Konuralp Journal of Mathematics, 8 (2) (2020) 313-321

Konuralp Journal of Mathematics

Journal Homepage: www.dergipark.gov.tr/konuralpjournalmath
e-ISSN: 2147-625X

(m_1, m_2)-GG Convex Functions and Related Inequalities

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Abstract

In this manuscript, we introduce and study the concept of (m_1, m_2)-GG convex functions and some algebraic properties of them. In addition, we obtain Hermite-Hadamard type inequalities for the newly introduced class of functions by using an identity and Hölder, Hölder-İşcan, power-mean and improwed power-mean integral inequalities.

Keywords: *Convex function, geometric convexity, GG-convex function, (m_1, m_2)-GG convex function, Hermite-Hadamard type inequalities.*
2010 Mathematics Subject Classification: *26A51, 26D10*

1. Preliminaries and Fundamentals

A function $f : I \to \mathbb{R}$ is said to be convex if the inequality

$$f(tx + (1-t)y) \leq t f(x) + (1-t) f(y)$$

is valid for all $x, y \in I$ and $t \in [0,1]$. If this inequality reverses, then f is said to be concave on interval $I \neq \emptyset$.
Convexity theory provides powerful principles and techniques to study a wide class of problems in both pure and applied mathematics. A convex function is a continuous function whose value at the midpoint of every interval in its domain does not exceed the arithmetic mean of its value at the ends of the considered interval. Readers can find more informations in the articles [2, 6, 7, 9, 13, 15, 16, 20, 21, 22] and the references therein.
Let $f : I \to \mathbb{R}$ be a convex function. Then the following inequalities hold:

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}$$

for all $a, b \in I$ with $a < b$ [1, 3]. Both inequalities hold in the reversed direction if the function f is concave. This inequality is well-known in the literature as Hermite-Hadamard inequality. This inequality gives us upper and lower bounds for the mean-value of a convex function. If the function f is concave both of the inequalities in above hold in reversed direction.

Definition 1.1 ([14]). *A function $f : I \subseteq \mathbb{R}_+ = (0,\infty) \to \mathbb{R}$ is said to be GA-convex function on I if*

$$f\left(x^\lambda y^{1-\lambda}\right) \leq \lambda f(x) + (1-\lambda) f(y)$$

holds for all $x, y \in I$ and $\lambda \in [0,1]$, where $x^\lambda y^{1-\lambda}$ and $\lambda f(x) + (1-\lambda) f(y)$ are respectively the weighted geometric mean of two positive numbers x and y and the weighted arithmetic mean of $f(x)$ and $f(y)$.

Definition 1.2 ([18]). *A function $f : [0,b] \to \mathbb{R}$ is said to be m-convex for $m \in (0,1]$ if the inequality*

$$f(\alpha x + m(1-\alpha)y) \leq \alpha f(x) + m(1-\alpha) f(y)$$

holds for all $x, y \in [0,b]$ and $\alpha \in [0,1]$.

Definition 1.3 ([11]). *The function $f : [0,b] \to \mathbb{R}$, $b > 0$, is said to be (m_1, m_2)-convex, if*

$$f(m_1 t x + m_2(1-t)y) \leq m_1 t f(x) + m_2(1-t) f(y)$$

for all $x, y \in I$, $t \in [0,1]$ and $(m_1, m_2) \in (0,1]^2$.

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Definition 1.4 ([14]). *The GG-convex functions (called in what follows multiplicatively convex functions) are those functions $f: I \to J$ (acting on subintervals of $(0,\infty)$) such that*

$$x,y \in I \text{ and } t \in [0,1] \Longrightarrow f\left(x^{1-t}y^{t}\right) \leq f(x)^{1-t}f(y)^{t},$$

i.e., for which $\log f$ *is convex.*

The class of all *GA*-convex functions is constituted by all functions $f: I \to \mathbb{R}$ (defined on subintervals of $(0,\infty)$) for which

$$x,y \in I \text{ and } t \in [0,1] \Longrightarrow f\left(x^{1-t}y^{t}\right) \leq tf(x)+(1-t)f(y).$$

For some recent results concerning Hermite-Hadamard type integral inequalities for GG-convex and GA-convex functions we refer interested reader to [4, 8, 10, 17, 23, 24].
In [10], Kadakal gave the concept of the (m_1,m_2)-GA convex function.

Definition 1.5 ([10]). *Let the function $f:[0,b]\to\mathbb{R}$ and $[m_1,m_2]\in[0,1]^2$. If*

$$f\left(a^{m_1t}b^{m_2(1-t)}\right) \leq m_1tf(a)+m_2(1-t)f(b).$$

for all $[a,b]\in[0,b]$ and $t\in[0,1]$, then the function f is called (m_1,m_2)-GA convex function, if this inequality reversed, then the function f is called (m_1,m_2)-GA concave function.

A refinement of Hölder integral inequality which has better approach than Hölder integral inequality can be given as follows:

Theorem 1.6 (Hölder-İşcan Integral Inequality [5]). *Let $p>1$ and $\frac{1}{p}+\frac{1}{q}=1$. If f and g are real functions defined on $[a,b]$ and if $|f|^p$, $|g|^q$ are integrable functions on interval $[a,b]$ then*

$$\int_a^b |f(x)g(x)|\,dx \leq \frac{1}{b-a}\left\{\left(\int_a^b (b-x)|f(x)|^p\,dx\right)^{\frac{1}{p}}\left(\int_a^b (b-x)|g(x)|^q\,dx\right)^{\frac{1}{q}}+\left(\int_a^b (x-a)|f(x)|^p\,dx\right)^{\frac{1}{p}}\left(\int_a^b (x-a)|g(x)|^q\,dx\right)^{\frac{1}{q}}\right\}.$$

A refinement of power-mean integral inequality which has better approach than power-mean inequality and obtained as a result of the Hölder-İşcan integral inequality can be given as follows:

Theorem 1.7 (Improved power-mean integral inequality [12]). *Let $q\geq 1$. If f and g are real functions defined on $[a,b]$ and if $|f|$, $|f||g|^q$ are integrable functions on $[a,b]$ then*

$$\begin{aligned}\int_a^b |f(x)g(x)|\,dx \leq \frac{1}{b-a}&\left\{\left(\int_a^b (b-x)|f(x)|\,dx\right)^{1-\frac{1}{q}}\left(\int_a^b (b-x)|f(x)||g(x)|^q\,dx\right)^{\frac{1}{q}}\right.\\ &\left.+\left(\int_a^b (x-a)|f(x)|\,dx\right)^{1-\frac{1}{q}}\left(\int_a^b (x-a)|f(x)||g(x)|^q\,dx\right)^{\frac{1}{q}}\right\}.\end{aligned}$$

The main purpose of this paper is to introduce the concept of (m_1,m_2)-GG convex functions and establish some results connected with new inequalities similar to the Hermite-Hadamard integral inequality for this class of functions.

2. Some Algebraic Properties of (m_1,m_2)-GG Convex Functions

In this section, we introduce a new concept, which is called (m_1,m_2)-GG convex functions and we give by setting some algebraic properties for the (m_1,m_2)-GG convex functions, as follows:

Definition 2.1. *Let the function $f:[0,b]\to\mathbb{R}$ and $(m_1,m_2)\in[0,1]^2$. If*

$$f\left(a^{m_1t}b^{m_2(1-t)}\right) \leq [f(a)]^{m_1t}[f(b)]^{m_2(1-t)}. \tag{2.1}$$

for all $[a,b]\in[0,b]$ and $t\in[0,1]$, then the function f is said to be (m_1,m_2)-GG convex function, if the inequality (2.1) reversed, then the function f is said to be (m_1,m_2)-geometric geometric concave function.

We discuss some connections between the class of the (m_1,m_2)-GG convex functions and other classes of generalized convex functions.

Remark 2.2. *When $m_1=m_2=1$, the (m_1,m_2)-GG convex (concave) function becomes a GG convex (concave) function defined in [14].*

Proposition 1. *The function $f:(0,\infty)\to\mathbb{R}$ is (m_1,m_2)-GG convex function on the interval I if and only if $\ln\circ f:(0,\infty)\to\mathbb{R}$ is (m_1,m_2)-GA convex function on the interval I.*

Proof. $(\Rightarrow)$ Suppose that $f:(0,\infty)\to\mathbb{R}$ is (m_1,m_2)-GG convex function. Then, we have

$$(\ln\circ f)\left(a^{m_1t}b^{m_2(1-t)}\right) \leq \ln\left\{[f(a)]^{m_1t}[f(b)]^{m_2(1-t)}\right\} \leq m_1t\ln f(a)+m_2(1-t)\ln f(b).$$

Hence, the function $\ln\circ f$ is (m_1,m_2)-convex function on the interval $\ln I$.
$(\Leftarrow)$ Let $\ln\circ f:(0,\infty)\to\mathbb{R}$, (m_1,m_2)-GA convex function on the interval $\ln I$. Then, we get

$$\begin{aligned}(\ln\circ f)\left(a^{m_1t}b^{m_2(1-t)}\right) &\leq m_1t\ln f(a)+m_2(1-t)\ln f(b)\\ e^{(\ln\circ f)\left(a^{m_1t}b^{m_2(1-t)}\right)} &\leq e^{m_1t\ln f(a)+m_2(1-t)\ln f(b)}\end{aligned}$$

which means that the function $f(x)$ is (m_1,m_2)-GG convex function on I. □

Theorem 2.3. *Let $f,g:(0,\infty)\to\mathbb{R}$. If f and g are (m_1,m_2)-GG convex functions, then fg is an (m_1,m_2)-GG convex function.*

Proof. Let f,g be (m_1,m_2)-GG convex functions, then

$$\begin{aligned}(fg)\left(a^{m_1t}b^{m_2(1-t)}\right) &= f\left(a^{m_1t}b^{m_2(1-t)}\right)g\left(a^{m_1t}b^{m_2(1-t)}\right)\\ &\le [f(a)]^{m_1t}[f(b)]^{m_2(1-t)}[g(a)]^{m_1t}[g(b)]^{m_2(1-t)}\\ &= [(fg)(a)]^{m_1t}[(fg)(b)]^{m_2(1-t)}.\end{aligned}$$

□

Theorem 2.4. *If $f:(0,\infty)\to(0,\infty)$ is a (m_1,m_2)-GG-convex and $g:(0,\infty)\to\mathbb{R}$ is a (m_1,m_2)-GG convex function and nondecreasing, then $g\circ f:(0,\infty)\to\mathbb{R}$ is a (m_1,m_2)-GG convex function.*

Proof. For $x,y\in I$ and $t\in[0,1]$, we get

$$\begin{aligned}(g\circ f)\left(a^{m_1t}b^{m_2(1-t)}\right) &= g\left(f\left(a^{m_1t}b^{m_2(1-t)}\right)\right)\\ &\le g\left([f(a)]^{m_1t}[f(b)]^{m_2(1-t)}\right)\\ &\le [g(f(a))]^{m_1t}[g(f(b))]^{m_2(1-t)}.\end{aligned}$$

This completes the proof of theorem. □

Theorem 2.5. *Let $m_1,m_2\in(0,1]$, $0<a^{m_1}<b^{m_2}$ and $f_\alpha:[a^{m_1},b^{m_2}]\to\mathbb{R}$ be an arbitrary family of (m_1,m_2)-GG convex functions and let $f(x)=\sup_\alpha f_\alpha(x)$. If $J=\{u\in[a^{m_1},b^{m_2}]:f(u)<\infty\}$ is nonempty, then J is an interval and f is an (m_1,m_2)-GG convex function on J.*

Proof. Let $t\in[0,1]$ and $a,b\in J$ be arbitrary. Then

$$\begin{aligned}f\left(a^{m_1t}b^{m_2(1-t)}\right) &= \sup_\alpha f_\alpha\left(a^{m_1t}b^{m_2(1-t)}\right)\\ &\le \sup_\alpha\left([f_\alpha(a)]^{m_1t}[f_\alpha(b)]^{m_2(1-t)}\right)\\ &\le \sup_\alpha[f_\alpha(a)]^{m_1t}\sup_\alpha[f_\alpha(b)]^{m_2(1-t)}\\ &= [f(a)]^{m_1t}[f(b)]^{m_2(1-t)}<\infty.\end{aligned}$$

This shows simultaneously that J is an interval, since it contains every point between any two of its points, and that f is an (m_1,m_2)-GG convex function on J. This completes the proof of theorem. □

Theorem 2.6. *If the function $f:[a^{m_1},b^{m_2}]\to\mathbb{R}$ is an (m_1,m_2)-GG convex function then f is bounded on the interval $[a^{m_1},b^{m_2}]$.*

Proof. Let $K=\max\{[f(a)]^{m_1},[f(b)]^{m_2}\}$ and $x\in[a,b]$ is an arbitrary point. Then there exists a $t\in[0,1]$ such that $x=a^{m_1t}b^{m_2(1-t)}$. Thus, since $m_1t+m_2(1-t)\le 1$ we have

$$f(x)\le f\left(a^{m_1t}b^{m_2(1-t)}\right)\le[f(a)]^{m_1t}[f(b)]^{m_2(1-t)}\le K=M.$$

Also, for every $x\in[a^{m_1},b^{m_2}]$ there exists a $\lambda\in\left[\sqrt{\frac{a^{m_1}}{b^{m_2}}},1\right]$ such that $x=\lambda\sqrt{a^{m_1}b^{m_2}}$ and $x=\frac{\sqrt{a^{m_1}b^{m_2}}}{\lambda}$. Without loss of generality we can suppose $x=\lambda\sqrt{a^{m_1}b^{m_2}}$. So, we have

$$f\left(\sqrt{a^{m_1}b^{m_2}}\right)=f\left(\sqrt{\left[\lambda\sqrt{a^{m_1}b^{m_2}}\right]\left[\frac{\sqrt{a^{m_1}b^{m_2}}}{\lambda}\right]}\right)\le\sqrt{f(x)f\left(\frac{\sqrt{a^{m_1}b^{m_2}}}{\lambda}\right)}.$$

Using M as the upper bound, we get

$$f(x)\ge\frac{f^2\left(\sqrt{a^{m_1}b^{m_2}}\right)}{f\left(\frac{\sqrt{a^{m_1}b^{m_2}}}{\lambda}\right)}\ge\frac{f^2\left(\sqrt{a^{m_1}b^{m_2}}\right)}{M}=m.$$

This completes the proof of theorem. □

3. Hermite-Hadamard Inequality for (m_1,m_2)-GG Convex Function

The goal of this section is to establish some inequalities of Hermite-Hadamard type for (m_1,m_2)-GG convex functions. In this section, we will denote by $L[a,b]$ the space of (Lebesgue) integrable functions on the interval $[a,b]$.

Theorem 3.1. *Let $f:[a^{m_1},b^{m_2}]\to\mathbb{R}$ be an (m_1,m_2)-GG convex function. If $a<b$ and $f\in L[a,b]$, then the following Hermite-Hadamard type integral inequalities hold:*

$$\begin{aligned} f\left(\sqrt{a^{m_1}b^{m_2}}\right) &\le \frac{1}{\ln b^{m_2}-\ln a^{m_1}}\int_{a^{m_1}}^{b^{m_2}}\frac{1}{u}f\left(\sqrt{u}\right)f\left(\sqrt{\frac{a^{m_1}b^{m_2}}{u}}\right)du \\ &\le \frac{1}{\ln b^{m_2}-\ln a^{m_1}}\int_{a^{m_1}}^{b^{m_2}}\frac{f(u)}{u}du \\ &\le \frac{[f(b)]^{m_2}-[f(a)]^{m_1}}{\ln[f(b)]^{m_2}-\ln[f(a)]^{m_1}}\le\frac{m_1f(a)+m_2f(b)}{2}. \end{aligned} \tag{3.1}$$

Proof. Firstly, since function f is (m_1,m_2)-GG convex function, we have

$$f\left(\sqrt{xy}\right)\le\sqrt{f(x)f(y)}\le\frac{f(x)+f(y)}{2}$$

for all $x,y\in I$. If we substitute $x=a^{m_1t}b^{m_2(1-t)}$ and $y=a^{m_1(1-t)}b^{m_2t}$ in the above inequalities for $t\in[0,1]$, we can write

$$\begin{aligned} f\left(\sqrt{a^{m_1}b^{m_2}}\right) &\le \sqrt{f\left(a^{m_1t}b^{m_2(1-t)}\right)f\left(a^{m_1(1-t)}b^{m_2t}\right)} \\ &\le \frac{f\left(a^{m_1t}b^{m_2(1-t)}\right)+f\left(a^{m_1(1-t)}b^{m_2t}\right)}{2}. \end{aligned}$$

Now, if we take integral in the last inequality with respect to $t\in[0,1]$, we deduce that

$$\begin{aligned} f\left(\sqrt{a^{m_1}b^{m_2}}\right) &\le \int_0^1\sqrt{f\left(a^{m_1t}b^{m_2(1-t)}\right)f\left(a^{m_1(1-t)}b^{m_2t}\right)}dt \\ &= \frac{1}{\ln b^{m_2}-\ln a^{m_1}}\int_{a^{m_1}}^{b^{m_2}}\frac{1}{u}f\left(\sqrt{u}\right)f\left(\sqrt{\frac{a^{m_1}b^{m_2}}{u}}\right)du \\ &\le \frac{1}{\ln b^{m_2}-\ln a^{m_1}}\int_{a^{m_1}}^{b^{m_2}}\frac{f(u)}{u}du. \end{aligned}$$

Secondly, by using the property of the (m_1,m_2)-GG convex function of f, we can write

$$f\left(a^{m_1t}b^{m_2(1-t)}\right)\le[f(a)]^{m_1t}[f(b)]^{m_2(1-t)}\le m_1tf(a)+m_2(1-t)f(b).$$

If the variable is changed as $u=a^{m_1t}b^{m_2(1-t)}$, then

$$\frac{1}{\ln b^{m_2}-\ln a^{m_1}}\int_{a^{m_1}}^{b^{m_2}}\frac{f(u)}{u}du=\frac{[f(b)]^{m_2}-[f(a)]^{m_1}}{\ln[f(b)]^{m_2}-\ln[f(a)]^{m_1}}\le\frac{m_1f(a)+m_2f(b)}{2}.$$

This completes the proof of theorem. □

Corollary 3.2. *If we take $m_1=m_2=1$ in the inequalities (3.1), then we have the following inequalities:*

$$\begin{aligned} f\left(\sqrt{ab}\right) &\le \frac{1}{\ln b-\ln a}\int_a^b\frac{1}{u}f\left(\sqrt{u}\right)f\left(\sqrt{\frac{ab}{u}}\right)du \\ &\le \frac{1}{\ln b-\ln a}\int_a^b\frac{f(u)}{u}du\le\frac{f(b)-f(a)}{\ln f(b)-\ln f(a)}\le\frac{f(a)+f(b)}{2}. \end{aligned}$$

This inequalities coincide with the inequalities in [4].

4. Some New Inequalities for (m_1,m_2)-GG Convex Functions

The main purpose of this section is to establish new estimates that refine Hermite-Hadamard integral inequality for functions whose first derivative in absolute value, raised to a certain power which is greater than one, respectively at least one, is (m_1,m_2)-GG convex function by using the Hölder and power-mean integral inequalities. In order to prove next theorems, we need the following identity for differentiable functions.

Lemma 4.1 ([23]). *Let $f:I\subseteq\mathbb{R}_+=(0,\infty)\to\mathbb{R}$ be a differentiable function and $a,b\in I$ with $a<b$. If $f'(x)\in L([a,b])$, then*

$$\frac{b^2f(a)-a^2f(b)}{2}-\int_a^bxf(x)dx=\frac{\ln b-\ln a}{2}\int_0^1a^{3(1-t)}b^{3t}f'\left(a^{1-t}b^t\right)dt.$$

Now, we will obtain some new Hermite-Hadamard type integral inequalities for (m_1,m_2)-GG-convex functions.

Theorem 4.2. *Let $f:\mathbb{R}_0=[0,\infty)\to\mathbb{R}$ be a differentiable function and $f'\in L([a,b])$ for $0<a<b<\infty$. If $|f'|$ is (m_1,m_2)-GG convex function on the interval $\left[0,\max\left\{a^{\frac{1}{m_1}},b^{\frac{1}{m_2}}\right\}\right]$ for $[m_1,m_2]\in(0,1]^2$, then the following integral inequality holds*

$$\left|\frac{b^2f(a)-a^2f(b)}{2}-\int_a^b xf(x)dx\right|\le\frac{\ln b-\ln a}{2}L\left(a^3\left|f'\left(a^{\frac{1}{m_1}}\right)\right|^{m_1},b^3\left|f'\left(b^{\frac{1}{m_2}}\right)\right|^{m_2}\right),\tag{4.1}$$

where L is the logarithmic mean.

Proof. By using Lemma 4.1 and the inequality

$$\left|f'\left(a^{1-t}b^t\right)\right|=\left|f'\left(a^{\frac{1}{m_1}}\right)^{m_1(1-t)}f'\left(b^{\frac{1}{m_2}}\right)^{m_2t}\right|\le\left|f'\left(a^{\frac{1}{m_1}}\right)\right|^{m_1(1-t)}\left|f'\left(b^{\frac{1}{m_2}}\right)\right|^{m_2t},$$

we get

$$\begin{aligned}\left|\frac{b^2f(a)-a^2f(b)}{2}-\int_a^b xf(x)dx\right| &\le \frac{\ln b-\ln a}{2}\int_0^1 a^{3(1-t)}b^{3t}\left|f'\left(a^{1-t}b^t\right)\right|dt\\ &\le \frac{\ln b-\ln a}{2}\int_0^1 a^{3(1-t)}b^{3t}\left|f'\left(\left(a^{\frac{1}{m_1}}\right)^{m_1(1-t)}\left(b^{\frac{1}{m_2}}\right)^{m_2t}\right)\right|dt\\ &\le \frac{\ln b-\ln a}{2}\int_0^1 a^{3(1-t)}b^{3t}\left|f'\left(a^{\frac{1}{m_1}}\right)\right|^{m_1(1-t)}\left|f'\left(b^{\frac{1}{m_2}}\right)\right|^{m_2t}dt\\ &= \frac{\ln b-\ln a}{2}\frac{b^3\left|f'\left(b^{\frac{1}{m_2}}\right)\right|^{m_2}-a^3\left|f'\left(a^{\frac{1}{m_1}}\right)\right|^{m_1}}{\ln\left(b^3\left|f'\left(b^{\frac{1}{m_2}}\right)\right|^{m_2}\right)-\ln\left(a^3\left|f'\left(a^{\frac{1}{m_1}}\right)\right|^{m_1}\right)}\\ &= \frac{\ln b-\ln a}{2}L\left(a^3\left|f'\left(a^{\frac{1}{m_1}}\right)\right|^{m_1},b^3\left|f'\left(b^{\frac{1}{m_2}}\right)\right|^{m_2}\right).\end{aligned}$$

This completes the proof of theorem. □

Corollary 4.3. *Under the assumption of Theorem 4.2, if we take $m_1=m_2=1$ in the inequality (4.1), then we have the following inequality:*

$$\left|\frac{b^2f(a)-a^2f(b)}{2}-\int_a^b xf(x)dx\right|\le\frac{\ln b-\ln a}{2}L\left(a^3\left|f'(a)\right|,b^3\left|f'(b)\right|\right).\tag{4.2}$$

Theorem 4.4. *Let $f:\mathbb{R}_0=[0,\infty)\to\mathbb{R}$ be a differentiable function and $f'\in L([a,b])$ for $0<a<b<\infty$ and assume that $q>1$. If $|f'|^q$ is a (m_1,m_2)-GG convex function on the interval $\left[0,\max\left\{a^{\frac{1}{m_1}},b^{\frac{1}{m_2}}\right\}\right]$ for $[m_1,m_2]\in(0,1]^2$, then the following integral inequality holds*

$$\left|\frac{b^2f(a)-a^2f(b)}{2}-\int_a^b xf(x)dx\right| \le \frac{\ln b-\ln a}{2}L^{\frac{1}{p}}\left(a^{3p},b^{3p}\right)L^{\frac{1}{q}}\left(\left|f'\left(a^{\frac{1}{m_1}}\right)\right|^{qm_1},\left|f'\left(b^{\frac{1}{m_2}}\right)\right|^{qm_2}\right),\tag{4.3}$$

where L is the arithmetic mean and $\frac{1}{p}+\frac{1}{q}=1$.

Proof. By using Lemma 4.1, Hölder's integral inequality and the inequality

$$\left|f'\left(a^{1-t}b^t\right)\right|^q=\left|f'\left(a^{\frac{1}{m_1}}\right)^{m_1(1-t)}f'\left(b^{\frac{1}{m_2}}\right)^{m_2t}\right|^q\le\left(\left|f'\left(a^{\frac{1}{m_1}}\right)\right|^q\right)^{m_1(1-t)}\left(\left|f'\left(b^{\frac{1}{m_2}}\right)\right|^q\right)^{m_2t}$$

which is the property of the (m_1,m_2)-GG convex function of $|f'|^q$, we get

$$\begin{aligned}\left|\frac{b^2f(a)-a^2f(b)}{2}-\int_a^b xf(x)dx\right| &\le \frac{\ln b-\ln a}{2}\int_0^1 a^{3(1-t)}b^{3t}\left|f'\left(a^{1-t}b^t\right)\right|dt\\ &\le \frac{\ln b-\ln a}{2}\left[\int_0^1\left(a^{3(1-t)}b^{3t}\right)^p dt\right]^{\frac{1}{p}}\left[\int_0^1\left|f'\left(\left(a^{\frac{1}{m_1}}\right)^{m_1(1-t)}\left(b^{\frac{1}{m_2}}\right)^{m_2t}\right)\right|^q dt\right]^{\frac{1}{q}}\\ &\le \frac{\ln b-\ln a}{2}\left[\int_0^1\left(a^{3(1-t)}b^{3t}\right)^p dt\right]^{\frac{1}{p}}\left[\int_0^1\left(\left|f'\left(a^{\frac{1}{m_1}}\right)\right|^q\right)^{m_1(1-t)}\left(\left|f'\left(b^{\frac{1}{m_2}}\right)\right|^q\right)^{m_2t}dt\right]^{\frac{1}{q}}\\ &= \frac{\ln b-\ln a}{2}L^{\frac{1}{p}}\left(a^{3p},b^{3p}\right)L^{\frac{1}{q}}\left(\left|f'\left(a^{\frac{1}{m_1}}\right)\right|^{qm_1},\left|f'\left(b^{\frac{1}{m_2}}\right)\right|^{qm_2}\right),\end{aligned}$$

where

$$\int_0^1\left(a^{3(1-t)}b^{3t}\right)^p dt=L\left(a^{3p},b^{3p}\right)$$

and

$$\int_0^1\left(\left|f'\left(a^{\frac{1}{m_1}}\right)\right|^q\right)^{m_1(1-t)}\left(\left|f'\left(b^{\frac{1}{m_2}}\right)\right|^q\right)^{m_2t}dt=L\left(\left|f'\left(a^{\frac{1}{m_1}}\right)\right|^{qm_1},\left|f'\left(b^{\frac{1}{m_2}}\right)\right|^{qm_2}\right).$$

This completes the proof of theorem. □

Corollary 4.5. *Under the assumption of Theorem 4.4, if we take $m_1 = m_2 = 1$ in the inequality (4.3), then we have the following inequality:*

$$\left|\frac{b^2f(a)-a^2f(b)}{2}-\int_a^b xf(x)dx\right| \leq \frac{\ln b-\ln a}{2}L^{\frac{1}{p}}\left(a^{3p},b^{3p}\right)L^{\frac{1}{q}}\left(\left|f'(a)\right|^q,\left|f'(b)\right|^q\right).$$

Corollary 4.6. *Under the assumption of Theorem 4.4, we can also write the following inequality:*

$$\left|\frac{b^2f(a)-a^2f(b)}{2}-\int_a^b xf(x)dx\right| \leq b^3\frac{\ln b-\ln a}{2}L^{\frac{1}{q}}\left(\left|f'\left(a^{\frac{1}{m_1}}\right)\right|^{qm_1},\left|f'\left(b^{\frac{1}{m_2}}\right)\right|^{qm_2}\right). \tag{4.4}$$

Proof. By using the property of $\left(\frac{b}{a}\right)^{3t} \leq \left(\frac{b}{a}\right)^3$ for $t \in [0,1]$ in the inequality (4.3), we obtain the desired result. □

Corollary 4.7. *Under the assumption of Theorem 4.4, if we take $m_1 = m_2 = 1$ in the inequality (4.4), then we have the following inequality:*

$$\left|\frac{b^2f(a)-a^2f(b)}{2}-\int_a^b xf(x)dx\right| \leq b^3\frac{\ln b-\ln a}{2}L^{\frac{1}{q}}\left(\left|f'(a)\right|^q,\left|f'(b)\right|^q\right).$$

Theorem 4.8. *Let $f:\mathbb{R}_0=[0,\infty)\to\mathbb{R}$ be a differentiable function and $f'\in L([a,b])$ for $0<a<b<\infty$ and assume that $q\geq 1$. If $|f'|^q$ is a (m_1,m_2)-GG convex function on the interval $[a,b]$, then the following inequality holds*

$$\left|\frac{b^2f(a)-a^2f(b)}{2}-\int_a^b xf(x)dx\right| \leq \frac{\ln b-\ln a}{2}L^{1-\frac{1}{q}}\left(a^3,b^3\right)L^{\frac{1}{q}}\left(a^3\left|f'\left(a^{\frac{1}{m_1}}\right)\right|^{qm_1},b^3\left|f'\left(b^{\frac{1}{m_2}}\right)\right|^{qm_2}\right), \tag{4.5}$$

where L is the arithmetic mean.

Proof. From Lemma 4.1, well known power-mean integral inequality and the property of the (m_1,m_2)-GG convex function of $|f'|^q$, we obtain

$$\begin{aligned}
\left|\frac{b^2f(a)-a^2f(b)}{2}-\int_a^b xf(x)dx\right| &\leq \frac{\ln b-\ln a}{2}\int_0^1 a^{3(1-t)}b^{3t}\left|f'\left(a^{1-t}b^t\right)\right|dt \\
&\leq \frac{\ln b-\ln a}{2}\left(\int_0^1 a^{3(1-t)}b^{3t}dt\right)^{1-\frac{1}{q}}\left(\int_0^1 a^{3(1-t)}b^{3t}\left|f'\left(\left(a^{\frac{1}{m_1}}\right)^{m_1(1-t)}\left(b^{\frac{1}{m_2}}\right)^{m_2t}\right)\right|^q dt\right)^{\frac{1}{q}} \\
&\leq \frac{\ln b-\ln a}{2}\left(\int_0^1 a^{3(1-t)}b^{3t}dt\right)^{1-\frac{1}{q}}\left(\int_0^1 a^{3(1-t)}b^{3t}\left(\left|f'\left(a^{\frac{1}{m_1}}\right)\right|^q\right)^{m_1(1-t)}\left(\left|f'\left(b^{\frac{1}{m_2}}\right)\right|^q\right)^{m_2t}dt\right)^{\frac{1}{q}} \\
&= \frac{\ln b-\ln a}{2}\left(\frac{b^3-a^3}{\ln b^3-\ln a^3}\right)^{1-\frac{1}{q}}\left(\frac{b^3\left|f'\left(b^{\frac{1}{m_2}}\right)\right|^{qm_2}-a^3\left|f'\left(a^{\frac{1}{m_1}}\right)\right|^{qm_1}}{\ln\left(b^3\left|f'\left(b^{\frac{1}{m_2}}\right)\right|^{qm_2}\right)-\ln\left(a^3\left|f'\left(a^{\frac{1}{m_1}}\right)\right|^{qm_1}\right)}\right)^{\frac{1}{q}} \\
&= \frac{\ln b-\ln a}{2}L^{1-\frac{1}{q}}\left(a^3,b^3\right)L^{\frac{1}{q}}\left(a^3\left|f'\left(a^{\frac{1}{m_1}}\right)\right|^{qm_1},b^3\left|f'\left(b^{\frac{1}{m_2}}\right)\right|^{qm_2}\right),
\end{aligned}$$

where

$$\int_0^1 a^{3(1-t)}b^{3t}dt = L\left(a^3,b^3\right)$$

and

$$\int_0^1 a^{3(1-t)}b^{3t}\left(\left|f'\left(a^{\frac{1}{m_1}}\right)\right|^q\right)^{m_1(1-t)}\left(\left|f'\left(b^{\frac{1}{m_2}}\right)\right|^q\right)^{m_2t}dt = L^{\frac{1}{q}}\left(a^3\left|f'\left(a^{\frac{1}{m_1}}\right)\right|^{qm_1},b^3\left|f'\left(b^{\frac{1}{m_2}}\right)\right|^{qm_2}\right)$$

This completes the proof of theorem. □

Corollary 4.9. *Under the assumption of Theorem 4.8 with $q=1$, we get the conclusion of Theorem 4.2 as follow:*

$$\left|\frac{b^2f(a)-a^2f(b)}{2}-\int_a^b xf(x)dx\right| \leq \frac{\ln b-\ln a}{2}L\left(a^3\left|f'\left(a^{\frac{1}{m_1}}\right)\right|^{m_1},b^3\left|f'\left(b^{\frac{1}{m_2}}\right)\right|^{m_2}\right).$$

Corollary 4.10. *Under the assumption of Theorem 4.8 with $q=1$ and $m_1=m_2=1$ in the inequality (4.5), we get the following inequality:*

$$\left|\frac{b^2f(a)-a^2f(b)}{2}-\int_a^b xf(x)dx\right| \leq \frac{\ln b-\ln a}{2}L\left(a^3\left|f'(a)\right|,b^3\left|f'(b)\right|\right).$$

This inequality coincides with the inequality (4.2).

Theorem 4.11. *Let $f:\mathbb{R}_0=[0,\infty)\to\mathbb{R}$ be a differentiable function and $f'\in L([a,b])$ for $0<a<b<\infty$. If $|f'|^q$ is (m_1,m_2)-GG convex function on the interval $\left[0,\max\left\{a^{\frac{1}{m_1}},b^{\frac{1}{m_2}}\right\}\right]$ for $[m_1,m_2]\in(0,1]^2$ and $q>1$, then the following integral inequality holds*

$$\begin{aligned}
&\left|\frac{b^2f(a)-a^2f(b)}{2}-\int_a^b xf(x)dx\right| \qquad (4.6)\\
\leq\ & \frac{\ln b-\ln a}{2}\left[\frac{L\left(a^{3p},b^{3p}\right)-a^{3p}}{3(\ln b-\ln a)}\right]^{\frac{1}{p}}\left[\frac{L\left(\left|f'\left(a^{\frac{1}{m_1}}\right)\right|^{qm_1},\left|f'\left(b^{\frac{1}{m_2}}\right)\right|^{qm_2}\right)-\left|f'\left(a^{\frac{1}{m_1}}\right)\right|^{qm_1}}{\ln\left|f'\left(b^{\frac{1}{m_2}}\right)\right|^{qm_2}-\ln\left|f'\left(a^{\frac{1}{m_1}}\right)\right|^{qm_1}}\right]^{\frac{1}{q}}\\
&+\frac{\ln b-\ln a}{2}\left[\frac{b^{3p}-L\left(a^{3p},b^{3p}\right)}{3(\ln b-\ln a)}\right]^{\frac{1}{p}}\left[\frac{\left|f'\left(b^{\frac{1}{m_2}}\right)\right|^{qm_2}-L\left(\left|f'\left(a^{\frac{1}{m_1}}\right)\right|^{qm_1},\left|f'\left(b^{\frac{1}{m_2}}\right)\right|^{qm_2}\right)}{\ln\left|f'\left(b^{\frac{1}{m_2}}\right)\right|^{qm_2}-\ln\left|f'\left(a^{\frac{1}{m_1}}\right)\right|^{qm_1}}\right]^{\frac{1}{q}},
\end{aligned}$$

where L is the logarithmic mean and $\frac{1}{p}+\frac{1}{q}=1$.

Proof. From Lemma 4.1, Hölder-İşcan integral inequality and the (m_1,m_2)-GG convexity of the function $|f'|^q$ on the interval $\left[0,\max\left\{a^{\frac{1}{m_1}},b^{\frac{1}{m_2}}\right\}\right]$, we obtain

$$\begin{aligned}
\left|\frac{b^2f(a)-a^2f(b)}{2}-\int_a^b xf(x)dx\right| \leq\ & \frac{\ln b-\ln a}{2}\left[\int_0^1(1-t)\left(a^{3(1-t)}b^{3t}\right)^p dt\right]^{\frac{1}{p}}\left[\int_0^1(1-t)\left|f'\left(\left(a^{\frac{1}{m_1}}\right)^{m_1(1-t)}\left(b^{\frac{1}{m_2}}\right)^{m_2t}\right)\right|^q dt\right]^{\frac{1}{q}}\\
&+\frac{\ln b-\ln a}{2}\left[\int_0^1 t\left(a^{3(1-t)}b^{3t}\right)^p dt\right]^{\frac{1}{p}}\left[\int_0^1 t\left|f'\left(\left(a^{\frac{1}{m_1}}\right)^{m_1(1-t)}\left(b^{\frac{1}{m_2}}\right)^{m_2t}\right)\right|^q dt\right]^{\frac{1}{q}}\\
\leq\ & \frac{\ln b-\ln a}{2}\left[\int_0^1(1-t)a^{3p(1-t)}b^{3pt}dt\right]^{\frac{1}{p}}\left[\int_0^1(1-t)\left|f'\left(a^{\frac{1}{m_1}}\right)\right|^{qm_1(1-t)}\left|f'\left(b^{\frac{1}{m_2}}\right)\right|^{qm_2t}dt\right]^{\frac{1}{q}}\\
&+\frac{\ln b-\ln a}{2}\left[\int_0^1 ta^{3p(1-t)}b^{3pt}dt\right]^{\frac{1}{p}}\left[\int_0^1 t\left|f'\left(a^{\frac{1}{m_1}}\right)\right|^{qm_1(1-t)}\left|f'\left(b^{\frac{1}{m_2}}\right)\right|^{qm_2t}dt\right]^{\frac{1}{q}}\\
=\ & \frac{\ln b-\ln a}{2}\left[\frac{L\left(a^{3p},b^{3p}\right)-a^{3p}}{3(\ln b-\ln a)}\right]^{\frac{1}{p}}\left[\frac{L\left(\left|f'\left(a^{\frac{1}{m_1}}\right)\right|^{qm_1},\left|f'\left(b^{\frac{1}{m_2}}\right)\right|^{qm_2}\right)-\left|f'\left(a^{\frac{1}{m_1}}\right)\right|^{qm_1}}{\ln\left|f'\left(b^{\frac{1}{m_2}}\right)\right|^{qm_2}-\ln\left|f'\left(a^{\frac{1}{m_1}}\right)\right|^{qm_1}}\right]^{\frac{1}{q}}\\
&+\frac{\ln b-\ln a}{2}\left[\frac{b^{3p}-L\left(a^{3p},b^{3p}\right)}{3(\ln b-\ln a)}\right]^{\frac{1}{p}}\left[\frac{\left|f'\left(b^{\frac{1}{m_2}}\right)\right|^{qm_2}-L\left(\left|f'\left(a^{\frac{1}{m_1}}\right)\right|^{qm_1},\left|f'\left(b^{\frac{1}{m_2}}\right)\right|^{qm_2}\right)}{\ln\left|f'\left(b^{\frac{1}{m_2}}\right)\right|^{qm_2}-\ln\left|f'\left(a^{\frac{1}{m_1}}\right)\right|^{qm_1}}\right]^{\frac{1}{q}}.
\end{aligned}$$

This completes the proof of theorem. □

Corollary 4.12. *Under the assumption of Theorem 4.11 with $m_1=m_2=1$ in the inequality (4.6), we get the following inequality:*

$$\begin{aligned}
\left|\frac{b^2f(a)-a^2f(b)}{2}-\int_a^b xf(x)dx\right| \leq\ & \frac{\ln b-\ln a}{2}\left[\frac{L\left(a^{3p},b^{3p}\right)-a^{3p}}{3(\ln b-\ln a)}\right]^{\frac{1}{p}}\left[\frac{L\left(|f'(a)|^q,|f'(b)|^q\right)-|f'(a)|^q}{\ln|f'(b)|^q-\ln|f'(a)|^q}\right]^{\frac{1}{q}}\\
&+\frac{\ln b-\ln a}{2}\left[\frac{b^{3p}-L\left(a^{3p},b^{3p}\right)}{3(\ln b-\ln a)}\right]^{\frac{1}{p}}\left[\frac{|f'(b)|^q-L\left(|f'(a)|^q,|f'(b)|^q\right)}{\ln|f'(b)|^q-\ln|f'(a)|^q}\right]^{\frac{1}{q}}. \qquad (4.7)
\end{aligned}$$

Corollary 4.13. *Under the assumption of Theorem 4.11, we can also write the following inequality:*

$$\begin{aligned}
\left|\frac{b^2f(a)-a^2f(b)}{2}-\int_a^b xf(x)dx\right| \leq\ & b^3\frac{\ln b-\ln a}{2}\left(\frac{1}{2}\right)^{\frac{1}{p}}\left[\frac{L\left(\left|f'\left(a^{\frac{1}{m_1}}\right)\right|^{qm_1},\left|f'\left(b^{\frac{1}{m_2}}\right)\right|^{qm_2}\right)-\left|f'\left(a^{\frac{1}{m_1}}\right)\right|^{qm_1}}{\ln\left|f'\left(b^{\frac{1}{m_2}}\right)\right|^{qm_2}-\ln\left|f'\left(a^{\frac{1}{m_1}}\right)\right|^{qm_1}}\right]^{\frac{1}{q}}\\
&+b^3\frac{\ln b-\ln a}{2}\left(\frac{1}{2}\right)^{\frac{1}{p}}\left[\frac{\left|f'\left(b^{\frac{1}{m_2}}\right)\right|^{qm_2}-L\left(\left|f'\left(a^{\frac{1}{m_1}}\right)\right|^{qm_1},\left|f'\left(b^{\frac{1}{m_2}}\right)\right|^{qm_2}\right)}{\ln\left|f'\left(b^{\frac{1}{m_2}}\right)\right|^{qm_2}-\ln\left|f'\left(a^{\frac{1}{m_1}}\right)\right|^{qm_1}}\right]^{\frac{1}{q}}.
\end{aligned}$$

Proof. By using the property of $\left(\frac{b}{a}\right)^{3t}\leq\left(\frac{b}{a}\right)^3$ for $t\in[0,1]$ in the inequality (4.6), we obtain the desired result. □

Corollary 4.14. *Under the assumption of Theorem 4.11 with $m_1 = m_2 = 1$ in the inequality (4.7), we get the following inequality:*

$$\begin{aligned}\left|\frac{b^2 f(a)-a^2 f(b)}{2}-\int_a^b xf(x)dx\right| &\leq b^3\frac{\ln b-\ln a}{2}\left(\frac{1}{2}\right)^{\frac{1}{p}}\left[\frac{L\left(|f'(a)|^q,\left|f'\left(b^{\frac{1}{m_2}}\right)\right|^q\right)-|f'(a)|^q}{\ln|f'(b)|^q-\ln|f'(a)|^q}\right]^{\frac{1}{q}}\\ &+b^3\frac{\ln b-\ln a}{2}\left(\frac{1}{2}\right)^{\frac{1}{p}}\left[\frac{|f'(b)|^q-L\left(|f'(a)|^q,|f'(b)|^q\right)}{\ln|f'(b)|^q-\ln|f'(a)|^q}\right]^{\frac{1}{q}}.\end{aligned}$$

Theorem 4.15. *Let $f:\mathbb{R}_0=[0,\infty)\to\mathbb{R}$ be a differentiable function and $f'\in L([a,b])$ for $0<a<b<\infty$. If $|f'|^q$ is (m_1,m_2)-GG convex function on the interval $\left[0,\max\left\{a^{\frac{1}{m_1}},b^{\frac{1}{m_2}}\right\}\right]$ for $[m_1,m_2]\in(0,1]^2$ and $q\geq 1$, then the following integral inequality holds*

$$\begin{aligned}&\left|\frac{b^2 f(a)-a^2 f(b)}{2}-\int_a^b xf(x)dx\right| \qquad (4.8)\\ \leq\ &\frac{\ln b-\ln a}{2}\left[\frac{L\left(a^3,b^3\right)-a^3}{3\left(\ln b-\ln a\right)}\right]^{1-\frac{1}{q}}\left[\frac{L\left(a^3\left|f'\left(a^{\frac{1}{m_1}}\right)\right|^{qm_1},b^3\left|f'\left(b^{\frac{1}{m_2}}\right)\right|^{qm_2}\right)-a^3\left|f'\left(a^{\frac{1}{m_1}}\right)\right|^{qm_1}}{\ln\left(b^3\left|f'\left(b^{\frac{1}{m_2}}\right)\right|^{qm_2}\right)-\ln\left(a^3\left|f'\left(a^{\frac{1}{m_1}}\right)\right|^{qm_1}\right)}\right]^{\frac{1}{q}}\\ &+\frac{\ln b-\ln a}{2}\left[\frac{b^3-L\left(a^3,b^3\right)}{3\left(\ln b-\ln a\right)}\right]^{1-\frac{1}{q}}\left[\frac{b^3\left|f'\left(b^{\frac{1}{m_2}}\right)\right|^{qm_2}-L\left(a^3\left|f'\left(a^{\frac{1}{m_1}}\right)\right|^{qm_1},b^3\left|f'\left(b^{\frac{1}{m_2}}\right)\right|^{qm_2}\right)}{\ln\left(b^3\left|f'\left(b^{\frac{1}{m_2}}\right)\right|^{qm_2}\right)-\ln\left(a^3\left|f'\left(a^{\frac{1}{m_1}}\right)\right|^{qm_1}\right)}\right]^{\frac{1}{q}},\end{aligned}$$

where L is the logarithmic mean and $\frac{1}{p}+\frac{1}{q}=1$.

Proof. By using Lemma 4.1, improved power-mean integral inequality and the property of (m_1,m_2)-GG convexity of the function $|f'|^q$ on the interval $\left[0,\max\left\{a^{\frac{1}{m_1}},b^{\frac{1}{m_2}}\right\}\right]$, we get

$$\begin{aligned}&\left|\frac{b^2 f(a)-a^2 f(b)}{2}-\int_a^b xf(x)dx\right|\\ \leq\ &\frac{\ln b-\ln a}{2}\left[\int_0^1(1-t)\left(a^{3(1-t)}b^{3t}\right)dt\right]^{1-\frac{1}{q}}\left[\int_0^1(1-t)\left(a^{3(1-t)}b^{3t}\right)\left|f'\left(\left(a^{\frac{1}{m_1}}\right)^{m_1(1-t)}\left(b^{\frac{1}{m_2}}\right)^{m_2t}\right)\right|^q dt\right]^{\frac{1}{q}}\\ &+\frac{\ln b-\ln a}{2}\left[\int_0^1 t\left(a^{3(1-t)}b^{3t}\right)dt\right]^{1-\frac{1}{q}}\left[\int_0^1 t\left(a^{3(1-t)}b^{3t}\right)\left|f'\left(\left(a^{\frac{1}{m_1}}\right)^{m_1(1-t)}\left(b^{\frac{1}{m_2}}\right)^{m_2t}\right)\right|^q dt\right]^{\frac{1}{q}}\\ \leq\ &\frac{\ln b-\ln a}{2}\left[\int_0^1(1-t)\left(a^{3(1-t)}b^{3t}\right)dt\right]^{1-\frac{1}{q}}\left[\int_0^1(1-t)\left(a^{3(1-t)}b^{3t}\right)\left|f'\left(a^{\frac{1}{m_1}}\right)\right|^{qm_1(1-t)}\left|f'\left(b^{\frac{1}{m_2}}\right)\right|^{qm_2t}dt\right]^{\frac{1}{q}}\\ &+\frac{\ln b-\ln a}{2}\left[\int_0^1 t\left(a^{3(1-t)}b^{3t}\right)dt\right]^{1-\frac{1}{q}}\left[\int_0^1 t\left(a^{3(1-t)}b^{3t}\right)\left|f'\left(a^{\frac{1}{m_1}}\right)\right|^{qm_1(1-t)}\left|f'\left(b^{\frac{1}{m_2}}\right)\right|^{qm_2t}dt\right]^{\frac{1}{q}}\\ =\ &\frac{\ln b-\ln a}{2}\left[\frac{L\left(a^3,b^3\right)-a^3}{3\left(\ln b-\ln a\right)}\right]^{1-\frac{1}{q}}\left[\frac{L\left(a^3\left|f'\left(a^{\frac{1}{m_1}}\right)\right|^{qm_1},b^3\left|f'\left(b^{\frac{1}{m_2}}\right)\right|^{qm_2}\right)-a^3\left|f'\left(a^{\frac{1}{m_1}}\right)\right|^{qm_1}}{\ln\left(b^3\left|f'\left(b^{\frac{1}{m_2}}\right)\right|^{qm_2}\right)-\ln\left(a^3\left|f'\left(a^{\frac{1}{m_1}}\right)\right|^{qm_1}\right)}\right]^{\frac{1}{q}}\\ &+\frac{\ln b-\ln a}{2}\left[\frac{b^3-L\left(a^3,b^3\right)}{3\left(\ln b-\ln a\right)}\right]^{1-\frac{1}{q}}\left[\frac{b^3\left|f'\left(b^{\frac{1}{m_2}}\right)\right|^{qm_2}-L\left(a^3\left|f'\left(a^{\frac{1}{m_1}}\right)\right|^{qm_1},b^3\left|f'\left(b^{\frac{1}{m_2}}\right)\right|^{qm_2}\right)}{\ln\left(b^3\left|f'\left(b^{\frac{1}{m_2}}\right)\right|^{qm_2}\right)-\ln\left(a^3\left|f'\left(a^{\frac{1}{m_1}}\right)\right|^{qm_1}\right)}\right]^{\frac{1}{q}}.\end{aligned}$$

This completes the proof of theorem. ☐

Corollary 4.16. *Under the assumption of Theorem 4.15 with $q=1$, we get the following inequality:*

$$\left|\frac{b^2 f(a)-a^2 f(b)}{2}-\int_a^b xf(x)dx\right|\leq\frac{\ln b-\ln a}{2}L\left(a^3\left|f'\left(a^{\frac{1}{m_1}}\right)\right|^{m_1},b^3\left|f'\left(b^{\frac{1}{m_2}}\right)\right|^{m_2}\right).$$

Corollary 4.17. *Under the assumption of Theorem 4.15 with $q=1$ and $m_1=m_2=1$ in the inequality (4.8), we get the following inequality:*

$$\left|\frac{b^2 f(a)-a^2 f(b)}{2}-\int_a^b xf(x)dx\right|\leq\frac{\ln b-\ln a}{2}L\left(a^3\left|f'(a)\right|,b^3\left|f'(b)\right|\right).$$

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