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AN ANALYTICAL APPROXIMATION FOR THE LOCAL FLOW BETWEEN COUNTER-ROTATING CYLINDER PAIR

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ABSTRACT

Flow around a pair of cylinders is of interest for various applications. Especially with the development of self-propelled, autonomous vehicles, this topic has gained further importance. There have been experimental, numerical and, to a much less extent, theoretical studies of this flow and its implications on the cylinders. However, the complexity of the problem does not allow a general solution, and case-by-case solutions have been produced. In this study, a theoretical approach was embraced, in which differential geometry is employed to model the local flow between a pair of counter-rotating cylinders. In order to obtain practical formulae for a steady, laminar, incompressible flow, Navier-Stokes equations were simplified and expressed in a new, parabolic coordinate system. Then, further simplifications due to the symmetrical nature of the problem were applied. Finally, boundary conditions were imposed while performing the integrals, and the desired equations for velocity and pressure were reached. Unlike the previous studies that are limited to very slow flows, the equations obtained in this study are applicable in the entire incompressible, laminar regime, albeit only between the cylinders. Using these equations, the effects of the rotation speed, cylinder spacing and cylinder radius were studied and presented graphically.

Keywords: Self-propulsion, Navier-Stokes, Differential Geometry, Parabolic Coordinates, Counter-Rotating Cylinder Pair

NOMENCLATURE

P : static pressure
P* $:=\frac{P-P_\infty}{\rho(wR_c)^2}$, non-dimensional pressure
R_c : radius of the cylinder
Re : $\frac{wR_c^2}{\nu}$, Reynolds number
s : half of the distance between the cylinder surfaces at the throat
s* : s/R_c , non-dimensional gap distance
U : uniform flow speed in potential flow formulation
V : velocity vector in the Navier-Stokes formulation
w : rotational speed of the cylinders
x-y : Cartesian coordinate systems
x*-y* : non-dimensional Cartesian coordinates
y_c : center of the inscribed circle at the apex of the reference parabola
Γ : circulation at the cylinder surface in potential flow
η-ξ : non-orthogonal, curvilinear coordinate system
θ : angular position on the cylinder in potential flow
ν : kinematic viscosity
Ω : vorticity
Ω* : Ω/w , non-dimensional vorticity

1. INTRODUCTION

Flow around cylinders has been the focus of research for more than a century. Variations of this flow have been tried by changing the number of cylinders and their relative motion with respect to the ambient fluid. Studies on non-rotating cylinders helped improve the industrial applications in heat exchangers, bridge columns exposed to water flow, suspension cables exposed to wind. Depending on the number of cylinders involved, analytical or experimental solutions and design guidelines have been developed for these non-rotationary cases. For the rotating cylinders, on the other hand,

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applications in aeronautics or heat exchangers were of interest. Again, analytical solutions for limited cases, and experimental and numerical solutions for various other applications have been developed.

Research on flow around a cylinder pair, thus, started long ago with non-rotationary or rotationary applications. However, due to the inherently unsteady and complicated nature of the problem and limited cases of applications, these studies did not become widespread. With the advent of computer technology, more studies were done to further and to refine the knowledge on this problem [1]. Nevertheless, a major push for the research on flow around a rotating cylinder pair came recently due to potential applications including robotics, biological flows [2] and autonomous vehicles with biomimetic features.

Specifically, one of the first studies on the flow around a rotating cylinder pair was by G.B. Jeffery in 1922 [3]. This was an analytical study, and it looked at two settings: first was that of two cylinders, one containing the other, and the second was the case with cylinders neighboring each other, as in the present study. Although he presented a solution for the first case, he sufficed with mathematically stating the complexity of the latter problem, since no steady state can be reached in the absolute sense.

On top of this work, significant progress in theoretical studies has been made towards the end of the last century [4,5,6]. Similar to the method of Ref.3, these studies assumed a very slow flow, Stokes Flow. In order to solve the entire flow field, they produced a solution with two parts. One was valid only in the vicinity of the cylinders. Outside of this region, they solved the full Navier-Stokes equations for steady, incompressible, two-dimensional (2D) flow conditions. As a result, they succeeded in resolving the entire flow field, but only for very slow velocities.

Aside from these, the stability of the wake flow from these cylinders was studied [7], again at low speeds. The wakes from the cylinders were found to interact less as the space between the cylinders increased. The same trend was also observed for increased rotational speed of the cylinders. These results were also confirmed by the experimental and numerical studies on vortex suppression and drag reduction [8,9].

A more recent study by Yoon et al. [10] also looked at the pair of cylinders counter-rotating in a stationary fluid. Performed with numerical simulations, their study showed that the ensuing flow has strong dependence on the ratio of the distance between the cylinders and their diameter. They also showed suppression of the unsteady vortex shedding regime at a critical rotation speed, after which a relatively steady flow regime is established.

An experimental study by Guo et al. [11] shed light on the details of the flow dynamics around the counter-rotating cylinder pair. In their study, the flow between the cylinders is ejected towards the oncoming flow, and the fluid is pushed in the downstream direction after being diverted backwards around the cylinders. Thus, the cylinder pair and the immediate flow around them work as a single unit, resembling an elliptical structure, around which the fluid flows. Aside from confirming the onset of a relatively steady state flow around the cylinders beyond the critical rotation speed, they reveal the drop of the critical rotation speed with increasing free stream Reynolds number.

Suwannasri [12] numerically studied the self-propulsion of a counter-rotating cylinder pair in a uniform flow. He showed two different modes of self-propulsion of the cylinder pair. For fixed spacing between the cylinders and the cylinder diameter, if the cylinders counter-rotate so that their mutually facing surfaces are moving into the oncoming flow, there is a certain value of rotational speed at which self-propulsion begins, as in the study of Guo et al. [11]. Aside from this, if the cylinders counter-rotate so that their mutually facing surfaces are moving along with the oncoming flow, then self-propulsion starts after a certain rotational speed due to the fast ejection of fluid from between the cylinders.

Van Rees et al. [13] performed a very detailed study of the self-propulsion of the counter-rotating cylinder pair in stationary fluid. Although they imposed no free stream flow velocity, their study also revealed the two different mechanisms of self-propulsion, depending on the distance between the cylinders and the rotational speed. At low rotational speeds with relatively large gap/diameter ratio, the cylinder pair moved in the direction of the jet ejected from between the cylinders (i.e. in the downstream direction). This was enabled by the blockage of the mass flow due to the co-presence of the cylinders, while no such blockage was present out of the pair. In the second mode of self-propulsion, which occurred at high rotational speeds with relatively low gap/diameter ratio, the pair moved opposite to the direction of the jet ejected from between the cylinders (i.e. into the oncoming flow). This mode of propulsion occurred due to the presence of the viscous resistance outside the pair, hence lower thrust compared to the thrust provided by the jet emanating from the gap.

In the present study, a counter-rotating cylinder pair was modeled in a stationary fluid. The goal was to obtain an analytical, open form expression for the velocity distribution along the mid horizontal plane and the mid vertical plane. Accordingly, in order to avoid the complexities with the complete flow field with unsteady features, a local flow analysis was performed by focusing on and around the gap between the cylinders. Furthermore, as is representative of the flow upstream of the cylinders, two-dimensional flow with steady state and laminar flow conditions was assumed.

2. METHOD

The cylinders modeled in this study are located in a stationary flow field. The only fluid motion is triggered by the viscous forces due to the rotation of the two cylinders. The upper cylinder is set to rotate counter-clockwise, whereas the lower one rotates clockwise (Fig. 1). In this figure, the distance between the cylinder surfaces is 2s, and the cylinder radius is R_c. With the assumption of steady, laminar flow, the flow field becomes symmetric with respect to the x-axis. The difficulty with modeling such a flow field is the juxtaposition of circular and linear features. Therefore, neither the polar coordinates nor the conventional Cartesian coordinates work in this case. Rather, there is a need for a hybrid system that can harbor both the curved and the linear features where necessary.

One solution to this problem is a parabolic coordinate system as described here. Since a parabola whose apex is located on the y-axis has the same radius of curvature as the inscribed circle (Fig. 2), and since the streamlines both upstream of the cylinders and between them resemble parabolas, the cylinders can be replaced by parabolas for local flow analysis purposes. Thus, while disabling the complete flow field analysis, a local analysis for the laminar flow immediately upstream of the cylinders and in the gap between them is enabled.

Having replaced the circles with parabolas, the next task is to describe the linear feature of the x-axis within this parabolic (curvilinear) system. X-axis is the central streamline across which the flow field is symmetric. So, focusing only on the upper half of the problem, if the arms of the parabola spread out as its apex descends towards the origin, a line can be obtained, which is the x-axis (Fig. 2). In fact, the generic formula below describes a family of parabolas covering the flow field exactly as explained above.

$$y = ax^2 + c \qquad a = \frac{1}{2R} \qquad c = \frac{s(s+R_c)}{s+R} = \frac{2as(s+R_c)}{2as+1} \tag{1}$$

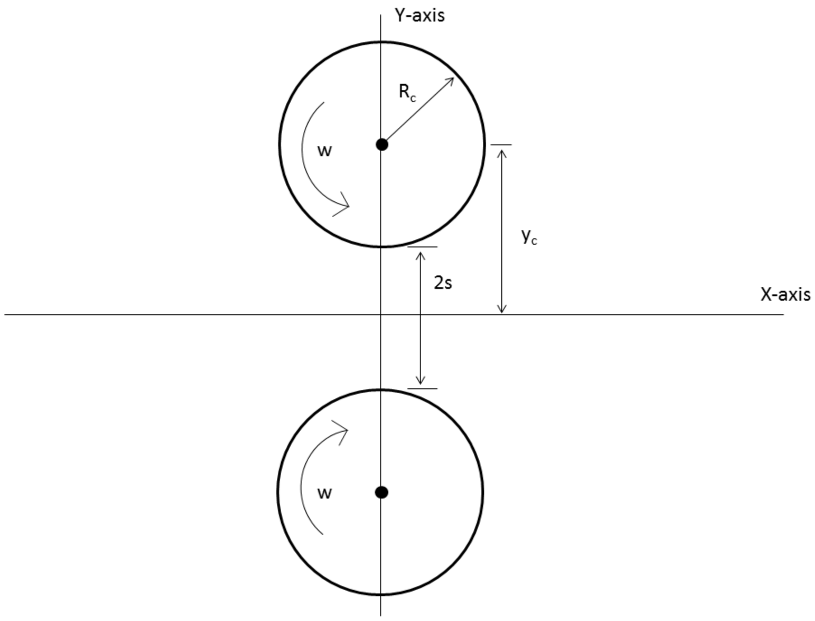

Figure 1. Basic geometry of the cylinders and the flow field.

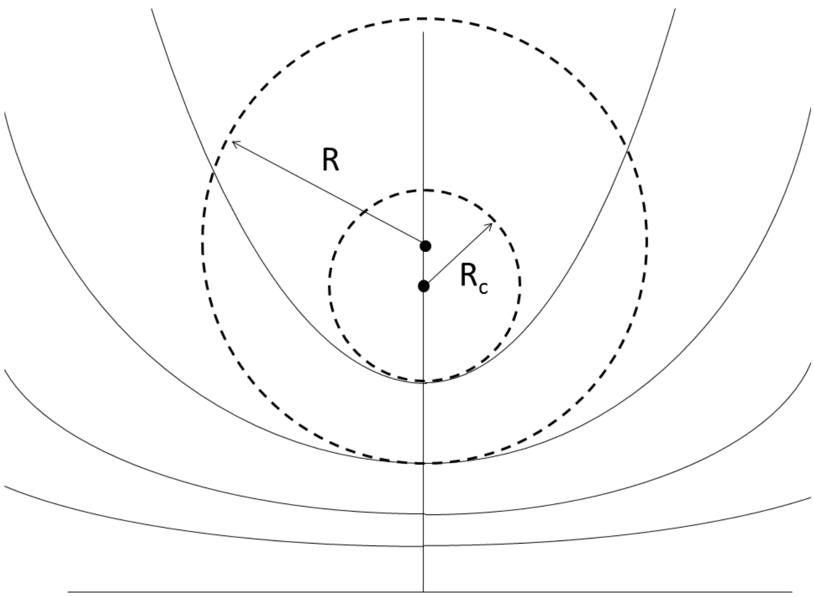

Figure 2. Illustration of the new parabolic coordinate system.

Note that in these formulas, R_c is the radius of the cylinder and the radius of curvature of the parabola inscribing it, whereas R is the functional radius of curvature of a parabola at its apex. Given a point in the x-y plane and the design parameters for the cylinder pair (i.e. s and R_c), using the formulae above, one can find the equation of the parabola that passes through that point, along with the radius of curvature at its apex.

Given the fact that each parabola in the flow field has a unique radius of curvature at its apex, this value can be used as one parameter in developing the new parabolic coordinate system. The second parameter that can be used for describing a point in the flow field is the distance along the parabola to get to that point. Accordingly, the new coordinate system is defined as (ξ,η), where,

$$\xi = a \tag{2}$$

$$\eta = \int_0^x \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx = \int_0^x \sqrt{1 + (2ax)^2} dx =$$

$$\frac{1}{4a}\left\{2ax\sqrt{1 + (2ax)^2} + \ln\left[\sqrt{1 + (2ax)^2} + 2ax\right]\right\} \tag{3}$$

The details of these derivations can also be found in the sections A1 and A2 of the Appendix.

Once on the parabolic surfaces, a straightforward consequence of this change is the fact that the unit vectors are not constant anymore (Fig. 3). That is, although the unit vectors in the Cartesian coordinate system show the same direction anywhere in the flow field, the directions shown by the unit vectors of the parabolic coordinate system change along the parabolas. Furthermore, although the unit vectors of the x-y system are normal to each other everywhere, the unit vectors of the ξ-η system are not necessarily normal to each other. In fact, they are normal to each other only along the x-axis and y-axis, and non-orthogonal elsewhere. More explicitly stated, the constant-ξ and the constant-η curves are not necessarily normal to each other. Therefore, in order to derive the formulae for the flow according to the new parabolic system, there is a need to relate the unit vectors of the two systems. This task is done in detail in section A3 of the Appendix.

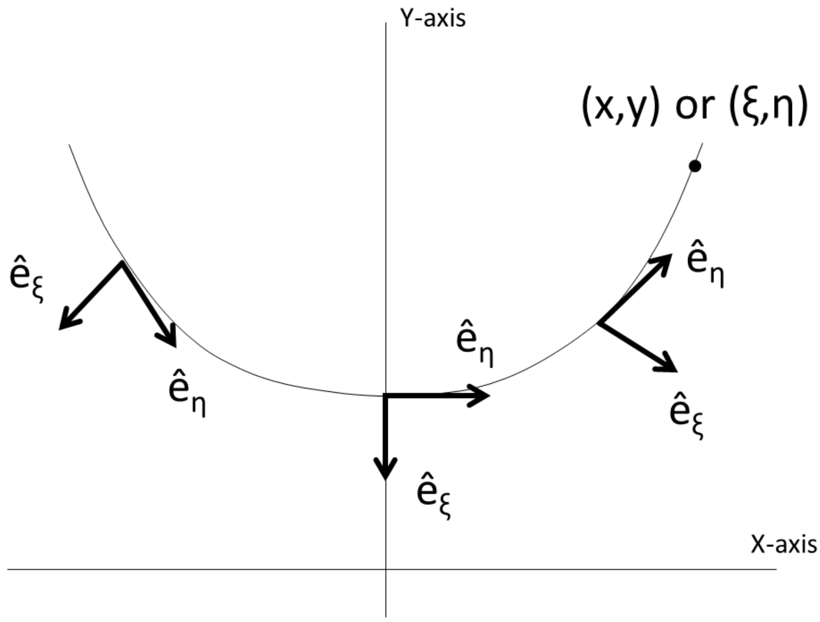

Figure 3. Depiction of the unit vectors over the new parabolic coordinate system, which is curvilinear and non-orthogonal.

As a consequence of the curvilinear and non-orthogonal nature of the parabolic coordinate system, all of the operators in the vector analysis, namely, the del operator, the divergence operator, the Laplacian operator, must be re-derived starting from their originals in the x-y system. These derivations are done in the Appendix in sections A4-A6. Using these results, in the theory section, conservation of mass and Navier-Stokes equations are, first, expressed in the parabolic coordinates, and then solved analytically according to the boundary conditions of the flow at hand. Hence, explicit formulae are obtained for the velocity distribution across the gap between the cylinders and along the symmetry plane separating the cylinders.

3. THEORY

For the development of the formulae describing the velocity variation between the cylinders, the derivation starts with the closed form of the conservation of mass and the Navier-Stokes equations. With the application of the steady-state and incompressibility assumptions, one gets:

$$\nabla.\vec{V} = 0 \tag{4}$$

$$\vec{V}.\nabla\,\vec{V} = -\frac{1}{\rho}\,\nabla P + \nu\,\nabla^2\vec{V} \tag{5}$$

The latter equation in the closed form has two distinct components due to its vectorial nature. Be reminded that in the present study, instead of the conventional Cartesian coordinate system, the new parabolic coordinate system is used. So, the terms composing the del, divergence and the Laplacian operators are not as straightforward as in the Cartesian coordinates. Nevertheless, by employing the geometrical and physical boundary conditions of two special application locations, namely mid horizontal plane (case1) and the mid vertical plane (case2), it is possible to simplify these operators. These conditions are mostly due to geometrical identities between the Cartesian and the new parabolic coordinate systems at the mentioned locations. There are also conditions due to flow symmetry. So, from now on, the analysis is going to be handled according to these two special cases.

3.1. Case 1 - Mid Horizontal Plane

This is the symmetry plane between the two counter-rotating cylinders (x-axis). It is horizontal, and the laminar flow along it has no component crossing it. These physical conditions are:

$$V_\xi = 0 \qquad \frac{\partial V_\xi}{\partial \eta} = 0 \qquad \frac{\partial V_\xi}{\partial \xi} = 0$$

Note that the last condition above is a reflection of the "pseudo one-dimensional flow" assumption along the mid horizontal plane, similar to the dominant one dimensional flow in the converging-diverging ducts. This assumption is acceptable, since the flow physically follows parabolic streamlines, hence a flow solely along the η-direction. A consequence of this stipulation is that there should not be a sudden change in the geometry. So, large values of the ratio between the cylinder spacing and cylinder radius, s/R_c, are beyond the coverage of the approach here.

Furthermore, by virtue of the symmetry across the mid horizontal plane:

$$\frac{\partial V_\eta}{\partial \xi} = 0$$

For the steady and incompressible conditions of the flow, conservation of mass equation (i.e. continuity equation) becomes "zero divergence of the velocity field" (see section A5.1 for the divergence equation). By further imposing the physical conditions of case 1, stated above, to the conservation of mass equation, one gets:

$$\nabla.\vec{V} = \frac{\partial V_\eta}{\partial \eta} + \frac{2x}{(x^2+2s^2+2sR_c)}V_\eta = 0 \tag{6}$$

By rearranging and integrating the above equation, one can get an expression for the velocity along the mid horizontal plane. Note that for the special case 1, η = x, and so, ∂η is equal to ∂x.

$$V_{\eta,case1} = \frac{c_1}{x^2+2s^2+2sR_c} \tag{7}$$

where c_1 is an integration constant. For the Navier-Stokes equations, one can import from the Appendix (A4.1, A6.1 and A7.1) the relevant forms of the operators, and get:

$$\left(-\frac{|\eta|}{\eta}\frac{1}{x^2+2s^2+2sR_c}V_\xi\frac{\partial}{\partial\xi}+V_\eta\frac{\partial}{\partial\eta}\right)\left(V_\xi\hat{e}_\xi+V_\eta\hat{e}_\eta\right)=$$

$$-\frac{1}{\rho}\left(-\frac{|\eta|}{\eta}\frac{1}{x^2+2s^2+2sR_c}\frac{\partial P}{\partial\xi}\hat{e}_\xi+\frac{\partial P}{\partial\eta}\hat{e}_\eta\right)+\nu\left[\frac{1}{(x^2+2s^2+2sR_c)^2}\frac{\partial^2}{\partial\xi^2}+\frac{\partial^2}{\partial\eta^2}+\frac{2x}{(x^2+2s^2+2sR_c)}\frac{\partial}{\partial\eta}\right]\left(V_\xi\hat{e}_\xi+V_\eta\hat{e}_\eta\right)$$

Because the ξ component of the velocity for case 1 is identically zero, the ξ component of the Navier-Stokes equations yields zero pressure gradient in the ξ direction for this case. Then, for the η component of momentum, after imposing the velocity boundary conditions stated above, one gets:

$$V_\eta\frac{\partial V_\eta}{\partial\eta}=-\frac{1}{\rho}\frac{\partial P}{\partial\eta}+\nu\left[-\frac{4x^2}{(x^2+2s^2+2sR_c)^2}V_\eta+\frac{2x}{(x^2+2s^2+2sR_c)}\frac{\partial V_\eta}{\partial\eta}+\frac{\partial^2 V_\eta}{\partial\eta^2}\right]$$

Using the V_η formula derived above, one can get the velocity derivatives in the Navier-Stokes equation. Again, note that for the special case 1, $\eta = x$, and so, $\partial\eta$ is equal to ∂x.

$$\frac{-2x}{(x^2+2s^2+2sR_c)}V_\eta^2=-\frac{1}{\rho}\frac{\partial P}{\partial\eta}-\frac{2\nu}{(x^2+2s^2+2sR_c)}V_\eta$$

From this, pressure can be found as:

$$\frac{1}{\rho}P=c_2-\frac{c_1^2}{2(x^2+2s^2+2sR_c)^2}-\frac{\nu c_1}{(2s^2+2sR_c)^{3/2}}\left[\frac{x/\sqrt{2s^2+2sR_c}}{\left(x/\sqrt{2s^2+2sR_c}\right)^2+1}+\tan^{-1}\left(x/\sqrt{2s^2+2sR_c}\right)\right]\tag{8}$$

where c_2 is an integration constant. Since at $x = -\infty$, pressure is equal to the ambient pressure, c_2 can be expressed in terms of c_1:

$$c_2=\frac{P_\infty}{\rho}-\frac{c_1\nu\pi}{2(2s^2+2sR_c)^{3/2}}$$

The details of the integration to obtain this pressure equation can be found in the A8 section of the Appendix. Nevertheless, the final expressions for c_1 and c_2 are going to be derived in the next section by use of velocity boundary conditions.

3.2. Case 2 - Mid Vertical Plane

This is the plane that passes from the throat between the two counter-rotating cylinders. It is vertical, and the laminar flow across it is only normal to it, as the flow is stipulated to be in the η-direction only. That is:

$$V_\xi=0\quad\frac{\partial V_\xi}{\partial\xi}=0\quad\frac{\partial V_\xi}{\partial\eta}=0\quad\frac{\partial^2 V_\xi}{\partial\xi^2}=0\quad\frac{\partial^2 V_\xi}{\partial\eta^2}=0$$

Furthermore, by virtue of covering the shortest distance between the two cylinders, this plane has the maximum velocity along any streamline. So:

$$\frac{\partial V_\eta}{\partial\eta}=0$$

For the Navier-Stokes equations, one can import from the Appendix (A4.2, A6.2 and A7.2) the relevant forms of the operators, and get:

$$\left(-\frac{|\eta|}{\eta}\frac{1+2s\xi}{2s^2+2sR_c-2sy}V_\xi\frac{\partial}{\partial\xi}+V_\eta\frac{\partial}{\partial\eta}\right)\left(V_\xi\hat{e}_\xi+V_\eta\hat{e}_\eta\right)=$$

$$-\frac{1}{\rho}\left(-\frac{|\eta|}{\eta}\frac{1+2s\xi}{2s^2+2sR_c-2sy}\frac{\partial P}{\partial\xi}\hat{e}_\xi+\frac{\partial P}{\partial\eta}\hat{e}_\eta\right)+\nu\left[\left(\frac{1+2s\xi}{2s^2+2sR_c-2sy}\right)^2\frac{\partial^2}{\partial\xi^2}+\frac{\partial^2}{\partial\eta^2}\right]\left(V_\xi\hat{e}_\xi+V_\eta\hat{e}_\eta\right)$$

Applying the simplifications for case 2 and working out the derivatives, the ξ and η components of the momentum equation become, respectively:

$$-2\xi V_\eta^2=\frac{1}{\rho}\left(\frac{(1+2s\xi)^2}{2s^2+2sR_c}\right)\frac{\partial P}{\partial\xi}$$

$$\frac{1}{\rho}\frac{\partial P}{\partial\eta}=\nu\left[-4\xi^2V_\eta+\left(\frac{(1+2s\xi)^2}{2s^2+2sR_c}\right)^2\frac{\partial^2V_\eta}{\partial\xi^2}+\frac{\partial^2V_\eta}{\partial\eta^2}\right]$$

Looking at the above equations, once the V_η is found from the second one, it can be inserted into the first one, yielding an expression for the change of pressure along the throat. Then, focusing on the second equation, it can be assumed that, other than the locations in the vicinity of the cylinders, values of ξ are small and the behavior of the flow is close to that for case 1. So, as an approximation, the η gradient of pressure is going to be imported from case 1with x=0, and changes in the η direction are going to be neglected in the face of changes in the ξ direction. Then:

$$-\frac{2\nu c_1}{(2s^2+2sR_c)^2}=\nu\left[\left(\frac{(1+2s\xi)^2}{2s^2+2sR_c}\right)^2\frac{\partial^2V_\eta}{\partial\xi^2}\right]$$

Simplifying and integrating:

$$V_{\eta,case2}=-\frac{c_1}{12s^2(1+2s\xi)^2}+c_3\xi+c_4 \tag{10}$$

This equation must give the cylinder's tangential speed when $R=R_c$, i.e. $V_{\eta,case2}=wR_c$ for $\xi=1/2R_c$, and must be equal to the estimate of the velocity for case 1 at y=0, i.e. $V_{\eta,case2}=V_{\eta,case1}$ for ξ=0. Furthermore, the ξ derivative of velocity must be zero at y=0, i.e. $\frac{\partial V_{\eta,case2}}{\partial\xi}=0$ for ξ=0. Accordingly, c_1, c_3 and c_4 are found as:

$$c_1=\frac{12swR_c^2(s+R_c)^2}{6R_c^2+3sR_c-2s^2}$$

$$c_3=\frac{-c_1}{3s}=\frac{-4wR_c^2(s+R_c)^2}{6R_c^2+3sR_c-2s^2}$$

$$c_4=\frac{c_1(7s+R_c)}{12s^2(s+R_c)}=\frac{wR_c^2\left(R_c^2+8sR_c+7s^2\right)}{s\left(6R_c^2+3sR_c-2s^2\right)}$$

Furthermore, using the relationship between y and ξ for x=0, one can formulate the same velocity in terms of y coordinate:

$$\xi_{x=0}=\frac{y}{2s(s+R_c-y)}$$

$$V_{\eta,case2}=-\frac{c_1(s+R_c-y)^2}{12s^2(s+R_c)^2}+c_3\frac{y}{2s(s+R_c-y)}+c_4 \tag{11}$$

Having found an expression for the velocity, pressure distribution in the throat can also be worked out. However, since the process is lengthy, the involved integrals and their results are presented in the A9 section of the Appendix.

Aside from this, knowledge of c_1 can also be used to finalize the formula for V_η for case 1 and to find c_2:

$$V_{\eta,case1} = \frac{12swR_c^2(s+R_c)^2}{(6R_c^2+3sR_c-2s^2)(x^2+2s^2+2sR_c)} \tag{12}$$

$$c_2 = \frac{P_\infty}{\rho} - \frac{3wR_c^2\nu\pi\sqrt{s+R_c}}{(6R_c^2+3sR_c-2s^2)\sqrt{2s}} \tag{13}$$

4. RESULTS & DISCUSSION

In this section, first, non-dimensionalization of the derived formulas is done, in order to show the relative importance of the terms. Next, the validation of these formulae is presented.

4.1. Non-Dimensionalization of the Formulas

In the theory section, formulae for the following have been developed: velocity and pressure variation along the mid horizontal plane, velocity and pressure variation along the mid vertical plane. As was done in other studies on a cylinder pair, a non-dimensional gap spacing can be defined as $s^*=s/R_c$, and these formulae can be rewritten accordingly:

$$\frac{V_{\eta,case1}}{wR_c} = \frac{c_1^*}{(x^{*2}+2s^*+2s^{*2})} \tag{14}$$

$$\frac{V_{\eta,case2}}{wR_c} = c_4^* + c_3^*\frac{y^*}{2s^*(s^*+1-y^*)} - c_1^*\frac{(s^*+1-y^*)^2}{12s^{*2}(s^*+1)^2} \tag{15}$$

where

$$x^*=x/R_c \qquad y^*=y/R_c \qquad c_1^* = c_1/wR_c^3 \qquad c_3^* = c_3/wR_c^2 \qquad c_4^* = c_4/wR_C$$

Note that both of these non-dimensional velocities are only function of the non-dimensional gap spacing, s^*, and the local coordinate, x^* or y^*. Also note that the actual velocities along the mid horizontal plane or mid vertical plane have a linear proportionality with the rotational speed of the cylinders. This means linear acceleration of the jet velocity with the rotational speed, as also confirmed by Ref.'s 12 and 13.

Studying the denominators of the formulae for the horizontal velocity for cases 1 and 2, it can be calculated that they have a singularity at around $s^*=2.64$. During the derivations, it was stated in section 3.1 that the velocity conditions were not valid for large values of s^*. This singularity is a manifestation of this limitation.

On top of the velocities, the pressure expressions for both cases can be non-dimensionalized. However, since the expression for the case 2 is extremely long, it is left to section A9. Below, only the formula for case 1 is going to be presented.

$$P^*_{case1} = -\frac{c_1^{*2}}{2\left(x^{*2}+d^{*2}\right)^2} - \frac{c_1^*}{Re\, d^{*3}}\left[\frac{x^*/_{d^*}}{\left(x^*/_{d^*}\right)^2+1} + \tan^{-1}\left(x^*/_{d^*}\right)\right] + \frac{\pi c_1^*}{2Re\, d^{*3}}$$

where $P^* = \frac{P-P_\infty}{\rho(wR_c)^2}$, $d^* = \sqrt{2s^{*2}+2s^*}$ and $Re = \frac{wR_c^2}{\nu}$. It is clear that with increasing Reynolds numbers, the last two terms become negligible. In fact, for most practical applications, this is the case, and the pressure variation for case 1, i.e. along the mid horizontal plane, can be approximated by the first term only.

4.2. Validation

Validation of the current formulae is done using two different methods. First, the rotating cylinder formulation in the potential flow theory is used with the assumption that the rotational speed is extremely larger than unity. The reason for this choice of the rotational value is that in the current study, the upstream flow is assumed to be stagnant, and this can be approximated in the mentioned potential flow case by stipulating a very high ratio of rotational speed to the uniform flow. Accordingly, one can simplify the pressure distribution according to the potential flow along the cylinder surface as follows:

$$P_{surf,pot.flow} = P_\infty + \frac{\rho U^2}{2}\left[1 - 4\sin^2\theta + \frac{2\Gamma}{\pi U R_c}\sin\theta - \left(\frac{\Gamma}{2\pi U R_c}\right)^2\right] \approx P_\infty - \left(\frac{\Gamma}{2\pi U R_c}\right)^2$$

Here, the circulation, Γ, can be taken as $2\pi R_c(wR_c)$. Then:

$$P_{surf,pot.flow} \approx P_\infty - \rho\frac{(wR_c)^2}{2}$$

So, the $P*$ becomes identically -0.5 along the cylinder surface according to the potential flow theory.

Returning to the current study, for very small gap distances between the cylinders, the flow must have the same velocity as the cylinder surfaces, and the pressure must be equal to that at the cylinder surface. So, to model a similar situation, very small values of gap is assumed in the formulas developed in the current study for case 1, and the pressure is plotted. As seen in Fig. 4, the non-dimensional pressure value becomes -0.5 at the throat of the gap, exactly the same as that predicted by the potential flow theory.

The second validation is done by comparison to the simulation results from an in-house flow simulation program that solves for the vorticity field around the rotating cylinder at low Reynolds number. For the comparison between the current formulation and this flow solver, vorticity distribution at the throat, i.e. at the shortest gap between the cylinders, is used. Derivation of the vorticity formula for case 2 is given in A10 section of the Appendix.

Three different gap spacings, s*, are used for the vorticity comparison: 0.1, 0.2 and 0.4. The results from the current formulation are shown in Fig. 5 and the results from the in-house flow solver are shown in Fig. 6. By comparing these two figures, it is seen that as the gap becomes narrower, the two calculations merge. For the largest gap, the difference is mostly felt in the vicinity of the cylinder surface.

By looking at the non-dimensional vorticity expression in section A10, it is clear that it is not affected by the rotation speed, but is only a function of the local coordinate and the non-dimensional gap space. Although not shown here, ineffectiveness of the change of rotational speed on the non-dimensional vorticity distribution is also seen in the simulations.

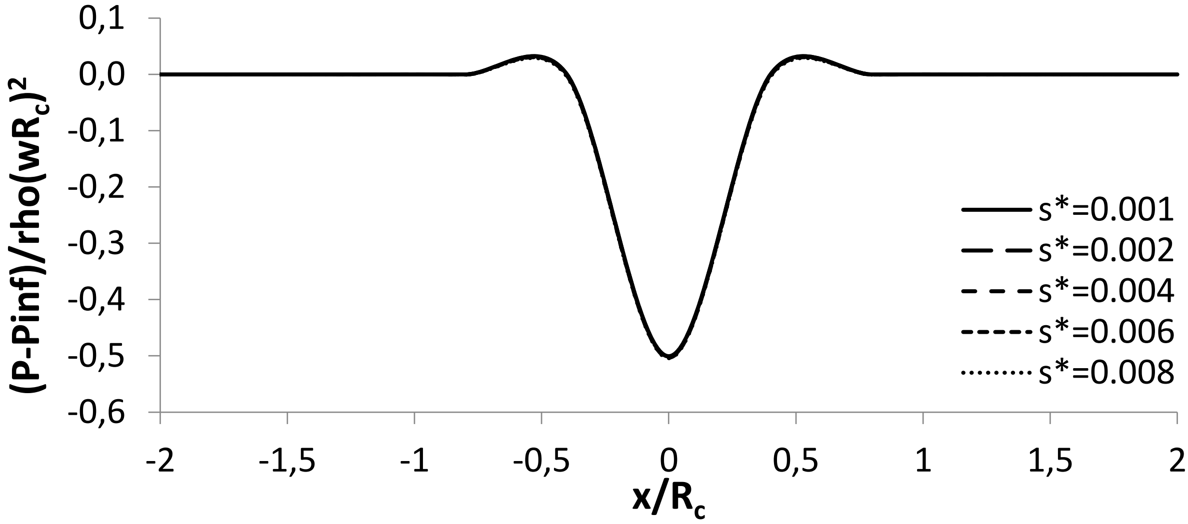

Figure 4. Comparison of pressure variation for case 1.

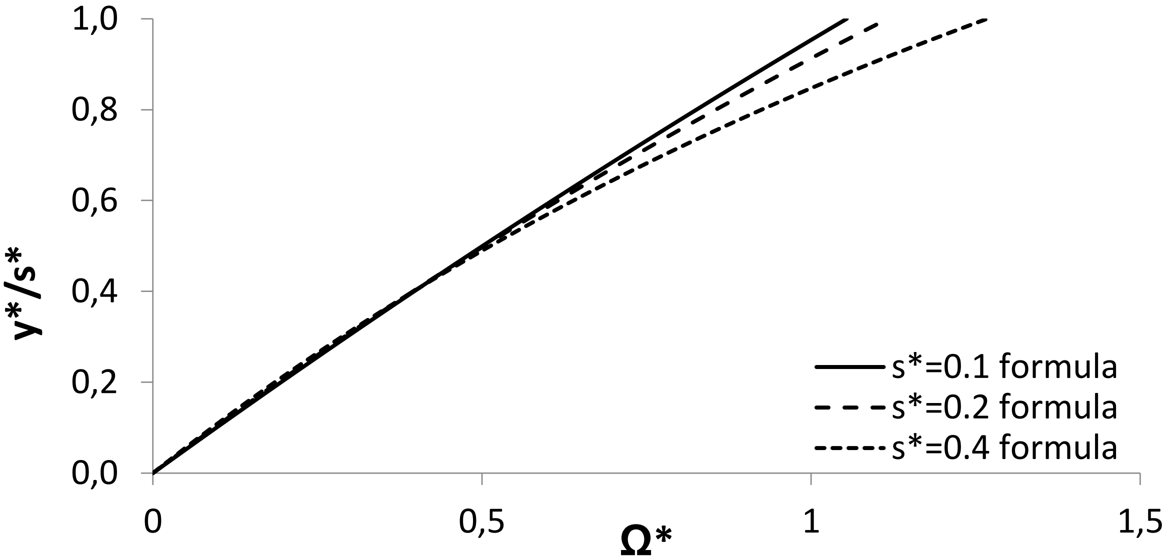

Figure 5. Vorticity distribution at the gap, case2; calculated using the current formulation.

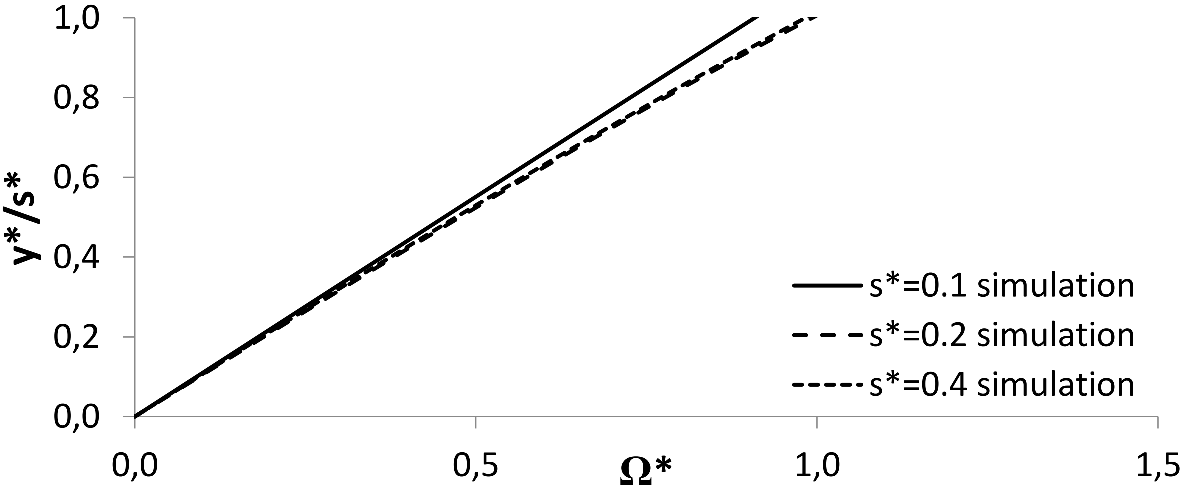

Figure 6. Vorticity distribution at the gap, case2; calculated using the in-house flow simulator.

Overall conclusion from the comparison of the flow solver and the current formulation is that it is valid for gap spacings smaller than 0.2Rc. Further validation by using more sophisticated flow solvers is necessary to see the validity of these formulas for spacings beyond 0.2Rc.

4.3. Flow Behavior

Velocity variation along the mid horizontal plane, i.e. case 1, is shown in Fig. 7. The increase of the gap between the cylinders leads to a higher velocity at the throat, hence more mass intake. Direct proportionality between the cross-sectional area and the velocity is contrary to the observations in the usual applications. Such counter-intuitive result is caused by the decrease of the blockage in front of the flow as cylinders move apart. This is similar to the initial opening of a valve; given a high enough pressure, as the valve opens, the maximum velocity also increases up to a certain value of the opening. Note that this relationship does not hold for all values of the spacing. After a maximum value, the interaction of the cylinders consistently weakens, and so, the relationship between the area and velocity become inversely proportional.

Looking at the values shown in Fig. 4, it is seen that the velocity at the throat surpasses that of the cylinder surface. A 10% excess is seen for s*=0.2, which is due to the accumulation of the mass upstream of the throat. It is also seen that the excess velocity increases with the increase of area.

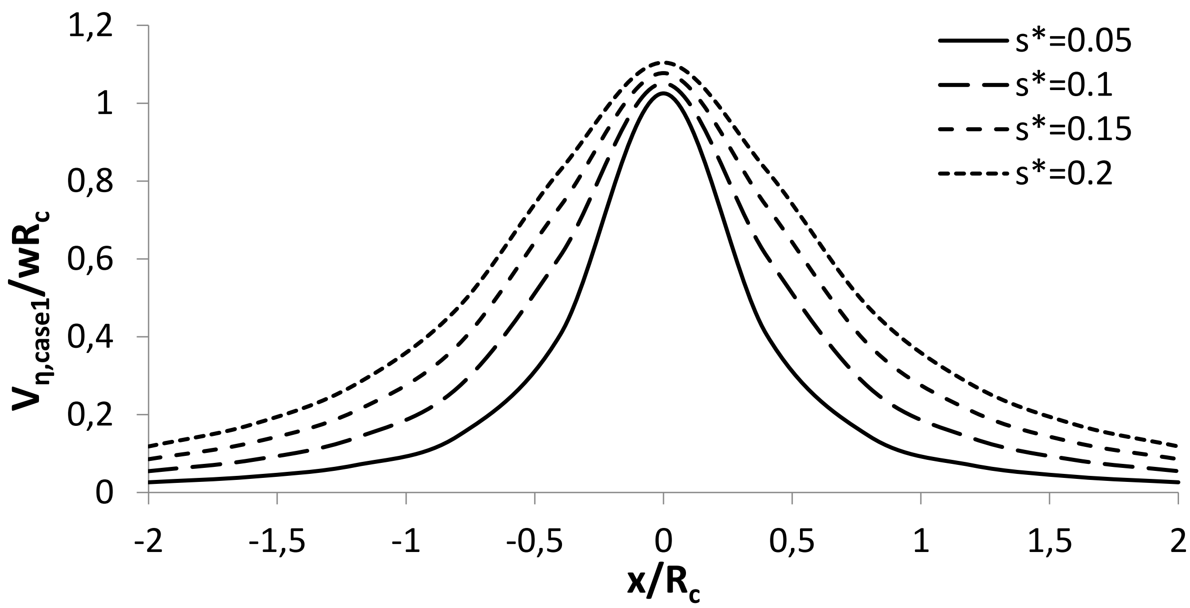

Figure 7. Variation of non-dimensional horizontal velocity in the mid horizontal plane at constant Re_Γ.

The observations stated above are also visible in the velocity distribution at the throat, Fig. 9. Increase of area leads to an excess velocity beyond the circumferential speed of the cylinders. The higher velocity and higher mass flow rate lead to increased thrust for applications looking at mobility. Increase of both the center velocity and the jet thrust due to the widening gap are also shown in a previous study (Figures 5 and 6 in Ref. 13). One implication of this result is the reversal of the overall thrust direction for the cylinder pair.

Besides these advantages, the excess velocity leads to excess drop in pressure, and so, can cause cavitation in certain applications, such as polymer extrusion. Such adverse situations can be predicted by using the formulation developed in the present study.

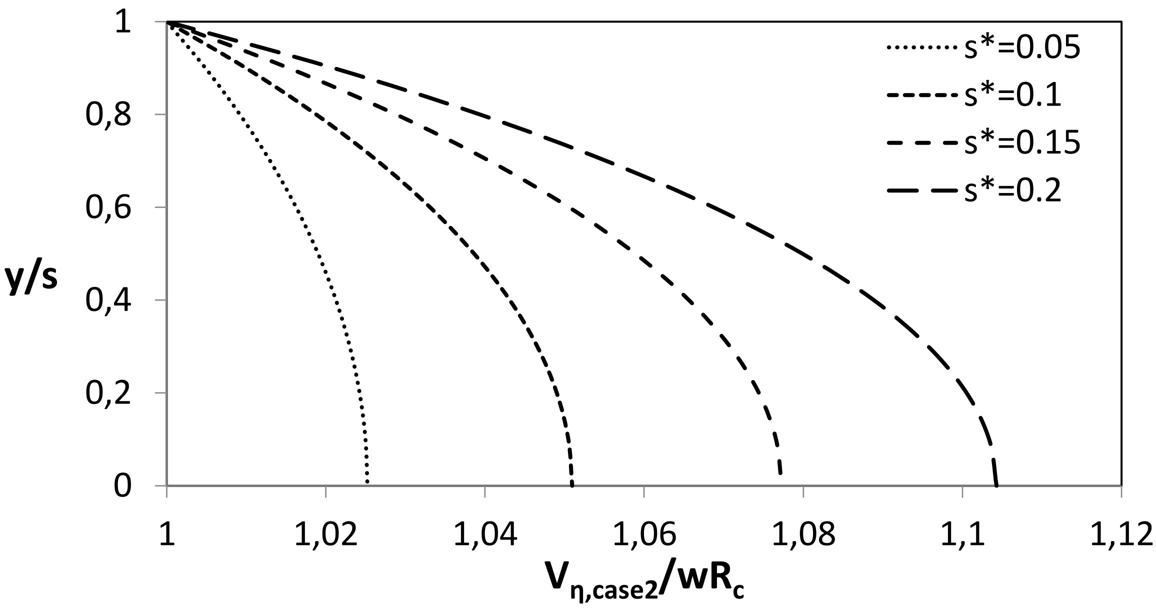

Figure 8. Variation of non-dimensional horizontal velocity in the mid vertical plane at constant Re_{Γ}.

5. CONCLUSIONS

In this study, a new coordinate system is developed for the analysis of the local flow between counter-rotating cylinder pair. Having expressed the vector operators in the new system, conservation of mass and Navier-Stokes equations are derived accordingly. By applying the geometrical and physical conditions of the locations in concern, i.e. the mid horizontal and the mid vertical planes, formulae for velocity and pressure are developed. Finally, the validity of these formulae is studied by comparing with the result from potential flow theory and from an in-house flow solver. Accordingly, for $s^* < 0.2$, these formulae are found to be useful approximations. Further validation is necessary to check the situation at larger gap spacing values. Unlike the previous studies that address the entire flow field but are limited to extremely low velocities, the formulae from the present study can be used for all velocities in the incompressible, laminar range. A key implication of the results is that the velocity in the mid vertical plane becomes faster than the tangential speed of the cylinder due to the acceleration of the fluid in the converging flow boundary. The increase of the distance between the cylinders leads to the further acceleration of this jet emanating from the gap between the cylinders, which implies an increase of the jet thrust, as also reported in the previous studies. The same effect can explain the undesired cavitation problem observed in the polymer extrusion processes, and the formulae developed in the present study can help with the improvements.

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APPENDIX

A1. Radius of Curvature

For a parabola defined by the general equation $y=ax^2+c$, the radius of curvature at any point is found as follows.

$$R=\frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{3/2}}{\left|\frac{d^2y}{dx^2}\right|}=\frac{[1+(2ax)^2]^{3/2}}{|2a|}$$

In the present study, the R value at the apex is of concern. More specifically, at (0,s) in the x-y plane, radius of curvature must match the radius of the cylinder. The arms of the parabolas spread from the cylinder surface towards to eventually form the x-axis, thereby covering the flow field, hence the need for a generic formula for the R at the apex. The radius of curvature at the apex of a parabola that is symmetric with respect to the y-axis is found by specifying x=0 in the above formula. Note that we are going to be solving the upper half of the flow field, where a>0.

$$R=\frac{1}{2a}$$

A2. New Parabolic Coordinates

For each point on the x-y plane, there exists a parabola that passes through this point and can be expressed as $y=ax^2+c$. In the present study, this parabola is selected such that a>0, and the coefficients a and c are related to each other through the following formulae:

$$a=\frac{1}{2R}$$

$$c=\frac{s(s+R_c)}{s+R}=\frac{2as(s+R_c)}{2as+1}$$

where R_c is the cylinder radius, s is half of the distance between the cylinder surfaces and R is the radius of curvature at the apex of the parabola that passes through the arbitrary point x-y. So, given a point and the design parameters R_c and s, one can easily calculate the parameters a, c and R.

The new parabolic coordinates (ξ-η) are defined through the following transformations:

$$\xi=a$$

$$\eta=\int_0^x\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx=\int_0^x\sqrt{1+(2ax)^2}\,dx$$

where η is the distance along the parabola from its apex to a generic point (x,y). Therefore, a point in the new parabolic coordinates is defined by two properties associated with the parabola that passes through it: one, the radius curvature of that parabola at its apex (not at that point!), and two, the distance along the parabola from its apex to that point.

The above integral for η can be solved by the transformation 2ax=tanθ and using the fact that a>0. Accordingly:

$$\eta = \frac{1}{2a}\int_0^\theta \sec^3\theta\, d\theta = \frac{1}{4a}\left\{\frac{sin\theta}{\cos^2\theta} + \ln\left|\frac{1+sin\theta}{cos\theta}\right|\right\}$$

$$\eta = \frac{1}{4a}\left\{2ax\sqrt{1+(2ax)^2} + \ln\left[\sqrt{1+(2ax)^2} + 2ax\right]\right\}$$

Note that the equation for the η yields positive values in the first quadrant and negative values in the second quadrant. Also note that the resultant parabolic coordinate system is a non-orthogonal system. That is, ξ-constant curves are not necessarily normal to the η-constant curves.

A3. Unit Vectors

Using the equation of the parabola, one can derive two derivatives of the parabola equation. First derivative is the one with respect to x while ξ is constant; and this derivative is going to give the tangent of the angle that is used for the unit vector in the η direction. Second derivative is the one with respect to x while η is constant, and this derivative is going to give the tangent of the angle that is used for the unit vector in the ξ direction. Be reminded that these two unit vectors in the parabolic coordinate system are not necessarily orthogonal, and so, knowing one does not help with finding the other.

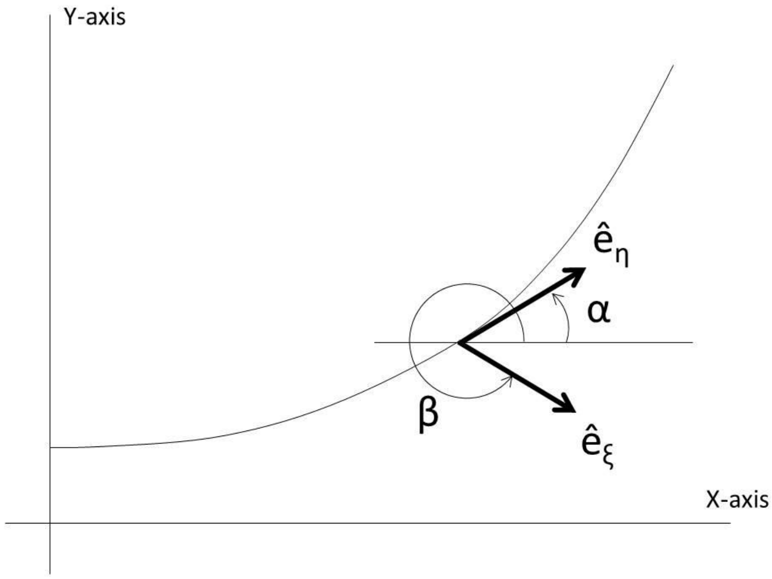

Figure A1. Representation of the unit vectors of the ξ-η system in terms of the x-y system.

According to the above description and to Fig. A1, the angle for the η direction can be found from:

$$\tan\alpha = \left(\frac{dy}{dx}\right)_{\xi=constant} = 2ax = \frac{x}{R}$$

So,

$$\alpha = \arctan(2ax)$$

Finding the angle for the ξ direction, however, is not as straightforward. The reason for this complication is the fact that both coefficients of the parabola equation, i.e. a and c, are functions of R, hence ξ.

$$\tan\beta = \left(\frac{dy}{dx}\right)_{\eta=constant} = \left(\frac{da}{dx}\right)_{\eta} x^2 + 2ax + \frac{dc}{da}\left(\frac{da}{dx}\right)_{\eta}$$

Using the definition of c and η, and setting dη=0, one can find:

$$\frac{dc}{da} = \frac{2s(s+R_c)}{(2as+1)^2}$$

$$\left(\frac{da}{dx}\right)_{\eta=constant} = \frac{4a^2\sqrt{1+(2ax)^2}}{\ln\left(\sqrt{1+(2ax)^2}+2ax\right) - 2ax\sqrt{1+(2ax)^2}}$$

So,

$$\beta = arctan\left\{2ax + \left[x^2 + \frac{2s(s+R_c)}{(2as+1)^2}\right]\left[\frac{4a^2\sqrt{1+(2ax)^2}}{\ln\left(\sqrt{1+(2ax)^2}+2ax\right) - 2ax\sqrt{1+(2ax)^2}}\right]\right\}$$

When inserting the value of β into the equation for the unit vector in the ξ direction, however, a minor correction is made in order to account for the sign changes for the portion of the parabola in the second quadrant (Fig. A2). That is, when the flow field in the first quadrant is considered, the β equation works as expected, but in the second quadrant, the values returned by the β equation need to be multiplied by -1.

$$\hat{e}_\eta = \cos\alpha\,\hat{\imath} + \sin\alpha\,\hat{\jmath}$$

$$\hat{e}_\xi = \frac{|\eta|}{\eta}(\cos\beta\,\hat{\imath} + \sin\beta\,\hat{\jmath})$$

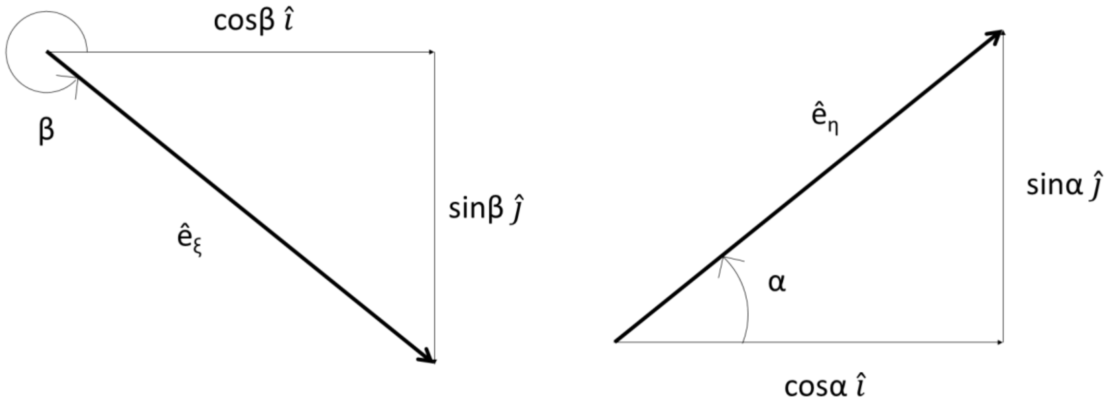

Figure A2. Components of the unit vectors of the ξ-η system in terms of the x-y system.

Using these equations, unit vectors in the x-y system can also be expressed in terms of the unit vectors in the ξ-η system.

$$\hat{\imath} = \frac{|\eta|}{\eta}\frac{\sin\alpha}{\sin(\alpha-\beta)}\hat{e}_\xi - \frac{\sin\beta}{\sin(\alpha-\beta)}\hat{e}_\eta$$

$$\hat{j} = -\frac{|\eta|}{\eta}\frac{\cos\alpha}{\sin(\alpha-\beta)}\hat{e}_\xi + \frac{\cos\beta}{\sin(\alpha-\beta)}\hat{e}_\eta$$

For the sake of saving space, these expressions are going to be shortened as follows where necessary.

$$\hat{\imath} = A\hat{e}_\xi + B\hat{e}_\eta \qquad\qquad \hat{j} = C\hat{e}_\xi + D\hat{e}_\eta$$

A4. Del Operator

The derivation of the del operator in the new parabolic coordinate system is simple in terms of its method, but is lengthy and complicated due to the non-orthogonal and curvilinear nature of the new system. Below, first the chain rule is employed to express the derivatives according to the ξ-η variables. Then, the unit vector definitions are imported, and the resultant expression is organized.

$$\nabla = \hat{\imath}\left(\frac{\partial}{\partial x}\right)_{y=constant} + \hat{j}\left(\frac{\partial}{\partial y}\right)_{x=constant} = \hat{\imath}\left(\frac{\partial\xi}{\partial x}\frac{\partial}{\partial\xi} + \frac{\partial\eta}{\partial x}\frac{\partial}{\partial\eta}\right)_y + \hat{j}\left(\frac{\partial\xi}{\partial y}\frac{\partial}{\partial\xi} + \frac{\partial\eta}{\partial y}\frac{\partial}{\partial\eta}\right)_x$$

Using the generic parabola formula and the definitions above:

$$y = ax^2 + c = \xi x^2 + \frac{2\xi s(s+R_c)}{2\xi s+1}$$

Differentiating each term with respect to x while keeping y constant, and reorganizing:

$$\frac{\partial\xi}{\partial x} = \frac{-2x\xi\,(1+2s\xi)}{4x^2\xi s + x^2 + 2s^2 + 2sR_c - 2sy}$$

Similarly, differentiating each term with respect to y while keeping x constant, and reorganizing:

$$\frac{\partial\xi}{\partial y} = \frac{1+2s\xi}{4x^2\xi s + x^2 + 2s^2 + 2sR_c - 2sy} = -\frac{1}{2x\xi}\frac{\partial\xi}{\partial x}$$

Also, utilizing the definition of η, and taking its derivative with respect to x, and reorganizing:

$$\frac{\partial\eta}{\partial x} = \frac{\partial\xi}{\partial x}\frac{\left(x\sqrt{1+(2ax)^2}-\eta\right)}{\xi} + \frac{2x^2\xi^2}{\sqrt{1+(2ax)^2}} + \sqrt{1+(2ax)^2}$$

Similarly, taking the y derivative of η, and reorganizing:

$$\frac{\partial\eta}{\partial y} = \frac{\partial\xi}{\partial y}\left(\frac{x\sqrt{1+(2ax)^2}}{\xi} - \frac{\eta}{\xi}\right)$$

Using the partial derivatives of ξ and η obtained above and the definitions of the unit vectors from the previous section, and reorganizing:

$$\nabla = \left[\left(\frac{|\eta|}{\eta}\frac{\sin\alpha}{\sin(\alpha-\beta)}\frac{\partial\xi}{\partial x} - \frac{|\eta|}{\eta}\frac{\cos\alpha}{\sin(\alpha-\beta)}\frac{\partial\xi}{\partial y}\right)\frac{\partial}{\partial\xi} + \left(\frac{|\eta|}{\eta}\frac{\sin\alpha}{\sin(\alpha-\beta)}\frac{\partial\eta}{\partial x} - \frac{|\eta|}{\eta}\frac{\cos\alpha}{\sin(\alpha-\beta)}\frac{\partial\eta}{\partial y}\right)\frac{\partial}{\partial\eta}\right]\hat{e}_\xi$$
$$+\left[\left(-\frac{\sin\beta}{\sin(\alpha-\beta)}\frac{\partial\xi}{\partial x} + \frac{\cos\beta}{\sin(\alpha-\beta)}\frac{\partial\xi}{\partial y}\right)\frac{\partial}{\partial\xi} + \left(-\frac{\sin\beta}{\sin(\alpha-\beta)}\frac{\partial\eta}{\partial x} + \frac{\cos\beta}{\sin(\alpha-\beta)}\frac{\partial\eta}{\partial y}\right)\frac{\partial}{\partial\eta}\right]\hat{e}_\eta$$

For the sake of saving space, this expression is going to be abbreviated as follows where necessary.

$$\nabla = \left[(\quad)_1 \frac{\partial}{\partial \xi} + (\quad)_2 \frac{\partial}{\partial \eta}\right] \hat{e}_\xi + \left[(\quad)_3 \frac{\partial}{\partial \xi} + (\quad)_4 \frac{\partial}{\partial \eta}\right] \hat{e}_\eta$$

In order to develop the necessary formulae for the flow field, two special cases are going to be analyzed.

A4.1. Case 1 - mid horizontal plane

The geometrical conditions for this case are as follows:

$$\xi = 0 \quad y = 0 \quad \alpha = 0 \quad \beta = -\frac{\pi}{2}$$

$$\eta = x \quad \frac{\partial \xi}{\partial x} = 0 \quad \frac{\partial \eta}{\partial y} = 0 \quad \frac{\partial \beta}{\partial \eta} = 0 \quad \frac{\partial \alpha}{\partial \eta} = 0$$

If these conditions are imposed on the del operator, the corresponding simplified del operator is found as:

$$(\quad)_1 = \left(\frac{|\eta|}{\eta} \frac{\sin\alpha}{\sin(\alpha-\beta)} \frac{\partial \xi}{\partial x} - \frac{|\eta|}{\eta} \frac{\cos\alpha}{\sin(\alpha-\beta)} \frac{\partial \xi}{\partial y}\right) = -\frac{|\eta|}{\eta} \frac{\partial \xi}{\partial y}$$

$$(\quad)_2 = \left(\frac{|\eta|}{\eta} \frac{\sin\alpha}{\sin(\alpha-\beta)} \frac{\partial \eta}{\partial x} - \frac{|\eta|}{\eta} \frac{\cos\alpha}{\sin(\alpha-\beta)} \frac{\partial \eta}{\partial y}\right) = 0$$

$$(\quad)_3 = \left(-\frac{\sin\beta}{\sin(\alpha-\beta)} \frac{\partial \xi}{\partial x} + \frac{\cos\beta}{\sin(\alpha-\beta)} \frac{\partial \xi}{\partial y}\right) = 0$$

$$(\quad)_4 = \left(-\frac{\sin\beta}{\sin(\alpha-\beta)} \frac{\partial \eta}{\partial x} + \frac{\cos\beta}{\sin(\alpha-\beta)} \frac{\partial \eta}{\partial y}\right) = 1$$

$$\frac{\partial \xi}{\partial y} = \frac{1}{x^2 + 2s^2 + 2sR_c}$$

$$\nabla = -\frac{|\eta|}{\eta} \frac{1}{x^2 + 2s^2 + 2sR_c} \frac{\partial}{\partial \xi} \hat{e}_\xi + \frac{\partial}{\partial \eta} \hat{e}_\eta$$

A4.2. Case 2 - mid vertical plane

The geometrical conditions for this case are as follows:

$$\eta = 0 \quad x = 0 \quad \alpha = 0 \quad \beta = -\frac{\pi}{2}$$

$$\frac{\partial \eta}{\partial x} = 1 \quad \frac{\partial \eta}{\partial y} = 0 \quad \frac{\partial \beta}{\partial \xi} = 0 \quad \frac{\partial \beta}{\partial \eta} = 0 \quad \frac{\partial \alpha}{\partial \xi} = 0 \quad \frac{\partial \alpha}{\partial \eta} = 2\xi \quad \frac{\partial \xi}{\partial x} = 0$$

If these conditions are imposed on the del operator, the corresponding simplified del operator is found as:

$$(\quad)_1 = \left(\frac{|\eta|}{\eta} \frac{\sin\alpha}{\sin(\alpha-\beta)} \frac{\partial \xi}{\partial x} - \frac{|\eta|}{\eta} \frac{\cos\alpha}{\sin(\alpha-\beta)} \frac{\partial \xi}{\partial y}\right) = -\frac{|\eta|}{\eta} \frac{\partial \xi}{\partial y}$$

$$(\quad)_2 = \left(\frac{|\eta|}{\eta}\frac{\sin\alpha}{\sin(\alpha-\beta)}\frac{\partial\eta}{\partial x} - \frac{|\eta|}{\eta}\frac{\cos\alpha}{\sin(\alpha-\beta)}\frac{\partial\eta}{\partial y}\right) = 0$$

$$(\quad)_3 = \left(-\frac{\sin\beta}{\sin(\alpha-\beta)}\frac{\partial\xi}{\partial x} + \frac{\cos\beta}{\sin(\alpha-\beta)}\frac{\partial\xi}{\partial y}\right) = 0$$

$$(\quad)_4 = \left(-\frac{\sin\beta}{\sin(\alpha-\beta)}\frac{\partial\eta}{\partial x} + \frac{\cos\beta}{\sin(\alpha-\beta)}\frac{\partial\eta}{\partial y}\right) = 1$$

$$\frac{\partial\xi}{\partial y} = \frac{1+2s\xi}{2s^2+2sR_c-2sy}$$

$$\nabla = -\frac{|\eta|}{\eta}\frac{1+2s\xi}{2s^2+2sR_c-2sy}\frac{\partial}{\partial\xi}\hat{e}_\xi + \frac{\partial}{\partial\eta}\hat{e}_\eta$$

A5. Divergence Operator

Divergence operator is the del operator acting on a vector. Although this operator seems straightforward in the conventional Cartesian coordinate system, it is not as easy in the non-orthogonal, curvilinear system of the present study. Unlike in the Cartesian coordinate system, when the derivative terms operate on the unit vectors, the result is non-zero, since the unit vectors change direction from point to point. So, in order to develop the expression for the divergence operator in the new parabolic coordinates, first the derivatives of the unit vectors are needed. In obtaining these derivatives, the basic definitions of the parabolic unit vectors are going to be used along with the fact that the unit vectors of the Cartesian system remain constant.

$$\frac{\partial\hat{e}_\xi}{\partial\xi} = \frac{\partial}{\partial\xi}\left[\frac{|\eta|}{\eta}(\cos\beta\,\hat{\imath} + \sin\beta\,\hat{\jmath})\right] = \frac{|\eta|}{\eta}\left[\left(A\frac{\partial\cos\beta}{\partial\xi} + C\frac{\partial\sin\beta}{\partial\xi}\right)\hat{e}_\xi + \left(B\frac{\partial\cos\beta}{\partial\xi} + D\frac{\partial\sin\beta}{\partial\xi}\right)\hat{e}_\eta\right]$$

$$\frac{\partial\hat{e}_\xi}{\partial\eta} = \frac{\partial}{\partial\eta}\left[\frac{|\eta|}{\eta}(\cos\beta\,\hat{\imath} + \sin\beta\,\hat{\jmath})\right] = \frac{|\eta|}{\eta}\left[\left(A\frac{\partial\cos\beta}{\partial\eta} + C\frac{\partial\sin\beta}{\partial\eta}\right)\hat{e}_\xi + \left(B\frac{\partial\cos\beta}{\partial\eta} + D\frac{\partial\sin\beta}{\partial\eta}\right)\hat{e}_\eta\right]$$

$$\frac{\partial\hat{e}_\eta}{\partial\xi} = \frac{\partial}{\partial\xi}[\cos\alpha\,\hat{\imath} + \sin\alpha\,\hat{\jmath}] = \left(A\frac{\partial\cos\alpha}{\partial\xi} + C\frac{\partial\sin\alpha}{\partial\xi}\right)\hat{e}_\xi + \left(B\frac{\partial\cos\alpha}{\partial\xi} + D\frac{\partial\sin\alpha}{\partial\xi}\right)\hat{e}_\eta$$

$$\frac{\partial\hat{e}_\eta}{\partial\eta} = \frac{\partial}{\partial\eta}[\cos\alpha\,\hat{\imath} + \sin\alpha\,\hat{\jmath}] = \left(A\frac{\partial\cos\alpha}{\partial\eta} + C\frac{\partial\sin\alpha}{\partial\eta}\right)\hat{e}_\xi + \left(B\frac{\partial\cos\alpha}{\partial\eta} + D\frac{\partial\sin\alpha}{\partial\eta}\right)\hat{e}_\eta$$

The letters A, B, C, D in the above expressions are the terms that were defined before in the Unit Vectors section. Another item that is essential for the current effort is the dot product of the unit vectors with each other. In order to obtain this expression, one can utilize the definition of the parabolic unit vectors in terms of the Cartesian unit vectors. Accordingly:

$$\hat{e}_\xi.\hat{e}_\xi = 1 \quad \hat{e}_\eta.\hat{e}_\eta = 1 \quad \hat{e}_\xi.\hat{e}_\eta = \frac{|\eta|}{\eta}\cos(\alpha-\beta)$$

Having obtained the derivatives of the unit vectors and their dot products with each other, now it is time to start the derivation of the divergence operator in the parabolic coordinate system.

$$\nabla.\vec{V} = \nabla.\left(V_\xi\hat{e}_\xi + V_\eta\hat{e}_\eta\right) = \left\{\left[(\quad)_1\frac{\partial}{\partial\xi} + (\quad)_2\frac{\partial}{\partial\eta}\right]\hat{e}_\xi + \left[(\quad)_3\frac{\partial}{\partial\xi} + (\quad)_4\frac{\partial}{\partial\eta}\right]\hat{e}_\eta\right\}\left(V_\xi\hat{e}_\xi + V_\eta\hat{e}_\eta\right)$$

Expanding the terms:

$$\nabla.\vec{V} = (\quad)_1 \frac{\partial V_\xi}{\partial \xi} + (\quad)_2 \frac{\partial V_\xi}{\partial \eta} + (\quad)_3 \frac{\partial V_\eta}{\partial \xi} + (\quad)_4 \frac{\partial V_\eta}{\partial \eta} +$$

$$\frac{|\eta|}{\eta}\cos(\alpha-\beta)\left[(\quad)_1 \frac{\partial V_\eta}{\partial \xi} + (\quad)_2 \frac{\partial V_\eta}{\partial \eta} + (\quad)_3 \frac{\partial V_\xi}{\partial \xi} + (\quad)_4 \frac{\partial V_\xi}{\partial \eta}\right] +$$

$$(\quad)_1 V_\xi \frac{\partial \hat{e}_\xi}{\partial \xi}.\hat{e}_\xi + (\quad)_1 V_\eta \frac{\partial \hat{e}_\eta}{\partial \xi}.\hat{e}_\xi + (\quad)_2 V_\xi \frac{\partial \hat{e}_\xi}{\partial \eta}.\hat{e}_\xi + (\quad)_2 V_\eta \frac{\partial \hat{e}_\eta}{\partial \eta}.\hat{e}_\xi +$$

$$(\quad)_3 V_\xi \frac{\partial \hat{e}_\xi}{\partial \xi}.\hat{e}_\eta + (\quad)_3 V_\eta \frac{\partial \hat{e}_\eta}{\partial \xi}.\hat{e}_\eta + (\quad)_4 V_\xi \frac{\partial \hat{e}_\xi}{\partial \eta}.\hat{e}_\eta + (\quad)_4 V_\eta \frac{\partial \hat{e}_\eta}{\partial \eta}.\hat{e}_\eta$$

At this point, the following trigonometric identities are going to be useful:

$$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \sin\beta\cos\alpha$$

$$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$$

$$\sin\alpha + \sin\beta = 2\sin\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right)$$

$$\cos\alpha + \cos\beta = 2\cos\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right)$$

$$\cos\alpha - \cos\beta = -2\sin\left(\frac{\alpha+\beta}{2}\right)\sin\left(\frac{\alpha-\beta}{2}\right)$$

$$\sin\alpha\cos\alpha + \sin\beta\cos\beta = \sin(\alpha+\beta)\cos(\alpha-\beta)$$

$$\sin^2\alpha + \sin^2\beta = 1 - \cos(\alpha+\beta)\cos(\alpha-\beta)$$

$$\cos^2\alpha + cos^2\beta = 1 + \cos(\alpha+\beta)\cos(\alpha-\beta)$$

$$1 - \cos(\alpha+\beta)\cos(\alpha-\beta) - 2\sin\alpha\sin\beta\cos(\alpha-\beta) = \sin^2(\alpha-\beta)$$

$$1 + \cos(\alpha+\beta)\cos(\alpha-\beta) - 2\cos\alpha\cos\beta\cos(\alpha-\beta) = \sin^2(\alpha-\beta)$$

By further expanding the terms in the del operator expression, importing the content of the abbreviated parentheses, utilizing the above trigonometric identities and reorganizing:

$$\nabla.\vec{V} = (\quad)_1 \frac{\partial V_\xi}{\partial \xi} + (\quad)_2 \frac{\partial V_\xi}{\partial \eta} + (\quad)_3 \frac{\partial V_\eta}{\partial \xi} + (\quad)_4 \frac{\partial V_\eta}{\partial \eta} +$$

$$\frac{|\eta|}{\eta}\cos(\alpha-\beta)\left[(\quad)_1 \frac{\partial V_\eta}{\partial \xi} + (\quad)_2 \frac{\partial V_\eta}{\partial \eta} + (\quad)_3 \frac{\partial V_\xi}{\partial \xi} + (\quad)_4 \frac{\partial V_\xi}{\partial \eta}\right] +$$

$$\frac{|\eta|}{\eta}\frac{V_\xi}{\sin^2(\alpha-\beta)}\left[\left(\frac{\partial\cos\beta}{\partial\xi}\frac{\partial\xi}{\partial x} + \frac{\partial\cos\beta}{\partial\eta}\frac{\partial\eta}{\partial x}\right)(1-\cos(\alpha+\beta)\cos(\alpha-\beta))\right] +$$

$$\frac{|\eta|}{\eta}V_\xi\left(\frac{\partial\sin\beta}{\partial\xi}\frac{\partial\xi}{\partial y} + \frac{\partial\sin\beta}{\partial\eta}\frac{\partial\eta}{\partial y}\right) + V_\eta\left[\frac{\partial\cos\alpha}{\partial\xi}\frac{\partial\xi}{\partial x} + \frac{\partial\cos\alpha}{\partial\eta}\frac{\partial\eta}{\partial x} + \frac{\partial\sin\alpha}{\partial\xi}\frac{\partial\xi}{\partial y} + \frac{\partial\sin\alpha}{\partial\eta}\frac{\partial\eta}{\partial y}\right]$$

In the present study, two simplified versions of the divergence operator are going to be derived. Each of these versions is simplified according to the geometrical conditions that are present in the local flow field between the counter rotating cylinder pair.

A5.1. Case 1 - mid horizontal plane

If the relevant conditions (see section A4.1) are imposed on the divergence operator, the corresponding simplified divergence operator becomes:

$$\nabla.\vec{V} = (\quad)_1 \frac{\partial V_\xi}{\partial \xi} + (\quad)_2 \frac{\partial V_\xi}{\partial \eta} + (\quad)_3 \frac{\partial V_\eta}{\partial \xi} + (\quad)_4 \frac{\partial V_\eta}{\partial \eta} + V_\eta \left[\frac{\partial \sin\alpha}{\partial \xi}\frac{\partial \xi}{\partial y}\right]$$

One can import the simplified versions of the abbreviated parentheses, the $\partial\xi/\partial y$ term (see section A4.1) and also work on the derivative of the sine term:

$$\frac{\partial \sin\alpha}{\partial \xi} = \cos\alpha \frac{\partial \alpha}{\partial \xi} = \frac{2x}{\sqrt{1+(2\xi x)^2}} = 2x$$

Then, the final form of the divergence operator in the special case 1, i.e. mid horizontal plane, becomes:

$$\nabla.\vec{V} = -\frac{|\eta|}{\eta}\frac{1}{x^2+2s^2+2sR_c}\frac{\partial V_\xi}{\partial \xi} + \frac{\partial V_\eta}{\partial \eta} + \frac{2x}{(x^2+2s^2+2sR_c)}V_\eta$$

A5.2. Case 2 - mid vertical plane

If the relevant conditions (see section A4.2) are imposed on the divergence operator, the corresponding simplified divergence operator becomes:

$$\nabla.\vec{V} = (\quad)_1 \frac{\partial V_\xi}{\partial \xi} + (\quad)_2 \frac{\partial V_\xi}{\partial \eta} + (\quad)_3 \frac{\partial V_\eta}{\partial \xi} + (\quad)_4 \frac{\partial V_\eta}{\partial \eta} + \frac{|\eta|}{\eta}V_\xi \frac{\partial \cos\beta}{\partial \eta}$$

One can import the simplified versions of the abbreviated parentheses, the $\partial\xi/\partial y$ term (see section A4.2) and also work on the derivative of the cosine term:

$$\frac{\partial \cos\beta}{\partial \eta} = 0$$

Then, the final form of the divergence operator in the special case 2, i.e. mid vertical plane, becomes:

$$\nabla.\vec{V} = -\frac{|\eta|}{\eta}\frac{1+2s\xi}{2s^2+2sR_c-2sy}\frac{\partial V_\xi}{\partial \xi} + \frac{\partial V_\eta}{\partial \eta}$$

A6. Laplacian Operator

Laplacian operator is a special case of the divergence operator, where the vector that is operated on is the del operator. The result from this operation is again an operator. Although this operator is simple in the Cartesian coordinates, the same is not true for the non-orthogonal and curvilinear parabolic coordinates.

$$\nabla.\nabla= \nabla^2= \nabla.\left\{\left[(\quad)_1\frac{\partial}{\partial\xi}+(\quad)_2\frac{\partial}{\partial\eta}\right]\hat{e}_\xi+\left[(\quad)_3\frac{\partial}{\partial\xi}+(\quad)_4\frac{\partial}{\partial\eta}\right]\hat{e}_\eta\right\}$$

In obtaining the Laplacian operator for the two special cases of the present study, the V_ξ and the V_η terms in the simplified versions of the divergence operator are going to be replaced by the ξ and η-components of the del operator, respectively.

A6.1. Case 1 - mid horizontal plane

According to the above description:

$$\nabla^2= -\frac{|\eta|}{\eta}\frac{1}{x^2+2s^2+2sR_c}\frac{\partial}{\partial\xi}\left[(\quad)_1\frac{\partial}{\partial\xi}+(\quad)_2\frac{\partial}{\partial\eta}\right]+$$
$$\frac{\partial}{\partial\eta}\left[(\quad)_3\frac{\partial}{\partial\xi}+(\quad)_4\frac{\partial}{\partial\eta}\right]+\frac{2x}{(x^2+2s^2+2sR_c)}\left[(\quad)_3\frac{\partial}{\partial\xi}+(\quad)_4\frac{\partial}{\partial\eta}\right]$$

Expanding the derivatives and using the content of the parenthesis for this simplified case:

$$\nabla^2= -\frac{|\eta|}{\eta}\frac{1}{x^2+2s^2+2sR_c}\left[\frac{\partial(\quad)_1}{\partial\xi}\frac{\partial}{\partial\xi}-\frac{|\eta|}{\eta}\frac{1}{x^2+2s^2+2sR_c}\frac{\partial^2}{\partial\xi^2}+\frac{\partial(\quad)_2}{\partial\xi}\frac{\partial}{\partial\eta}\right]+$$
$$\left[\frac{\partial(\quad)_3}{\partial\eta}\frac{\partial}{\partial\xi}+\frac{\partial(\quad)_4}{\partial\eta}\frac{\partial}{\partial\eta}+\frac{\partial^2}{\partial\eta^2}\right]+\frac{2x}{(x^2+2s^2+2sR_c)}\frac{\partial}{\partial\eta}$$

At this point, there is a need for simplifying the derivatives of the terms in the parentheses. For this, aside from those listed before for the special case 1, the following geometrical conditions are going to be useful, which are derived from the fact that for case 1, x=η and y=ξ.

$$\frac{\partial^2\xi}{\partial y\partial\xi}=0\qquad\frac{\partial^2\xi}{\partial x\partial\xi}=0\qquad\frac{\partial^2\xi}{\partial x\partial\eta}=0\qquad\frac{\partial^2\xi}{\partial y\partial\eta}=0$$
$$\frac{\partial^2\eta}{\partial x\partial\xi}=0\qquad\frac{\partial^2\eta}{\partial y\partial\xi}=0\qquad\frac{\partial^2\eta}{\partial x\partial\eta}=0\qquad\frac{\partial^2\eta}{\partial y\partial\eta}=0$$

Then,

$$\frac{\partial(\quad)_1}{\partial\xi}=\frac{\partial}{\partial\xi}\left(\frac{|\eta|}{\eta}\frac{\sin\alpha}{\sin(\alpha-\beta)}\frac{\partial\xi}{\partial x}-\frac{|\eta|}{\eta}\frac{\cos\alpha}{\sin(\alpha-\beta)}\frac{\partial\xi}{\partial y}\right)=0$$
$$\frac{\partial(\quad)_2}{\partial\xi}=\frac{\partial}{\partial\xi}\left(\frac{|\eta|}{\eta}\frac{\sin\alpha}{\sin(\alpha-\beta)}\frac{\partial\eta}{\partial x}-\frac{|\eta|}{\eta}\frac{\cos\alpha}{\sin(\alpha-\beta)}\frac{\partial\eta}{\partial y}\right)=0$$
$$\frac{\partial(\quad)_3}{\partial\eta}=\frac{\partial}{\partial\eta}\left(-\frac{\sin\beta}{\sin(\alpha-\beta)}\frac{\partial\xi}{\partial x}+\frac{\cos\beta}{\sin(\alpha-\beta)}\frac{\partial\xi}{\partial y}\right)=0$$
$$\frac{\partial(\quad)_4}{\partial\eta}=\frac{\partial}{\partial\eta}\left(-\frac{\sin\beta}{\sin(\alpha-\beta)}\frac{\partial\eta}{\partial x}+\frac{\cos\beta}{\sin(\alpha-\beta)}\frac{\partial\eta}{\partial y}\right)=0$$

Inserting these into the divergence expression, the Laplacian operator for the special case 1 becomes:

$$\nabla^2=\frac{1}{(x^2+2s^2+2sR_c)^2}\frac{\partial^2}{\partial\xi^2}+\frac{\partial^2}{\partial\eta^2}+\frac{2x}{(x^2+2s^2+2sR_c)}\frac{\partial}{\partial\eta}$$

A6.2. Case 2 - mid vertical plane

Again according to the above description of the Laplacian operator:

$$\nabla^2 = -\frac{|\eta|}{\eta}\frac{1+2s\xi}{2s^2+2sR_c-2sy}\frac{\partial}{\partial\xi}\left[(\quad)_1\frac{\partial}{\partial\xi}+(\quad)_2\frac{\partial}{\partial\eta}\right]+\frac{\partial}{\partial\eta}\left[(\quad)_3\frac{\partial}{\partial\xi}+(\quad)_4\frac{\partial}{\partial\eta}\right]$$

Expanding the derivatives and using the content of the parenthesis for this simplified case:

$$\nabla^2 = -\frac{|\eta|}{\eta}\frac{1+2s\xi}{2s^2+2sR_c-2sy}\left[\frac{\partial(\quad)_1}{\partial\xi}\frac{\partial}{\partial\xi}-\frac{|\eta|}{\eta}\frac{1+2s\xi}{2s^2+2sR_c-2sy}\frac{\partial^2}{\partial\xi^2}+\frac{\partial(\quad)_2}{\partial\xi}\frac{\partial}{\partial\eta}\right]+\left[\frac{\partial(\quad)_3}{\partial\eta}\frac{\partial}{\partial\xi}+\frac{\partial(\quad)_4}{\partial\eta}\frac{\partial}{\partial\eta}+\frac{\partial^2}{\partial\eta^2}\right]$$

At this point, there is a need for simplifying the derivatives of the terms in the parentheses. For this, aside from those listed before for the special case 2, the following geometrical conditions are going to be useful, which are derived from the fact that x=η for case 2:

$$\frac{\partial^2\xi}{\partial x\partial\xi}=0 \quad \frac{\partial^2\xi}{\partial y\partial\xi}=0 \quad \frac{\partial^2\xi}{\partial x\partial\eta}=0 \quad \frac{\partial^2\xi}{\partial y\partial\eta}=0$$

$$\frac{\partial^2\eta}{\partial x\partial\xi}=0 \quad \frac{\partial^2\eta}{\partial x\partial\eta}=0 \quad \frac{\partial^2\eta}{\partial y\partial\xi}=0 \quad \frac{\partial^2\eta}{\partial y\partial\eta}=0$$

Then,

$$\frac{\partial(\quad)_1}{\partial\xi}=\frac{\partial}{\partial\xi}\left(\frac{|\eta|}{\eta}\frac{\sin\alpha}{\sin(\alpha-\beta)}\frac{\partial\xi}{\partial x}-\frac{|\eta|}{\eta}\frac{\cos\alpha}{\sin(\alpha-\beta)}\frac{\partial\xi}{\partial y}\right)=0$$

$$\frac{\partial(\quad)_2}{\partial\xi}=\frac{\partial}{\partial\xi}\left(\frac{|\eta|}{\eta}\frac{\sin\alpha}{\sin(\alpha-\beta)}\frac{\partial\eta}{\partial x}-\frac{|\eta|}{\eta}\frac{\cos\alpha}{\sin(\alpha-\beta)}\frac{\partial\eta}{\partial y}\right)=0$$

$$\frac{\partial(\quad)_3}{\partial\eta}=\frac{\partial}{\partial\eta}\left(-\frac{\sin\beta}{\sin(\alpha-\beta)}\frac{\partial\xi}{\partial x}+\frac{\cos\beta}{\sin(\alpha-\beta)}\frac{\partial\xi}{\partial y}\right)=0$$

$$\frac{\partial(\quad)_4}{\partial\eta}=\frac{\partial}{\partial\eta}\left(-\frac{\sin\beta}{\sin(\alpha-\beta)}\frac{\partial\eta}{\partial x}+\frac{\cos\beta}{\sin(\alpha-\beta)}\frac{\partial\eta}{\partial y}\right)=0$$

Inserting these into the divergence expression, the Laplacian operator for the special case 2 becomes:

$$\nabla^2 = \left(\frac{1+2s\xi}{2s^2+2sR_c-2sy}\right)^2\frac{\partial^2}{\partial\xi^2}+\frac{\partial^2}{\partial\eta^2}$$

A7. Convective Derivative of Velocity

Convective derivative of velocity is the term to the left of the equal sign in the momentum equation after the steady state and incompressibility conditions are imposed. Its content needs to be re-derived in terms of the parabolic coordinates in this study. So:

$$\vec{V}.\nabla\vec{V} = \left\{\left(V_\xi\hat{e}_\xi+V_\eta\hat{e}_\eta\right).\left(\left[(\quad)_1\frac{\partial}{\partial\xi}+(\quad)_2\frac{\partial}{\partial\eta}\right]\hat{e}_\xi+\left[(\quad)_3\frac{\partial}{\partial\xi}+(\quad)_4\frac{\partial}{\partial\eta}\right]\hat{e}_\eta\right)\right\}\left(V_\xi\hat{e}_\xi+V_\eta\hat{e}_\eta\right)$$

$$=\left\{V_\xi\left[(\quad)_1\frac{\partial}{\partial\xi}+(\quad)_2\frac{\partial}{\partial\eta}\right]+V_\eta\left[(\quad)_3\frac{\partial}{\partial\xi}+(\quad)_4\frac{\partial}{\partial\eta}\right]\right\}\left(V_\xi\hat{e}_\xi+V_\eta\hat{e}_\eta\right)$$

At this point, due to the derivatives of the unit vectors and the involvement of the abbreviated parentheses, this equation is very lengthy and complicated. So, for the sake of simplicity, the convective derivative is going to be studied according to the two special cases.

A7.1. Case 1 - mid horizontal plane

By importing the velocity conditions, the unit vector angles and the simplified versions of the abbreviated parentheses pertaining to the case 1, the convective derivative expression becomes:

$$\vec{V}.\nabla\vec{V} = V_\eta \hat{e}_\eta \frac{\partial}{\partial\eta}\left(V_\eta \hat{e}_\eta\right) = V_\eta \hat{e}_\eta.\left[\frac{\partial V_\eta}{\partial\eta}\hat{e}_\eta + \frac{\partial \hat{e}_\eta}{\partial\eta}V_\eta\right] = V_\eta \frac{\partial V_\eta}{\partial\eta}$$

A7.2. Case 2 - mid vertical plane

By importing the velocity conditions, the unit vector angles and the simplified versions of the abbreviated parentheses pertaining to the case 2, the convective derivative expression takes a form similar to that for case 1:

$$\vec{V}.\nabla\vec{V} = V_\eta \hat{e}_\eta \frac{\partial}{\partial\eta}\left(V_\eta \hat{e}_\eta\right) = V_\eta \hat{e}_\eta.\left[\frac{\partial V_\eta}{\partial\eta}\hat{e}_\eta + \frac{\partial \hat{e}_\eta}{\partial\eta}V_\eta\right] = V_\eta \frac{\partial V_\eta}{\partial\eta}$$

A8. Derivation of the Pressure Formula for Case 1

Having found a formula for the velocity, and having inserted its derivatives into the simplified Navier-Stokes equation, the resultant momentum equation is as follows:

$$\frac{-2x}{(x^2+2s^2+2sR_c)}V_\eta^2 = -\frac{1}{\rho}\frac{\partial P}{\partial\eta} + \nu V_\eta\left[\frac{4x^2}{(x^2+2s^2+2sR_c)^2} - \frac{2}{(x^2+2s^2+2sR_c)}\right]$$

By inserting the velocity and rearranging, one can get:

$$\frac{1}{\rho}\frac{\partial P}{\partial\eta} = \frac{2c_1^2 x}{(x^2+2s^2+2sR_c)^3} + \nu c_1\left[\frac{4x^2}{(x^2+2s^2+2sR_c)^3} - \frac{2}{(x^2+2s^2+2sR_c)^2}\right]$$

Noting that for case 1, η is equal to x, and incorporating the:

$$\frac{d}{dx}\left[\frac{x}{(x^2+2s^2+2sR_c)^2}\right] = \frac{1}{(x^2+2s^2+2sR_c)^2} - \frac{4x^2}{(x^2+2s^2+2sR_c)^3}$$

the last term in the brackets can be separated into two, and the following can be obtained after integration:

$$\frac{1}{\rho}P = \frac{-c_1^2}{2(x^2+2s^2+2sR_c)^2} - \frac{\nu c_1 x}{(x^2+2s^2+2sR_c)^2} - \nu c_1\int\frac{1}{(x^2+2s^2+2sR_c)^2}dx + c_2$$

where c_2 is an integration constant. In order to solve the remaining integral, define a constant q such that,

$$q^2 = 2s^2 + 2sR_c$$

Then, the integral can be shortened as,

$$\int\frac{1}{(x^2+q^2)^2}dx$$

Then, by imposing the following transformation:

$$\frac{x}{q} = \tan\theta$$

$$\int\frac{1}{(x^2+q^2)^2}dx = \frac{1}{q^3}\int\frac{1}{(1+\tan^2\theta)}d\theta = \frac{1}{q^3}\int\cos^2\theta\, d\theta =$$

$$\frac{1}{q^3}\int\frac{\cos 2\theta+1}{2}d\theta=\frac{1}{2q^3}\left(\frac{\sin 2\theta}{2}+\theta\right)=\frac{1}{2q^3}(\sin\theta\cos\theta+\theta)$$

Then, using the transformation function, one can insert the sine and cosine terms to obtain:

$$\frac{1}{2q^3}\left(\frac{x}{\sqrt{x^2+q^2}}\frac{q}{\sqrt{x^2+q^2}}+\tan^{-1}\left(\frac{x}{q}\right)\right)=\frac{1}{2q^3}\left[\frac{xq}{x^2+q^2}+\tan^{-1}\left(\frac{x}{q}\right)\right]=\frac{1}{2q^3}\left[\frac{\left(\frac{x}{q}\right)}{\left(\frac{x}{q}\right)^2+1}+\tan^{-1}\left(\frac{x}{q}\right)\right]$$

A9. Derivation of the Pressure Formula for Case 2

In section 3.2, the differential equation for the pressure distribution in the throat was found as:

$$-2\xi V_\eta^2=\frac{1}{\rho}\left(\frac{(1+2s\xi)^2}{2s^2+2sR_c}\right)\frac{\partial P}{\partial\xi}$$

Importing the velocity expression for V_η from the same section and working out the algebra, the following is obtained:

$$\frac{1}{\rho}\frac{\partial P}{\partial\xi}=-[4s(s+R_c)c_4^2]\frac{\xi}{(1+2s\xi)^2}-[8s(s+R_c)c_4c_3]\frac{\xi^2}{(1+2s\xi)^2}$$

$$-\left[\frac{(s+R_c)c_1^2}{36s^3}\right]\frac{\xi}{(1+2s\xi)^6}-[4s(s+R_c)c_3^2]\frac{\xi^3}{(1+2s\xi)^2}+\left[\frac{2(s+R_c)c_1c_4}{3s}\right]\frac{\xi}{(1+2s\xi)^4}+\left[\frac{2(s+R_c)c_1c_3}{3s}\right]\frac{\xi^2}{(1+2s\xi)^4}$$

Next, the integrals can be worked out one by one:

$$\int\frac{\xi}{(1+2s\xi)^2}d\xi=\frac{1}{4s^2}\ln(1+2s\xi)+\frac{1}{4s^2(1+2s\xi)}$$

$$\int\frac{\xi^2}{(1+2s\xi)^2}d\xi=\frac{\xi}{4s^2}+\frac{1}{8s^3(1+2s\xi)}-\frac{1}{s}\left[\frac{1}{4s^2}\ln(1+2s\xi)+\frac{1}{4s^2(1+2s\xi)}\right]$$

$$\int\frac{\xi^3}{(1+2s\xi)^2}d\xi=\frac{\xi^2}{8s^2}-\frac{1}{4s^2}\left[\frac{1}{4s^2}\ln(1+2s\xi)+\frac{1}{4s^2(1+2s\xi)}\right]$$

$$-\frac{1}{s}\left\{\frac{\xi}{4s^2}+\frac{1}{8s^3(1+2s\xi)}-\frac{1}{s}\left[\frac{1}{4s^2}\ln(1+2s\xi)+\frac{1}{4s^2(1+2s\xi)}\right]\right\}$$

$$\int\frac{\xi}{(1+2s\xi)^4}d\xi=-\frac{\xi}{6s(1+2s\xi)^3}-\frac{1}{24s^2(1+2s\xi)^2}$$

$$\int\frac{\xi^2}{(1+2s\xi)^4}d\xi=\frac{-\xi^2}{6s(1+2s\xi)^3}-\frac{1}{3s}\left[\frac{\xi}{4s(1+2s\xi)^2}+\frac{1}{8s^2(1+2s\xi)}\right]$$

$$\int\frac{\xi}{(1+2s\xi)^6}d\xi=-\frac{\xi}{10s(1+2s\xi)^5}-\frac{1}{80s^2(1+2s\xi)^4}$$

Inserting these integrals into the pressure expression and rearranging, one gets:

$$\frac{1}{\rho}P=c_5-\frac{(s+R_c)c_4^2}{s}\left[\ln(1+2s\xi)+\frac{1}{(1+2s\xi)}\right]-\frac{2(s+R_c)c_4c_3}{s}\left[\xi-\frac{1}{2s(1+2s\xi)}-\frac{1}{s}\ln(1+2s\xi)\right]$$

$$-\frac{(s+R_c)c_3^2}{s}\left[\frac{\xi^2}{2}+\frac{3}{4s^2}\ln(1+2s\xi)+\frac{1}{4s^2(1+2s\xi)}-\frac{\xi}{s}\right]+\frac{(s+R_c)c_1^2}{360s^4(1+2s\xi)^4}\left[\frac{\xi}{(1+2s\xi)}+\frac{1}{8s}\right]$$

$$-\frac{(s+R_c)c_4c_1}{9s^2(1+2s\xi)^2}\left[\frac{\xi}{(1+2s\xi)}+\frac{1}{4s}\right]-\frac{(s+R_c)c_3c_1}{9s^2(1+2s\xi)}\left[\frac{\xi^2}{(1+2s\xi)^2}+\frac{\xi}{2s(1+2s\xi)}+\frac{1}{4s^2}\right]$$

In order to find out c_5, the equivalence of the two pressure expressions at the origin, i.e. case 1 for x=0 and case 2 for ξ=0. Then:

$$c_5=\frac{P_\infty}{\rho}-\frac{c_1^2}{8s^2(s+R_c)^2}+\frac{(s+R_c)c_4^2}{s}-\frac{(s+R_c)c_4c_3}{s^2}+\frac{(s+R_c)c_3^2}{4s^3}-\frac{(s+R_c)c_1^2}{2880s^5}+\frac{(s+R_c)c_4c_1}{36s^3}+\frac{(s+R_c)c_3c_1}{36s^4}$$

Expressing all the coefficients in terms of c_1:

$$c_5=\frac{P_\infty}{\rho}+\frac{c_1^2\left[4(8s+R_c)(s+R_c)^2-9sR_c^2\right]}{72s^5(s+R_c)^2}$$

Inserting c_1, the final expression for c_5 becomes:

$$c_5=\frac{P_\infty}{\rho}+\frac{2w^2R_c^4\left[4(8s+R_c)(s+R_c)^2-9sR_c^2\right](s+R_c)^2}{s^3\left(-2s^2+3sR_c+6R_c^2\right)^2}$$

Then, the non-dimensional version of the pressure equation for case 2 is:

$$P_{case2}^*=c_5^*-\frac{(s^*+1)c_4^{*2}}{s^*}\left[\ln(1+2s^*\xi^*)+\frac{1}{(1+2s^*\xi^*)}\right]$$

$$-\frac{2(s^*+1)c_4^*c_3^*}{s^*}\left[\xi^*-\frac{1}{2s^*(1+2s^*\xi^*)}-\frac{1}{s^*}\ln(1+2s^*\xi^*)\right]$$

$$-\frac{(s^*+1)c_3^{*2}}{s^*}\left[\frac{\xi^{*2}}{2}+\frac{3}{4s^{*2}}\ln(1+2s^*\xi^*)+\frac{1}{4s^{*2}(1+2s^*\xi^*)}-\frac{\xi^*}{s^*}\right]$$

$$+\frac{(s^*+1)c_1^{*2}}{360s^{*4}(1+2s^*\xi^*)^4}\left[\frac{\xi^*}{(1+2s^*\xi^*)}+\frac{1}{8s^*}\right]-\frac{(s^*+1)c_4^*c_1^*}{9s^{*2}(1+2s^*\xi^*)^2}\left[\frac{\xi^*}{(1+2s^*\xi^*)}+\frac{1}{4s^*}\right]$$

$$-\frac{(s^*+1)c_3^*c_1^*}{9s^{*2}(1+2s^*\xi^*)}\left[\frac{\xi^{*2}}{(1+2s^*\xi^*)^2}+\frac{\xi^*}{2s^*(1+2s^*\xi^*)}+\frac{1}{4s^{*2}}\right]$$

where $\xi^*=\xi R_c$ and $c_5^*=\left(c_5-\frac{P_\infty}{\rho}\right)/(wR_c)^2$ and $P_{case2}^*=(P-P_\infty)/\rho(wR_c)^2$.

A10. Curl Operator and Vorticity for Case 2

Before working on the vorticity, first, the curl of a vector is going to be done. Bringing the simplified version of the del operator for case 2, curl of $\vec{V}$ becomes:

$$\nabla x\vec{V}=\left[-\frac{|\eta|}{\eta}\frac{1+2s\xi}{2s^2+2sR_c-2sy}\frac{\partial}{\partial\xi}\hat{e}_\xi+\frac{\partial}{\partial\eta}\hat{e}_\eta\right]x\left(A\hat{e}_\xi+B\hat{e}_\eta\right)$$

$$=\left(-\frac{|\eta|}{\eta}\frac{1+2s\xi}{2s^2+2sR_c-2sy}\right)\left[\frac{\partial A}{\partial\xi}\left(\hat{e}_\xi x\hat{e}_\xi\right)+A\left(\hat{e}_\xi x\frac{\partial\hat{e}_\xi}{\partial\xi}\right)+\frac{\partial B}{\partial\xi}\left(\hat{e}_\xi x\hat{e}_\eta\right)+B\left(\hat{e}_\xi x\frac{\partial\hat{e}_\eta}{\partial\xi}\right)\right]$$

$$+\left[\frac{\partial A}{\partial\eta}\left(\hat{e}_\eta x\hat{e}_\xi\right)+A\left(\hat{e}_\eta x\frac{\partial\hat{e}_\xi}{\partial\eta}\right)+\frac{\partial B}{\partial\eta}\left(\hat{e}_\eta x\hat{e}_\eta\right)+B\left(\hat{e}_\eta x\frac{\partial\hat{e}_\eta}{\partial\eta}\right)\right]$$

Inserting the geometrical identities expressed in the previous sections for case 2, this expression becomes:

$$\nabla x\vec{V} = \left(-\frac{|\eta|}{\eta}\frac{1+2s\xi}{2s^2+2sR_c-2sy}\right)\left[\frac{\partial A}{\partial \xi}\left(\hat{e}_\xi x\hat{e}_\xi\right)+\frac{\partial B}{\partial \xi}\left(\hat{e}_\xi x\hat{e}_\eta\right)\right]+\left[\frac{\partial A}{\partial \eta}\left(\hat{e}_\eta x\hat{e}_\xi\right)+\frac{\partial B}{\partial \eta}\left(\hat{e}_\eta x\hat{e}_\eta\right)\right]$$

Using the definitions of the units vectors of the new coordinate system in terms of the Cartesian coordinate system, one can work out their cross products.

$$\hat{e}_\xi x\hat{e}_\xi = \cos\beta\sin\beta-\sin\beta\cos\beta = 0$$

$$\hat{e}_\eta x\hat{e}_\eta = \cos\alpha\sin\alpha-\sin\alpha\cos\alpha = 0$$

$$\hat{e}_\xi x\hat{e}_\eta = \frac{|\eta|}{\eta}(\cos\beta\sin\alpha-\sin\beta\cos\alpha) = \frac{|\eta|}{\eta}\sin(\alpha-\beta)$$

Plugging these into the curl operator expression, and using the values of α and β for case 2, one finds the simplified version of the curl operator for case 2.

$$\nabla x\vec{V} = \left(-\frac{1+2s\xi}{2s^2+2sR_c-2sy}\right)\frac{\partial B}{\partial \xi}+\frac{\partial A}{\partial \eta}$$

Note that in this formula, A is the component of the velocity vector in the ξ direction, and it is assumed as zero for case 2. Then, vorticity, which is equal to the curl of velocity, is obtained as:

$$\nabla x\vec{V} = \Omega = \left(-\frac{1+2s\xi}{2s^2+2sR_c-2sy}\right)\frac{\partial V_{\eta,case2}}{\partial \xi}$$

Taking the derivative of the velocity and using the transformation between the Cartesian coordinates and the new coordinate system, vorticity is found as:

$$\Omega^* = \frac{\Omega}{w} = \frac{2y^*(3s^{*2}+6s^*+3-3y^*s^*-3y^*+y^{*2})}{s^*(-2s^{*2}+3s^*+6)(s^*+1-y^*)^2}$$

where $y^*=y/R_c$ and $s^*=s/R_c$.