\dots\}$. Therefore, we have

$$S^{\theta}_{m,r}\left(|t-x|^2e^{Bt};x\right)\le e^{Bx}M\sum_{k=0}^{\infty}\frac{1}{k!}\left(\frac{B^2x}{m}\right)^kS^{\theta}_{m,r}(\phi^2_x;x)$$
$$=Me^{Bx}e^{\frac{B^2x}{m}}S^{\theta}_{m,r}(\phi^2_x;x).$$

Since $0<B<x<\frac{m}{B^2}$,

$$S^{\theta}_{m,r}\left(|t-x|^2e^{Bt};x\right)\le C_a(B,x)S^{\theta}_{m,r}(\phi^2_x;x), \tag{33}$$

where $C_a(B,x)=Me^{Bx+1}$. By employing Cauchy-Schwarz inequality, we obtain the next inequalities

$$S^{\theta}_{m,r}\left(|t-x|^3e^{Bt};x\right)\le\sqrt{S^{\theta}_{m,r}(|t-x|^2e^{2Bt};x)}\sqrt{S^{\theta}_{m,r}(|t-x|^4;x)}$$
$$\le\sqrt{C_a(2B,x)S^{\theta}_{m,r}(\phi^2_x;x)}\sqrt{S^{\theta}_{m,r}(\phi^4_x;x)}. \tag{34}$$
$$S^{\theta}_{m,r}(|t-x|^3;x)\le\sqrt{S^{\theta}_{m,r}(|t-x|^4;x)}\sqrt{S^{\theta}_{m,r}(|t-x|^2;x)}$$
$$\le\sqrt{S^{\theta}_{m,r}(\phi^4_x;x)}\sqrt{S^{\theta}_{m,r}(\phi^2_x;x)}. \tag{35}$$

Thus, by using the inequalities (33), (34) and (35) in (30), we write

$$\left|S^{\theta}_{m,r}(f;x)-f(x)-f'(x)S^{\theta}_{m,r}(\phi^1_x;x)-\frac{1}{2}f''(x)S^{\theta}_{m,r}(\phi^2_x;x)\right|$$
$$\le\left(\frac{1}{2h}\sqrt{C_a(2B,x)S^{\theta}_{m,r}(\phi^2_x;x)}\sqrt{S^{\theta}_{m,r}(\phi^4_x;x)}+\frac{1}{2}C_a(B,x)S^{\theta}_{m,r}(\phi^2_x;x)\right.$$
$$\left.+\frac{e^{2Bx}}{2h}\sqrt{S^{\theta}_{m,r}(\phi^4_x;x)}\sqrt{S^{\theta}_{m,r}(\phi^2_x;x)}+\frac{e^{2Bx}}{2}S^{\theta}_{m,r}(\phi^2_x;x)\right)\omega_1(f'',h,B). \tag{36}$$

Lastly, when $h=\sqrt{\frac{S^{\theta}_{m,r}(\phi^4_x;x)}{S^{\theta}_{m,r}(\phi^2_x;x)}}$ is chosen and substituted in (36), we get

$$\left|S^{\theta}_{m,r}(f;x)-f(x)-f'(x)S^{\theta}_{m,r}(\phi^1_x;x)-\frac{1}{2}f''(x)S^{\theta}_{m,r}(\phi^2_x;x)\right|$$
$$\le S^{\theta}_{m,r}(\phi^2_x;x)\left(\frac{\sqrt{C_a(2B,x)}}{2}+\frac{C_a(B,x)}{2}+e^{2Bx}\right)\omega_1\left(f'',\sqrt{\frac{S^{\theta}_{m,r}(\phi^4_x;x)}{S^{\theta}_{m,r}(\phi^2_x;x)}},B\right).$$

It must be notices that for fixed $x\in(0,\infty)$, $\frac{S^{\theta}_{m,r}(\phi^4_x;x)}{S^{\theta}_{m,r}(\phi^2_x;x)}=\frac{6x}{m}+\mathcal{O}(m^{-2})\to 0$ as $m\to\infty$, guarantees the convergence of Theorem 11.

In section 6, in order to investigate the asymptotic behaviour of the constructed operators (2), the Voronovskaya-type theorem is given.

6. VORONOVSKAYA-TYPE THEOREM

Theorem 12. For $f, f', f'' \in C^*[0,\infty)$ and $x \in [0,\infty)$, we get

$$\left|m\left(S_{m,r}^{\theta}(f;x) - f(x)\right) + 2axf'(x) - xf''(x)\right| \le |r_m(x)||f'(x)| + |t_m(x)||f''(x)| \\ +2(2t_m(x) + 2x + z_m(x))\omega^*(f'', m^{-1/2}),$$

where

$$r_m(x) = mS_{m,r}^{\theta}(\phi_x^1;x) + 2ax,$$

$$t_m(x) = \frac{m}{2}S_{m,r}^{\theta}(\phi_x^2;x) - x,$$

$$z_m(x) = m^2\sqrt{S_{m,r}^{\theta}((e^{-x} - e^{-t})^4;x)}\sqrt{S_{m,r}^{\theta}(\phi_x^4;x)}.$$

Proof. By considering the Taylor expansion, we get

$$f(t) = f(x) + (t-x)f'(x) + \frac{(t-x)^2}{2}f''(x) + k(t,x)(t-x)^2. \tag{37}$$

Here, the remainder term $k(t,x)$ can be written as

$$k(t,x) := \frac{1}{2}(f''(\xi) - f''(x)).$$

Also, the remainder term is $k(t,x)$ and ξ is a number between x and t. When we apply the $S_{m,r}^{\theta}$ operators to (37), we have

$$S_{m,r}^{\theta}(f;x) - f(x) = f'(x)S_{m,r}^{\theta}(\phi_x^1;x) + \frac{1}{2}f''(x)S_{m,r}^{\theta}(\phi_x^2;x) + S_{m,r}^{\theta}(k(t,x)\phi_x^2;x).$$

Then

$$\left|m\left[S_{m,r}^{\theta}(f;x) - f(x)\right] + 2axf'(x) - xf''(x)\right| \le \left|mS_{m,r}^{\theta}(\phi_x^1;x) + 2ax\right||f'(x)| \\ +\frac{1}{2}\left|mS_{m,r}^{\theta}(\phi_x^2;x) - 2x\right||f''(x)| + \left|mS_{m,r}^{\theta}(k(t,x)\phi_x^2;x)\right|.$$

It is briefly symbolized that $r_m(x) := mS_{m,r}^{\theta}(\phi_x^1;x) + 2ax$ and $t_m(x) := \frac{m}{2}S_{m,r}^{\theta}(\phi_x^2;x) - x$. Thus,

$$\left|m\left[S_{m,r}^{\theta}(f;x) - f(x)\right] + 2axf'(x) - xf''(x)\right| \le |r_m(x)||f'(x)| + |t_m(x)||f''(x)| \\ +\left|mS_{m,r}^{\theta}(k(t,x)\phi_x^2;x)\right|.$$

Note that from (5) and (6), we see that $r_m(x)$ and $t_m(x)$ go to zero as $m \to \infty$. Now, we study the term $\left|mS_{m,r}^{\theta}(k(t,x)\phi_x^2;x)\right|$.

$$|f(t) - f(x)| \le \left(1 + \frac{(e^{-x} - e^{-t})^2}{\eta^2}\right)\omega^*(f,\eta).$$

By employing this inequality, we get

$$|k(t,x)| \le \left(1 + \frac{(e^{-x} - e^{-t})^2}{\eta^2}\right)\omega^*(f'',\eta).$$

For $\eta > 0$, if $|e^{-x} - e^{-t}| \le \eta$, then $|k(t,x)| \le 2\omega^*(f'',\eta)$ and if $|e^{-x} - e^{-t}| > \eta$, then $|k(t,x)| \le \frac{2(e^{-x}-e^{-t})^2}{\eta^2}\omega^*(f'',\eta)$. Thusly, we have $|k(t,x)| \le 2\left(\frac{(e^{-x}-e^{-t})^2}{\eta^2}+1\right)\omega^*(f'',\eta)$. Accordingly,

$$\begin{aligned}\left|mS_{m,r}^{\theta}(k(t,x)\phi_x^2;x)\right| &\le mS_{m,r}^{\theta}(|k(t,x)|\phi_x^2;x)\\ &\le 2m\omega^*(f'',\eta)S_{m,r}^{\theta}(\phi_x^2;x)+\frac{2m}{\eta^2}\omega^*(f'',\eta)S_{m,r}^{\theta}((e^{-x}-e^{-t})^2\phi_x^2;x)\\ &\le 2m\omega^*(f'',\eta)S_{m,r}^{\theta}(\phi_x^2;x)\\ &\quad+\frac{2m}{\eta^2}\omega^*(f'',\eta)\sqrt{S_{m,r}^{\theta}((e^{-x}-e^{-t})^4;x)}\sqrt{S_{m,r}^{\theta}(\phi_x^4;x)}.\end{aligned}$$

If we choose $\eta = 1/\sqrt{m}$ and $z_m := \sqrt{m^2S_{m,r}^{\theta}((e^{-x}-e^{-t})^4;x)}\sqrt{m^2S_{m,r}^{\theta}(\phi_x^4;x)}$, we get

$$\begin{aligned}\left|m\left(S_{m,r}^{\theta}(f;x)-f(x)\right)+2axf'(x)-xf''(x)\right| &\le |r_m(x)||f'(x)|+|t_m(x)||f''(x)|\\ &\quad+(4t_m(x)+4x+2z_m(x))\omega^*(f'',m^{-1/2}).\end{aligned}$$

Remark 13. After some calculations the following limit result is obtained:

$$\lim_{m\to\infty} m^2S_{m,r}^{\theta}(\phi_x^4;x) = 12x^2. \tag{38}$$

In addition, we get the result as follows:

$$\lim_{m\to\infty} m^2S_{m,r}^{\theta}((e^{-t}-e^{-x})^4;x) = 12x^2e^{-4x}. \tag{39}$$

Proof. We have after some calculations

$$\begin{aligned}m^2S_{m,r}^{\theta}(\phi_x^4;x) &= 12x^2+\frac{12x\left(1-r-8ax-2\beta x+4a^2x^2-2\alpha\right)}{m}\\ &\quad+\frac{3r^2-96a^3x^3+16a^4x^4-4r(3-26ax-9\beta x+18a^2x^2-3\alpha)}{m^2}\\ &\quad+\frac{-72a^2x^2(-1+2\beta x+2\alpha)}{m^2}\\ &\quad+\frac{8ax(7+36\beta x+24\alpha)+3(-5+12\beta^2x^2-4\alpha+4\alpha^2+12\beta x(-1+2\alpha))}{m^2}+\mathcal{O}(m^{-3}).\end{aligned}$$

So,

$$\lim_{m\to\infty} m^2S_{m,r}^{\theta}(\phi_x^4;x) = 12x^2.$$

In the same manner, we have

$$\begin{aligned}m^2S_{m,r}^{\theta}((e^{-t}-e^{-x})^4;x) &= 12x^2e^{-4x}+\frac{4xe^{-4x}(3r^2-6(5+2a+\beta)x+(65+60a+12a^2)x^2)}{m}.\\ &\quad+\frac{4xe^{-4x}(3r(-3+2(5+2a)x)-6(1+\alpha))}{m}+\mathcal{O}(m^{-2}).\end{aligned}$$

Thus,

$$\lim_{m\to\infty} m^2S_{m,r}^{\theta}((e^{-t}-e^{-x})^4;x) = 12x^2e^{-4x}.$$

The next corollary is given as a consequence of Theorem 12 and Remark 13 as follows:

Corollary 14. Assume that $x \in [0,\infty)$ and $f, f'' \in C^*[0,\infty)$. Thus,

$$\lim_{m\to\infty} m\left(S_{m,r}^{\theta}(f;x) - f(x)\right) = -2axf'(x) + xf''(x) \tag{40}$$

holds.

Now, we investigate that our new constructed Szász-Mirakyan-Durrmeyer-Stancu operators which reproduce e^{2ax} for $a > 0$ approximate better than Szász-Mirakyan operators preserving e^{2ax} which is taken into consideration by Acar et al. [2].

Theorem 15. Let $f \in C^2[0,\infty)$ be an increasing and convex function. Assume that for all $m \geq m_0$, $x \in [0,\infty)$ there is a number $m_0 \in \mathbb{N}$ such that

$$f(x) \leq S_{m,r}^{\theta}(f;x) \leq R_m^*(f;x). \tag{41}$$

Then

$$xf''(x) \geq 2axf'(x) \geq 0. \tag{42}$$

Contrarily, if inequality (42) holds with strict inequalities at $x \in [0,\infty)$, then there is a number $m_0 \in \mathbb{N}$ such that for $m \geq m_0$

$$f(x) < S_{m,r}^{\theta}(f;x) < R_m^*(f;x). \tag{43}$$

Proof. From the inequality (41) we have for all $m \geq m_0$ and $x \in [0,\infty)$ that

$$0 \leq m\left(S_{m,r}^{\theta}(f;x) - f(x)\right) \leq m(R_m^*(f;x) - f(x)). \tag{44}$$

By using the Voronovskaya-type theorem for Szász-Mirakyan operators preserving e^{2ax}, $a > 0$ which is obtained by Acar et al. [2], we get

$$\lim_{m\to\infty} m(R_m^*(f;x) - f(x)) = -axf'(x) + \frac{x}{2}f''(x). \tag{45}$$

After that, by taking the limit of the inequality (44) as $m \to \infty$ and using Equation (40) and Equation (45), we get

$$0 \leq -2axf'(x) + xf''(x) \leq -axf'(x) + \frac{x}{2}f''(x). \tag{46}$$

Thus, we directly achieve inequality (42). Contrarily, if inequality (42) holds with strict at $x \in [0,\infty)$, then

$$0 < -2axf'(x) + xf''(x) < -axf'(x) + \frac{x}{2}f''(x). \tag{47}$$

Finally, by using Equation (40) and Equation (45) we obtain the desired result.

CONFLICTS OF INTEREST

No conflict of interest was declared by the authors.

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