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Structures and $\mathscr{D}$-isometric warping

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Abstract
We introduce the notion of $\mathscr{D}$-isometric warping and we use it to construct a 1-parameter family of Kählerian structures from a single Sasakian structure and also a quaternionic Kählerian structure from a Sasakian 3-structure.

Keywords and 2010 Mathematics Subject Classification
Sasakian structure, kählerian structure, 3-Sasakian structure, Quaternionic kählerian structure.
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1. Introduction

The product of an almost contact manifold M and the real line $\mathbb{R}$ carries a natural almost complex structure. When this structure is integrable the almost contact structure is said to be normal.

In 1985, using the warped product $M \times_f \mathbb{R}$ where $f \in C^\infty(\mathbb{R}_+)$, Oubiña showed that there is a one-to-one correspondence between Sasakian and Kähler structures [11].

In 2013, building on the work of Tanno [12] (the homothetic deformation on contact metric manifold), Blair [8] introduced the notion of $\mathscr{D}$-homothetic warping. He used it to show by another way that there is a one-to-one correspondence between Sasakian and Kählerian structures too.

Recently, Beldjilali and Belkhelfa [2] have generalized the idea of Blair, they introduced the notion of generalized $\mathscr{D}$-homothetic bi-warping and they proved that every Sasakian manifold M generates a 1-parameter family of Kählerian manifolds. After that, they gives the notion of generalized doubly $\mathscr{D}$-homothetic bi-warping [3].

By a similar techniques of Oubiña, Bär [1] and Tshikuna-Matamba [14] pointed out that there is one-to-one correspondence between Sasakian 3-structures and hyperKähler structures. In [15] and [16] we find the construction of quaternionic kählerian structure from 3-Sasakian structures.

Here, after giving preliminary background on almost Hermitian structures and almost contact metric manifolds in Section 2, we introduce in Section 3 the notion of $\mathscr{D}$-isometric warping and prove some basic properties. In Section 4 we give the first application for this product. Starting from a Sasakian manifold M, we construct a 1-parameter family of Kählerian structures on the product of a $\mathbb{R} \times M$ which is different from that in [2] and we construct an example. In Section 5, we give the second application, we constructed a quaternionic kählerian structure from 3-Sasakian structures.

2. Preliminaries on manifolds

Recall that an almost Hermitian manifold is a Riemannian manifold (M^{2n}, g) equipped with a tensor field J of type $(1,1)$ such that for all vectors fiels X, Y on M the following two conditions are satisfied:

$$J^2(X) = -X, \qquad g(JX, JY) = g(X,Y).$$

An almost complex stucture J is integrable, and hence the manifold is a complex manifold, if and only if its Nijenhuis tensor N_j vanishes, with

$$N_j(X,Y) = [JX,JY] - [X,Y] - J[X,JY] - J[JX,Y].$$

Any almost Hermitian manifold (M,g,J) possesses a differential 2-form Ω, called the fundamental 2-form or the Kähler 2-form, defined by

$$\Omega(X,Y) = g(X,JY). \tag{1}$$

(M,J,g) is then called almost Kähler if Ω is closed i.e. $d\Omega = 0$. An almost Kähler manifold with integrable J is called a Kähler manifold, and thus is characterized by the conditions: $d\Omega = 0$ and $N = 0$. One can prove that these both conditions combined are equivalent with the single condition

$$\nabla J = 0.$$

An almost quaternionic metric manifold is a quintuple (M,g,J_1,J_2,J_3), where

$$\begin{cases} (1) : (M,g) \textit{ is a Riemannian manifold}; \\ (2) : (g,J_\alpha) \textit{ is an almost Hermitian structure on } M \textit{ for } \alpha = 1,2,3; \\ (3) : J_1J_2 = J_3,\ J_2J_3 = J_1,\ J_3J_1 = J_2. \end{cases} \tag{2}$$

Almost quaternionic metric manifolds are of dimension $4m$ and their nomenclature is related to that of almost Hermitian structures. According Calabi [9], for a structure to be hyperkählerian, it is sufficient that in (g,J_1,J_2,J_3) two of these structures are kählerians. A differential 4-form is defined by

$$\tilde{\Omega} = \sum_{\alpha=1}^{3} \Omega_\alpha \wedge \Omega_\alpha.$$

An almost quaternion metric manifold is quaternion kählerian manifold if and only if $\nabla\tilde{\Omega} = 0$ [17].

Proposition 1. *([17], p 161) An almost quaternionic Hermitian manifold is called a quaternionic kähler manifold if an almost hypercomplex structure J_α, $\alpha = 1,2,3$ in any local coordinate neighborhood U satisfies*

$$\begin{cases} \nabla_X J_1 = \qquad\qquad \omega_3(X)J_2 - \omega_2(X)J_3 \\ \nabla_X J_2 = -\omega_3(X)J_1 \qquad\qquad + \omega_1(X)J_3 \\ \nabla_X J_3 = \omega_2(X)J_1 - \omega_1(X)J_2 \end{cases} \tag{3}$$

for any vector field X on U, where ∇ is the Levi-Civita connection of the Riemannian metric, and ω_α are certain local 1-forms defined in U. In particular, if all ω_α for each U are vanishing, then the structure is called hyper-Kähler. Remark that if $dimM > 4$, a quaternionic Kähler manifold is an Einstein manifold.

By an almost contact metric manifold, one understands a quintuple (M,g,φ,ξ,η), where

(1) ξ is a characteristic vector field ;

(2) η is a differential 1-form such that $\eta(\xi) = 1$;

(3) φ is a tensor field of type $(1,1)$ satisfying $\varphi^2 X = -X + \eta(X)\xi$;

(4) $g(\varphi X, \varphi Y) = g(X,Y) - \eta(X)\eta(Y)$.

Replacing J by φ, the fundamental 2-form ϕ is defined by

$$\phi(X,Y) = g(X,\varphi Y). \tag{4}$$

Denoting by ∇ the Levi-Civita connection of g, the covariant derivative of η and the exterior differential of η are defined, respectively, by

$$(\nabla_X \eta)Y = g(Y, \nabla_X \xi), \tag{5}$$

$$2d\eta(X,Y) = (\nabla_X \eta)Y - (\nabla_Y \eta)X, \tag{6}$$

An almost contact metric manifold is said to be almost cosymplectic if the forms ϕ and η are closed, that is, $d\phi = d\eta = 0$.

Such a manifold is said to be a contact metric manifold if $d\eta = \phi$. If, in addition, ξ is a Killing vector field, then M is said to be a K-contact manifold. It is well-known that a contact metric manifold is a K-contact manifold if and only if $\nabla_X \xi = -\varphi X$, for any vector field X on M. On the other hand, the almost contact metric structure of M is said to be normal if $[\varphi,\varphi](X,Y) = -2d\eta\ (X,Y)\xi$, for any X and Y where $[\varphi,\varphi]$ denotes the Nijenhuis torsion of φ, given by

$$[\varphi,\varphi](X,Y) = \varphi^2[X,Y] + [\varphi X, \varphi Y] - \varphi[\varphi X, Y] - \varphi[X, \varphi Y].$$

A normal almost cosymplectic manifold is called a cosymplectic manifold. It is well-known that a necessary and sufficient condition for M to be cosymplectic is $\nabla\varphi = 0$.

A normal contact metric manifold is called a Sasakian manifold. It can be proved that a Sasakian manifold is K-contact, and that an almost contact metric manifold is Sasakian if and only if

$$(\nabla_X \varphi)Y = g(X,Y)\xi - \eta(Y)X, \tag{7}$$

for any X,Y. Moreover, for a Sasakian manifold we have the following identities:

$$\nabla_X \xi = -\varphi X, \qquad (\nabla_X \eta)(Y) = -g(\varphi X, Y). \tag{8}$$

Let $(\varphi_i, \xi_i, \eta_i)_{i=1}^3$ be three almost contact structures such that each of them is compatible with the Riemannian structure g (i.e. $g(\varphi_i X, \varphi_i Y) = g(X,Y) - \eta_i(X)\eta_i(Y)$, $i = 1,2,3$). We say that $\left(M, g, (\varphi_i, \xi_i, \eta_i)_{i=1}^3\right)$ is an almost contact metric manifold 3-structure if for any cyclic permutation (i,j,k) of $\{1,2,3\}$ the following conditions are satisfied :

$$\begin{cases} (1): \eta_i(\xi_j) = \eta_j(\xi_i) = 0; \\ (2): \varphi_i \xi_j = -\varphi_j \xi_i = \xi_k; \\ (3): \varphi_i \circ \varphi_j - \eta_j \otimes \xi_i = -\varphi_j \circ \varphi_i + \eta_i \otimes \xi_j = \varphi_k; \\ (4): \eta_i \circ \varphi_j = -\eta_j \circ \varphi_i = \eta_k. \end{cases} \tag{9}$$

Almost contact metric manifolds 3-structure are of odd dimension $4m+3$. If each $(\varphi_i, \xi_i, \eta_i)_{i=1}^3$ is a Sasakian structure then almost contact metric manifolds 3-structure $(\varphi_i, \xi_i, \eta_i)_{i=1}^3$ is called a Sasakian 3-structure and ξ_1, ξ_2, ξ_3 are orthonormal vector fields, satisfying

$$[\xi_i, \xi_j] = 2\xi_k$$

for any cyclic permutation (i,j,k) of $\{1,2,3\}$ ([6], p.294). Such a manifold with a Sasakian 3-structure is called a 3-Sasakian manifold. Remark that a 3-Sasakian manifold is an Einstein manifold.

3. $\mathscr{D}$-isometric warping

Let $(M, \varphi, \xi, \eta, g)$ be an almost contact metric manifold with $dimM = 2n+1$. The equation $\eta = 0$ defines a $2n$-dimensional distribution $\mathscr{D}$ on M. By an $2n$-isometric deformation or $\mathscr{D}$-isometric deformation we mean a change of structure tensors of the form

$$\overline{\varphi} = \varphi, \qquad \overline{\eta} = a\eta, \qquad \overline{\xi} = \frac{1}{a}\xi, \qquad \overline{g} = g + (a^2-1)\eta\otimes\eta, \quad a \neq 0.$$

If $(M, \varphi, \xi, \eta, g)$ is an almost contact metric structure , then $(\overline{M}, \overline{\varphi}, \overline{\xi}, \overline{\eta}, \overline{g})$ is also an almost contact metric structure.

The notion of $\mathscr{D}$-homothetic warping is very well known [2], [3], [4] and [8]. Given a Riemannian manifolds (M_1, g_1) and an almost contact metric manifold $(M_2, \varphi_2, \xi_2, \eta_2, g_2)$, and a positive function f on M_1, the Riemannian metric $g = g_1 + fg_2 + f(f-1)\eta_2\otimes\eta_2$ on $M_1 \times M_2$ is known as a $\mathscr{D}$-homothetically warped metric.

Now consider the product of a Riemannian manifold (M_1, g_1) and an almost contact metric manifold $(M_2, \varphi_2, \xi_2, \eta_2, g_2)$. On $M_1 \times M_2$ define a metric g by

$$g = g_1 + g_2 + (f^2-1)\eta_2 \otimes \eta_2$$

where f is a function non-zero everywhere on M_1.
Notice that, for all X vectors field on M_2 orthogonal to ξ_2 we have $g(X,X) = g_2(X,X)$. That is why, we refer to this construction as *$\mathscr{D}$-isometric warping*.

Using the Koszul formula for the Levi-Civita connection of a Riemannian metric

$$2g(\nabla_X Y, Z) = Xg(Y,Z) + Yg(X,Z) - Zg(X,Y) + g([X,Y],Z) + g([Z,X],Y) - g([Y,Z],X),$$

where $X = (X_1, X_2)$, $Y = (Y_1, Y_2)$ and $Z = (Z_1, Z_2)$, one can compute the Levi-Civita connection of the $\mathscr{D}$-isometrically warped metric.

Proposition 2. *Let ∇^1, ∇^2 and ∇ be connections of g_1, g_2 and g respectively. For all vectors field X_1, Y_1, Z_1 tangent to M_1 and independent of M_2 and similarly for X_2, Y_2, Z_2 we have:*

$$\nabla_{X_1} Y_1 = \nabla^1_{X_1} Y_1,$$

$$\nabla_{X_1} Y_2 = \nabla_{Y_2} X_1 = \frac{X_1(f)}{f}\eta_2(Y_2)\xi_2,$$

$$g(\nabla_{X_2} Y_2, Z_2) = g(\nabla^2_{X_2} Y_2, Z_2) + (f^2-1)\Big(\tfrac{1}{2}\big(g_2(\nabla^2_{X_2}\xi_2, Y_2) + g_2(\nabla^2_{Y_2}\xi_2, X_2)\big)\eta_2(Z_2) + d\eta_2(X_2,Z_2)\eta_2(Y_2) + d\eta_2(Y_2,Z_2)\eta_2(X_2)\Big),$$

which in turn can be used to find $g(\nabla_{X_2} Y_2, Z_1) = -g(\nabla_{X_2} Z_1, Y_2)$.

Let σ denote the second fundamental form of M_2 in $M_1 \times M_2$ and while f is a function on M_1, for emphasis we denote its gradient by $grad_1 f$. Then we have the following Theorem.

Theorem 3. *For an almost contact metric manifold $(M_2, \varphi_2, \xi_2, \eta_2, g_2)$ and a $\mathscr{D}$-isometrically warped metric on $M_1 \times M_2$ we have the following:*

1. *M_1 is a totally geodesic submanifold.*
2. *M_2 is a cylindrical submanifold and its second fundamental form is given by*

$$\sigma_2(X_2, Y_2) = -\frac{1}{2}\eta_2(X_2)\eta_2(Y_2) grad_1 f^2.$$

3. *The mean curvature vector of M_2 in $M_1 \times M_2$ is*

$$\mathscr{H} = -\frac{1}{2(2n+1)} grad_1 f^2.$$

4. *If in addition, $d\eta_2(\xi_2, X_2) = 0$ for every X_2 (equivalently the integral curves of ξ_2 are geodesics), then the Reeb vector field ξ_2 is g-Killing if and only if it is g_2-Killing.*

Proof. Recall that a submanifold N of a Riemannian manifold (M^{2n+1}, g) is called quasi-umbilical [10] if its second fundamental tensor has the form

$$\omega(X,Y) = \alpha g(X,Y)\rho + \beta\eta(X)\eta(Y)\rho$$

where α, β are scalars, X, Y are vectors fields on N and ρ is the unit normal vector field

- If $\alpha = 0$, then N is cylindrical.
- If $\beta = 0$, then N is umbilical.
- If $\alpha = \beta = 0$, then N is geodesic.

1. Let σ_1 be the second fundamental form of M_1 in $M_1 \times M_2$. Since $\nabla_{X_1} Y_1 = \nabla^1_{X_1} Y_1$, then

$$\sigma_1 = \nabla_{X_1} Y_1 - \nabla^1_{X_1} Y_1 = 0.$$

2. Let σ_2 be the second fundamental form of M_2 in $M_1 \times M_2$. We have

$$\begin{aligned} g\big(\nabla_{X_2} Y_2, Z_1\big) &= -fZ_1(f)\eta_2(X_2)\eta_2(Y_2) \\ &= g_1\Big(-\frac{1}{2}\eta_2(X_2)\eta_2(Y_2) grad_1 f^2, Z_1\Big), \end{aligned}$$

since $g\left(\nabla^2_{X_2}Y_2, Z_1\right)=0$ then

$$\sigma_2(X_2,Y_2)=-\frac{1}{2}\eta_2(X_2)\eta_2(Y_2)grad_1 f^2.$$

3. Knowing that The mean curvature vector of M_2 in $M_1\times M_2$ is defined by

$$\mathscr{H}=\frac{1}{2n+1}tr_{g_2} \quad and \quad \sigma_2=\frac{1}{2n+1}\sum_{i=1}^{2n+1}\sigma_2(e_i,e_i)$$

where $\{e_i\}_{i=1,2n+1}$ orthonormal basis of M_2 then,

$$\begin{aligned}
\mathscr{H} &= \frac{1}{2n+1}\sum_{i=1}^{i=2n+1}\sigma_2(e_i,e_i)\\
&= -\frac{1}{2(2n+1)}grad_1 f^2\sum_{i=1}^{i=2n+1}\eta_2(e_i)\eta_2(e_i)\\
&= -\frac{1}{2(2n+1)}grad_1 f^2.
\end{aligned}$$

4. For all $X=X_1+X_2$ and $Y=Y_1+Y_2$ two vectors fields on $M_1\times M_2$ we have

$$\xi_2 \; is \; g-Killing \Leftrightarrow g(\nabla_X\xi_2,Y)+g(\nabla_Y\xi_2,X)=0.$$

So,

$$\begin{aligned}
g(\nabla_X\xi_2,Y)+g(\nabla_Y\xi_2,X) &= g(\nabla_{X_1+X_2}\xi_2,Y_1+Y_2)+g(\nabla_{Y_1+Y_2}\xi_2,X_1+X_2)\\
&= g(\nabla_{X_1}\xi_2,Y_2)+g(\nabla_{X_2}\xi_2,Y_1)+g(\nabla_{X_2}\xi_2,Y_2)\\
&\quad +g(\nabla_{Y_1}\xi_2,X_2)+g(\nabla_{Y_2}\xi_2,X_1)+g(\nabla_{Y_2}\xi_2,X_2) \qquad (10)
\end{aligned}$$

suppose that $d\eta_2(\xi_2,X_2)=0$ equivalent to $\xi_2\eta_2(X_2)=\eta_2\left(\nabla^2_{\xi_2}X_2\right)$ (i.e. $\nabla^2_{\xi_2}\xi_2=0$) then, we can easily verify the following statements:

$$g(\nabla_{X_1}\xi_2,Y_2)=\frac{1}{2}X_1(f^2)\eta_2(Y_2),$$

$$\tilde{g}(\nabla_{X_2}\xi_2,Y_1)=-\frac{1}{2}Y'(f^2)\eta_2(X_2),$$

$$g(\nabla_{X_2}\xi_2,Y_2)=g(\nabla^2_{X_2}\xi_2,Y_2)+(f^2-1)d\eta_2(X_2,Y_2).$$

Replacing in formula (10), we get

$$\begin{aligned}
g(\nabla_X\xi_2,Y)+g(\nabla_Y\xi_2,X) &= g(\nabla_{X_2}\xi_2,Y_2)+g(\nabla^2_{Y_2}\xi_2,X_2)\\
&= g_2(\nabla^2_{X_2}\xi_2,Y_2)+g_2(\nabla^2_{Y_2}\xi_2,X_2).
\end{aligned}$$

This completes the proof. ■

4. From a single Sasakian structure to a 1-parameter family of Kählerian structures.

For our first application of the idea of $\mathscr{D}$-isometric warping we consider the case where $M_1=\mathbb{R}$, $M_2=M$ is a Sasakian manifold and the metric

$$\tilde{g}=h^2\left(dt^2+g+(f^2-1)\eta\otimes\eta\right), \qquad (11)$$

where f,h are two functions non-zero everywhere on $\mathbb{R}$. For brevity, we denote the unit tangent field to $\mathbb{R}$ by ∂_t. In this case the proposition (2) becomes:

Proposition 4. *Let $(M, \varphi, \xi, \eta, g)$ be a Sasakian manifold. Let ∇ and $\tilde{\nabla}$ denote the Riemannian connections of g, and $\tilde{g}$ respectively. For all X, Y vector fields tangent to M and independent of $\mathbb{R}$, we have*

$$\tilde{\nabla}_{\partial_t}\partial_t = \frac{h'}{h}\partial_t,$$

$$\tilde{\nabla}_{\partial_t}X = \tilde{\nabla}_X\partial_t = \frac{h'}{h}X + \frac{f'}{f}\eta(X)\xi,$$

$$\tilde{\nabla}_X Y = \nabla_X Y + (1-f^2)\big(\eta(X)\varphi Y + \eta(Y)\varphi X\big) - \frac{1}{h}\Big(h' g(X,Y) + \big(f(fh)' - h'\big)\eta(X)\eta(Y)\Big)\partial_t.$$

Next, we introduce a class of almost complex structure $\tilde{J}$ on manifold $\tilde{M}$:

$$\tilde{J}(a\partial_t, X) \quad = \quad \Big(f\eta(X)\partial_t\ ,\ \varphi X - \frac{a}{f}\xi\Big), \tag{12}$$

for any vector filds X of M where f, h are functions on $\mathbb{R}$ and $fh \neq 0$ everywhere.

That $J^2 = -I$ is easily checked and for all $\tilde{X} = (a\partial_t, X), \tilde{Y} = (b\partial_t, Y)$ on $\tilde{M}$ we can see that $\tilde{g}$ is almost Hermitian with respect to $\tilde{J}$ i.e.

$$\tilde{g}\big(\tilde{J}\tilde{X}, \tilde{J}\tilde{Y}\big) = \tilde{g}(\tilde{X}, \tilde{Y}).$$

Knowing that $(\nabla_{\tilde{X}}J)\tilde{Y} = \nabla_{\tilde{X}}(\tilde{J}\tilde{Y}) - \tilde{J}\nabla_{\tilde{X}}\tilde{Y}$ with using the proposition (4) and formulas (7) and (8), we get the following proposition:

Proposition 5. *Let $(M, \varphi, \xi, \eta, g)$ be a Sasakian manifold. Let ∇ and $\tilde{\nabla}$ denote the Riemannian connections of g and $\tilde{g}$ respectively. For all X, Y vector fields tangent to M and independent element of $\mathbb{R}$, we have*

$$(\tilde{\nabla}_X\tilde{J})\partial_t \quad = \quad \Big(f - \frac{h'}{h}\Big)\varphi X,$$

$$(\tilde{\nabla}_X\tilde{J})Y \quad = \quad \Big(f - \frac{h'}{h}\Big)\Big(\frac{1}{f}g(X,Y)\xi - f\eta(Y)X - \big(\frac{1}{f} - f\big)\eta(X)\eta(Y)\xi + g(X, \varphi Y)\partial_t\Big).$$

Therefore, summing up the arguments above, we have the following main theorem:

Theorem 6. *Let $(M, \varphi, \xi, \eta, g)$ be a Sasakian manifold. The almost Hermitian structure constructed in (11) and (12) is Kählerian if and only if $f = \frac{h'}{h}$.*

Remark 7. *In this theorem, for $h = ce^t$ where $c > 0$ i.e. $f = 1$ we get the result of Oubiña (see [11]).*

Remark 8. *In [11], Oubiña showed that there is a one-to-one correspondence between Sasakian and Kählerian structures and in [8], Blair showed by another way this correspondence. Here again, we generalized this correspondence by building another 1-parameter family of Kählérian structures from a single Sasakian structure (see [2]).*

Example 9. *For this example, we rely on the example of Blair [5]. We know that $\mathbb{R}^3$ with coordinates (x, y, z), admits the Sasakian structure*

$$g = \frac{1}{4}\begin{pmatrix} 1+y^2 & 0 & -y \\ 0 & 1 & 0 \\ -y & 0 & 1 \end{pmatrix}, \qquad \varphi = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & y & 0 \end{pmatrix}, \qquad \xi = 2\Big(\frac{\partial}{\partial z}\Big), \qquad \eta = \frac{1}{2}(dz - ydx).$$

So, using this structure, we can define a family of Kählerian structures $(\tilde{J}, \tilde{g})$ on $\mathbb{R}^4$ as follows

$$\tilde{g} = \frac{1}{4}\begin{pmatrix} 4h^2 & 0 & 0 & 0 \\ 0 & (h^2 + h'^2y^2) & 0 & -h'^2y \\ 0 & 0 & h^2 & 0 \\ 0 & -h'^2y & 0 & h'^2 \end{pmatrix}$$

$$\tilde{J} = \begin{pmatrix} 0 & -\frac{1}{2}yh & 0 & \frac{1}{2}h \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -\frac{2}{h} & 0 & y & 0 \end{pmatrix}$$

5. From 3-Sasakian structure to quaternionic Kählerian structure

For a second application of the idea of $\mathscr{D}$-isometric warping we consider a three almost contact structures $(\varphi_i, \xi_i, \eta_i)_{i=1}^3$ on a manifold M of dimension $4n+3$ and we define an almost hypercomplex structure $\tilde{J}_\alpha$, $\alpha = 1,2,3$ on $\tilde{M}^{4n+4} = M \times \mathbb{R}$ by

$$\tilde{J}_\alpha(a\partial_t, X) = (f\eta_\alpha(X)\partial_t, \varphi_\alpha X - \frac{a}{f}\xi_\alpha), \tag{13}$$

then we give a Riemannian metric on $\tilde{M}$ by

$$\tilde{g} = h^2\big(dt^2 + g + (f^2-1)\sum_{i=1}^{i=3}\eta_i \otimes \eta_i\big), \tag{14}$$

where f, h are functions on $\mathbb{R}$ such that $fh \neq 0$ everywhere and dt^2 is the usual metric on $\mathbb{R}$. Then by (2) and (9) one can showed the following:

Proposition 10. *Let $(\varphi_i, \xi_i, \eta_i)_{i=1}^3$ be an almost contact metric 3-structure on a manifold M of dimension $4n+3$ and f, h are functions on $\mathbb{R}$ such that $fh \neq 0$ everywhere. Then $\big(\tilde{M}^{4n+1}, (\tilde{J}_\alpha)_{\alpha=1}^3, \tilde{g}\big)$ constructed as above is an almost quaternionic Hermitian manifold.*

Proof. Obvious. ■

Next, let $\big(M^{4n+3}, (\varphi_i, \xi_i, \eta_i)_{i=1}^3, g\big)$ be a 3-Sasakian manifold then, from proposition (4) we can conclude that

$$\tilde{\nabla}_{\partial_t}\partial_t = \frac{h'}{h}\partial_t,$$
$$\tilde{\nabla}_{\partial_t}X = \tilde{\nabla}_X\partial_t = \frac{h'}{h}X + \frac{f'}{f}\eta_i(X)\xi_i,$$
$$\tilde{\nabla}_X Y = \nabla_X Y + (1-f^2)\big(\eta_i(X)\varphi_i Y + \eta_i(Y)\varphi_i X\big) - \frac{1}{h}\Big(h'g(X,Y) + \big(f(fh)' - h'\big)\eta_i(X)\eta_i(Y)\Big)\partial_t.$$

Note: we will use the convention of Einstein. (Whenever an index is repeated, it is a dummy index and is summed from 1 to 3).

Now, we compute directly $\tilde{\nabla}\tilde{J}_\alpha$, $\alpha = 1,2,3$ we get

Proposition 11. *Let $\big(M^{4n+3}, (\varphi_i, \xi_i, \eta_i)_{i=1}^3, g\big)$ be 3-Sasakian manifold. Let ∇ and $\tilde{\nabla}$ denote the Riemannian connections of g, and $\tilde{g}$ respectively. For all X, Y vector fields tangent to M and independent of $\mathbb{R}$, we have*

$$\begin{aligned}
(\tilde{\nabla}_X\tilde{J}_\alpha)\partial_t &= \Big(f - \frac{h'}{h}\Big)\varphi_\alpha X + \frac{1}{f}(1-f^2-f')\eta_i(X)\varphi_\alpha\xi_i, \qquad \alpha = 1,2,3\\
(\tilde{\nabla}_X\tilde{J}_1)Y &= \Big(f - \frac{h'}{h}\Big)A_1 + (1-f^2+f')B_1 + 2(1-f^2)\big(\eta_3(X)\varphi_2 Y - \eta_2(X)\varphi_3 Y\big)\\
&\quad - \frac{1}{h}\big(f(fh)' - h'\big)\big(\eta_3(X)\eta_2(Y) - \eta_2(X)\eta_3(Y)\big)\partial_t,\\
(\tilde{\nabla}_X\tilde{J}_2)Y &= \Big(f - \frac{h'}{h}\Big)A_2 + (1-f^2+f')B_2 + 2(1-f^2)\big(\eta_1(X)\varphi_3 Y - \eta_3(X)\varphi_1 Y\big)\\
&\quad - \frac{1}{h}\big(f(fh)' - h'\big)\big(\eta_1(X)\eta_3(Y) - \eta_3(X)\eta_1(Y)\big)\partial_t,\\
(\tilde{\nabla}_X\tilde{J}_3)Y &= \Big(f - \frac{h'}{h}\Big)A_3 + (1-f^2+f')B_3 + 2(1-f^2)\big(\eta_2(X)\varphi_1 Y - \eta_1(X)\varphi_2 Y\big),\\
&\quad - \frac{1}{h}\big(f(fh)' - h'\big)\big(\eta_2(X)\eta_1(Y) - \eta_1(X)\eta_2(Y)\big)\partial_t,
\end{aligned}$$

others =0, and

$$\begin{aligned}
A_\alpha &= \Big(f - \frac{h'}{h}\Big)\Big(\frac{1}{f}g(X,Y)\xi_\alpha - f\eta_\alpha(Y)X + \big(f - \frac{1}{f}\big)\eta_i(X)\eta_i(Y)\xi_\alpha\Big),\\
B_\alpha &= (1-f^2+f')\eta_\alpha(X)\big(\eta_\alpha(X)\xi_i - \eta_i(X)\xi_\alpha\big).
\end{aligned}$$

On the other hand, we have

$$\begin{cases} \big(\omega_3(X)J_2-\omega_2(X)J_3\big)\partial_t = \frac{1}{f}\big(\omega_2(X)\xi_3-\omega_3(X)\xi_2\big), \\ \big(-\omega_3(X)J_1+\omega_1(X)J_3\big)\partial_t = \frac{1}{f}\big(\omega_3(X)\xi_1-\omega_1(X)\xi_3\big), \\ \big(\omega_2(X)J_1-\omega_1(X)J_2\big)\partial_t = \frac{1}{f}\big(\omega_1(X)\xi_2-\omega_2(X)\xi_1\big) \end{cases}$$

and

$$\begin{cases} \big(\omega_3(X)J_2-\omega_2(X)J_3\big)Y &= \omega_3(X)\varphi_2Y-\omega_2(X)\varphi_3Y+f\big(\omega_3(X)\eta_2(Y)-\omega_2(X)\eta_3(Y)\big)\partial_t, \\ \big(-\omega_3(X)J_1+\omega_1(X)J_3\big)Y &= -\omega_3(X)\varphi_1Y+\omega_1(X)\varphi_3Y+f\big(-\omega_3(X)\eta_1(Y)+\omega_1(X)\eta_3(Y)\big)\partial_t, \\ \big(\omega_2(X)J_1-\omega_1(X)J_2\big)Y &= \omega_2(X)\varphi_1Y-\omega_1(X)\varphi_2Y+f\big(\omega_2(X)\eta_1(Y)-\omega_1(X)\eta_2(Y)\big)\partial_t. \end{cases}$$

Now, we will make a comparison using the proposition (1) we get the following equations:

$$f=\frac{h'}{h}, \qquad 1-f^2+f'=0,$$

$$\omega_\alpha=(1-f'-f^2)\eta_\alpha=2(1-f^2)\eta_\alpha=-\frac{1}{h}\big(f(fh)'-h'\big)\eta_\alpha.$$

and moreover that these equations are equivalent to the OED system

$$f=\frac{h'}{h}, \qquad 1-f^2+f'=0, \qquad \omega_\alpha=2(1-f^2)\eta_\alpha,$$

Solving the differential equation system, we obtain the following theorem:

Theorem 12. *Let $(\varphi_i,\xi_i,\eta_i)_{i=1}^3$ be a 3-Sasakian manifold. Then the almost quaternionic Hermitian structure constructed in (13) and (14) is:*

1. *Hyper-Kählerian structure if and only if $f=1$ and $h=ce^t$ where $c>0$.*

2. *Quaternionic Kählerian structure if and only if*

$$f(t)=-\tanh(t+c_1), \qquad \textit{and} \qquad h(t)=\frac{c_2}{\cosh(t+c_1)},$$

where c_1 and c_2 are two arbitrary constants.

Remark 13. *In [14], T. Tshikuna-Matamba showed that the method of Oubiña [11], serves to define an hyperKählerian manifold using a 3-Sasakian manifold. Here, for $f=1$ and $h=ce^t$, $(c>0)$, we can see immediatly that the idea of Tshikuna-Matamba is a particular case.*

6. Doubly D-isometric warping

Finally recall the notion of a doubly warped product metric, namely

$$g=Fg_1+fg_2,$$

where f is a positive function on M_1 and F is a positive function on M_2. If now both $(M_1,\varphi_1,\xi_1,\eta_1,g_1)$ and $(M_2,\varphi_2,\xi_2,\eta_2,g_2)$ are almost contact metric manifolds we can define a doubly $\mathscr{D}$-isometrically warped metric by

$$g=g_1+(F^2-1)\eta_1\otimes\eta_1+g_2+(f^2-1)\eta_2\otimes\eta_2,$$

where F and f are two functions non-zero everywhere on M_1 and M_2 respectively. On the other hand, we can introduce a class of almost complex structure J on the product manifold $M_1\times M_2$:

$$\tilde{J}(X_1,X_2)=\Big(\varphi_1X_1-\frac{f}{F}\eta_1(X_1)\xi_2\ ,\ \varphi_2X_2+\frac{F}{f}\eta_2(X_2)\xi_1\Big),$$

then it is easily seen that (J,g) is an almost Hermitian structure on the product $M_1\times M_2$. While this is an area of possible future research.

7. Conclusion

We know that through a conformal and related changes of the metric we can build several bridges between the various known structures (almost complex, almost contact, almost Golden,...). Here, we introduced a certain deformation called " D-isometric warping" and we studied some basic properties. As applications, we constructed a 1-parameter family of Kahlerian structures from a single Sasakian structure with this deformation. Then, a quaternionic Kahlerian structure from a 3-Sasakian structures.

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