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The application domain of infinite matrices with algorithms

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Abstract

The purpose of this paper is twofold. First, we define the new spaces and investigate some topological and structural properties. Also, we compute dual spaces of new spaces which are help us in the characterization of matrix mappings. Second, we give some examples related to new spaces. A flow chart of the stages of the newly constructed sequence spaces and the algorithms of the workings at each step are given.

1. Introduction

It is well known that, the ω denotes the family of all real (or complex)-valued sequences. ω is a linear space and each linear subspace of ω (with the included addition and scalar multiplication) is called a *sequence space* such as the spaces c, c_0 and ℓ_∞, where c, c_0 and ℓ_∞ denote the set of all convergent sequences in fields $\mathbb{R}$ or $\mathbb{C}$, the set of all null sequences and the set of all bounded sequences, respectively. It is clear that the sets c, c_0 and ℓ_∞ are the subspaces of the ω. Thus, c, c_0 and ℓ_∞ equipped with a vector space structure, from a sequence space. By bs and cs, we define the spaces of all bounded and convergent series, respectively.

A coordinate space (or $K-space$) is a vector space of numerical sequences, where addition and scalar multiplication are defined pointwise. That is, a sequence space X with a linear topology is called a K-space provided each of the maps $p_i : X \to \mathbb{C}$ defined by $p_i(x) = x_i$ is continuous for all $i \in \mathbb{N}$. A $K-$space is called an $FK-space$ provided X is a complete linear metric space. An $FK-space$ whose topology is normable is called a $BK-$ *space*.

Let X be a $BK-$space. Then X is said to have monotone norm if $\|x^{[m]}\| \geq \|x^{[n]}\|$ for $m > n$ and $\|x\| = \sup \|x^{[m]}\|$. The spaces c_0, c, ℓ_∞, cs, bs have monotone norms.

If a normed sequence space X contains a sequence (b_n) with the property that for every $x \in X$ there is unique sequence of scalars (α_n) such that

$$\lim_{n\to\infty} \|x - (\alpha_0 b_0 + \alpha_1 b_1 + \ldots + \alpha_n b_n)\| = 0$$

then (b_n) is called *Schauder basis* for X. The series $\sum \alpha_k b_k$ which has the sum x is then called the expansion of x with respect to (b_n), and written as $x = \sum \alpha_k b_k$. An $FK-space$ X is said to have AK property, if $\phi \subset X$ and $\{e^k\}$ is a basis for X, where e^k is a sequence whose only non-zero term is a 1 in k^{th} place for each $k \in \mathbb{N}$ and $\phi = span\{e^k\}$, the set of all finitely non-zero sequences. An $FK-$space $X \supset \phi$ is said to have AB, if $(x^{[n]})$ is a bounded set in X for each $x \in X$.

Let $A = (a_{nk})$ be an infinite matrix of complex numbers a_{nk} and $x = (x_k) \in \omega$, where $k, n \in \mathbb{N}$. Then the sequence Ax is called as the $A-$transform of x defined by the usual matrix product. Hence, we transform the sequence x into the sequence $Ax = \{(Ax)_n\}$ where

$$(Ax)_n = \sum_k a_{nk} x_k \tag{1.1}$$

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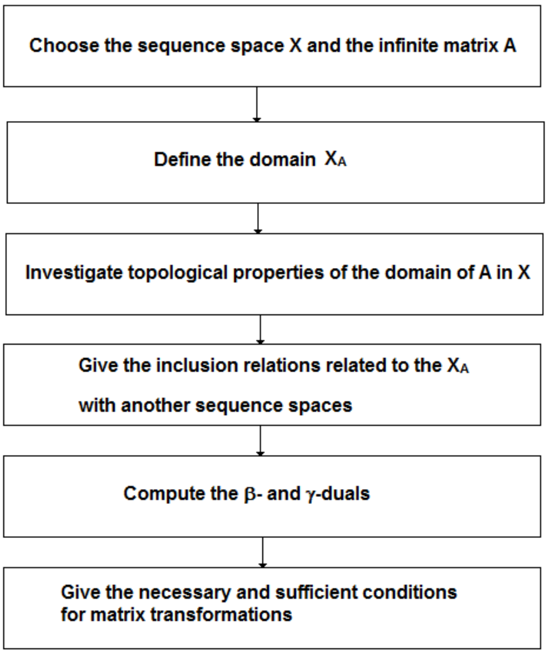

Figure 1.1: Flowchart of constructing a new sequence space

for each $n \in \mathbb{N}$, provided the series on the right hand side of (1.1) converges for each $n \in \mathbb{N}$. Let X and Y be two sequence spaces. If Ax exists and is in Y for every sequence $x = (x_k) \in X$, then we say that A defines a matrix mapping from X into Y, and we denote it by writing $A : X \to Y$ if and only if the series on the right hand side of (1.1) converges for each $n \in \mathbb{N}$ and every $x \in X$, and we have $Ax = \{(Ax)_n\}_{n\in\mathbb{N}} \in Y$ for all $x \in X$. A sequence x is said to be A-summable to l if Ax converges to l which is called the A-limit of x. Let X be a sequence space and A be an infinite matrix. The sequence space

$$X_A = \{x = (x_k) \in \omega : Ax \in X\} \tag{1.2}$$

is called the domain of A in X which is a sequence space.

The matrix $\Omega = (a_{nk})$ defined by $a_{nk} = k$, $(1 \le k \le n)$ and $a_{nk} = 0$, $(k > n)$, and the matrix $\Gamma = (b_{nk})$ defined by by $b_{nk} = 1/k$, $(1 \le k \le n)$ and $b_{nk} = 0$, $(k > n)$, respectively, i.e.,

$$a_{nk} = \begin{bmatrix} 1 & 0 & 0 & 0 & \cdots \\ 1 & 2 & 0 & 0 & \cdots \\ 1 & 2 & 3 & 0 & \cdots \\ 1 & 2 & 3 & 4 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \quad \text{and} \quad b_{nk} = \begin{bmatrix} 1 & 0 & 0 & 0 & \cdots \\ 1 & 1/2 & 0 & 0 & \cdots \\ 1 & 1/2 & 1/3 & 0 & \cdots \\ 1 & 1/2 & 1/3 & 1/4 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

We can give the matrices $\Omega^{-1} = (c_{nk})$ and $\Gamma^{-1} = (d_{nk})$ which are inverse of the above matrices by $c_{nk} = 1/n$, $(n = k)$, $c_{nk} = -1/n$, $(n-1 = k)$, $c_{nk} = 0$, $(other)$ and $d_{nk} = n$, $(n = k)$, $d_{nk} = -n$, $(n-1 = k)$, $d_{nk} = 0$, $(other)$, respectively, i.e.,

$$c_{nk} = \begin{bmatrix} 1 & 0 & 0 & 0 & \cdots \\ -1/2 & 1/2 & 0 & 0 & \cdots \\ 0 & -1/3 & 1/3 & 0 & \cdots \\ 0 & 0 & -1/4 & 1/4 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \quad \text{and} \quad d_{nk} = \begin{bmatrix} 1 & 0 & 0 & 0 & \cdots \\ -2 & 2 & 0 & 0 & \cdots \\ 0 & -3 & 3 & 0 & \cdots \\ 0 & 0 & -4 & 4 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

Now, we show that the matrices Ω and Γ preserve the limits on the set of all convergent sequences.

Theorem 1.1. *The matrices Ω and Γ are regular.*

Proof. Take a sequence $x = (x_k)$. We must show that if for $n \to \infty$ and some L, $\lim_n |x_k - L| \to 0$, then, $\lim_n |b_{nk}x_k - L| \to 0$, where b_{nk} is Γ matrix. Suppose that for $n \to \infty$ and some L, $\lim_n |x_k - L| \to 0$, and choose $\varepsilon > 0$. Then, there exists a positive integer N such that $\lim_n |x_k - L| < \varepsilon$ for $n \ge N$. Then, for $n \ge N$ and $N \in \mathbb{N}$, $\lim_n |b_{nk}x_k - L| = \lim_n |\sum_{k=1}^{n} (k^{-1}x_k - L)| < \varepsilon$. Therefore the matrix Γ is regular.

Similarly, we can show that the matrix Ω is regular. □

The paper is organized into six sections. After the introduction in Section 1, new sequence spaces are constructed in Section 2. Also, some topological properties of these new spaces are investigated in Section 2 . Section 3 describes and computes the dual spaces. The dual spaces are very important for matrix transformations. Section 4 is dedicated to characterization of matrix mappings. Examples related to the new spaces are in Section 5. Finally, Section 6 presents the conclusion(Figure 1.1).

2. New spaces and topological properties

Take an infinite matrix A
Apply to the sequence space X
If the matrix A is a triangle and $f : X_A \to X$ is bijective, **then**
X_A and X are linearly isomorphic
Investigate the topological properties of X_A
If A is triangle, **then**
X_A is a BK-space
Compute the beta- and gamma-duals
do
Characterize the matrix mappings
while(exist beta- and gamma-duals)

Table 1: Algorithm related to the constructing a new space

Now, we introduce the new sequence spaces derived by the $\Omega-$ and $\Gamma-$ matrices as follows:

$$\ell_\infty(\Omega) = \Big\{x = (x_k) \in \omega : \Omega x \in \ell_\infty\}$$
$$c(\Omega) = \{x = (x_k) \in \omega : \Omega x \in c\}$$
$$c_0(\Omega) = \{x = (x_k) \in \omega : \Omega x \in c_0\}$$

and

$$\ell_\infty(\Gamma) = \{x = (x_k) \in \omega : \Gamma x \in \ell_\infty\}$$
$$c(\Gamma) = \{x = (x_k) \in \omega : \Gamma x \in c\}$$
$$c_0(\Gamma) = \{x = (x_k) \in \omega : \Gamma x \in c_0\}$$

Let us define the sequences $u = (u_n)$ and $v = (v_n)$, as the $\Omega-$transform and $\Gamma-$transform of a sequence $x = (x_k)$, respectively, that is, for $k,n \in \mathbb{N}$, $u_n = (\Omega x)_n = \sum_{k=1}^n |kx_k|$ and $v_n = (\Gamma x)_n = \sum_{k=1}^n \left|k^{-1}x_k\right|$.

Theorem 2.1. *The new bounded, convergent and null sequence spaces are norm isomorphic to the classical sets consisting of the bounded, convergent and null sequences.*

Proof. We will show that there is a linear isometry between new bounded, convergent, null sequence spaces and classical bounded, convergent and null convergent sequence space. We consider the transformation defined Φ, from $X(\Omega)$ to X by $x \mapsto u = \Phi x = \sum_{k=1}^n |kx_k|$, where $X = \{\ell_\infty, c, c_0\}$. Then, it is clear that the equality $\Phi(a+b) = \Phi(a) + \Phi(b)$ holds. Choose $\lambda \in \mathbb{R}$. Then,

$$\Phi(\lambda a) = \Phi(\lambda a_k) = \sum_{k=1}^n |\lambda k a_k| = \lambda \sum_{k=1}^n |ka_k| = \lambda \Phi a.$$

Therefore, we can say that Φ is linear.

Choose a sequence $y = (y_k)$ in $X(\Omega)$ and define the sequence $x = (x_k)$ such that $x = (c_{nk}y_k)$, where c_{nk} is inverse of $\Omega = (a_{nk})$ matrix. Then,

$$\|x\|_{\ell_\infty(\Omega)} = \sup_k |a_{nk}x_k| = \sup_k |a_{nk}c_{nk}y_k|_{\ell_\infty} = \|y\|_{\ell_\infty}.$$

Therefore, we can say that Φ is norm preserving.

Similarly, we can also show that the other spaces are norm isomorphic to classical sequence spaces. □

Theorem 2.2. *The new bounded, convergent and null sequence spaces are BK−spaces with the norms defined by* $\|x\|_{X(\Omega)} = \|\Omega x\|_{\ell_\infty}$ *and* $\|x\|_{X(\Gamma)} = \|\Gamma x\|_{\ell_\infty}$*, respectively, where* $X = \{\ell_\infty, c, c_0\}$.

Proof. Take a sequence $x = (x_k)$ in $X(\Omega)$, where $X = \{\ell_\infty, c, c_0\}$ and define $f_k(x) = x_k$ for all $k \in \mathbb{N}$. Then, we have

$$\|x\|_{X(\Omega)} = \sup\{1|x_1| + 2|x_2| + 3|x_3| + \cdots + k|x_k| + \cdots\}$$

Therefore, $k|x_k| \le \|x\|_{X(\Omega)} \Rightarrow |x_k| \le K\|x\|_{X(\Omega)} \Rightarrow |f_k(x)| \le K\|x\|_{X(\Omega)}$. From this result, we say that f_k is a continuous linear functional for each k. Then, $X(\Omega)$ is a $BK-$space.

In the same idea, we can prove that the space $X(\Gamma)$ is a $BK-$space. □

Remark 2.3. *We can give the proof of Theorem 2.2 in a different way: From 4.3.1 of [9], we know that if a sequence space X is $BK-$space with respective norm and A is a triangular infinite matrix, then the matrix domain X_A is also $BK-$space with respective norm.*

Theorem 2.4. *The spaces $X(\Omega)$ and $X(\Gamma)$ have $AK-$property.*

Theorem 2.5. *The spaces $X(\Omega)$ and $X(\Gamma)$ have monotone norm.*

Theorem 2.4 and 2.5 can be proved as Theorem 2.4., Theorem 2.6. of [5].

Remark 2.6. *Any space with a monotone norm has AB(10.3.12 of [9]).*

Corollary 2.7. *The spaces $X(\Omega)$ and $X(\Gamma)$ have AB.*

Theorem 2.8. *The following statements hold:*

(i) Define a sequence $t^{(k)} := \{t_n^{(k)}\}_{n\in\mathbb{N}}$ of elements of the space $X(\Omega)$ for every fixed $k \in \mathbb{N}$ by

$$t_n^{(k)} = \begin{cases} (-1)^{n-k}k^{-1} & , \quad (n-1 \le k \le n) \\ 0 & , \quad (1 \le k \le n-1) \quad or \quad (k > n) \end{cases}$$

Then the sequence $\{t^{(k)}\}_{k\in\mathbb{N}}$ is a basis for the space $X(\Omega)$ and if we choose $E_k = (\Omega x)_k$ for all $k \in \mathbb{N}$, then any $x \in X(\Omega)$ has a unique representation of the form

$$x := \sum_k E_k t^{(k)}.$$

(ii) Define a sequence $s^{(k)} := \{s_n^{(k)}\}_{n\in\mathbb{N}}$ of elements of the space $X(\Gamma)$ for every fixed $k \in \mathbb{N}$ by

$$s_n^{(k)} = \begin{cases} (-1)^{n-k}k & , \quad (n-1 \le k \le n) \\ 0 & , \quad (1 \le k \le n-1) \quad or \quad (k > n) \end{cases}$$

Then the sequence $\{s^{(k)}\}_{k\in\mathbb{N}}$ is a basis for the space $X(\Gamma)$ and if we choose $F_k = (\Gamma x)_k$ for all $k \in \mathbb{N}$, then any $x \in X(\Gamma)$ has a unique representation of the form

$$x := \sum_k F_k s^{(k)}.$$

Remark 2.9. *If a space has a Schauder basis, then it is separable.*

Corollary 2.10. *The spaces $X(\Omega)$ and $X(\Gamma)$ are separable.*

1:	Take X_A
2:	Define $f : X_A \to X$
3:	**If** f is an isomorphic and surjective, **then**
4:	the inverse image of basis of X is the basis of X_A
5:	**If** X has a Schauder basis, **then**
6:	X_A is separable

Table 2: Algorithm for basis and separability

In this section, we have defined the new spaces derived by infinite matrices and examined some structural and topological properties.

3. Dual spaces

In this section, we compute dual spaces of new defined spaces. The beta-, gamma-duals of new defined spaces will help us in the characterization of the matrix mappings.

From Lemma 5.3 of [4] and Theorem 3.1 of [1], we will give an algorithm, which provides convenience to compute $\alpha-$, $\beta-$ and $\gamma-$ duals of these new spaces and characterize some matrix transformations.

Let x and y be sequences, X and Y be subsets of ω and $A = (a_{nk})_{n,k=0}^{\infty}$ be an infinite matrix of complex numbers. We write $xy = (x_k y_k)_{k=0}^{\infty}$, $x^{-1} * Y = \{a \in \omega : ax \in Y\}$ and $M(X,Y) = \bigcap_{x\in X} x^{-1} * Y = \{a \in \omega : ax \in Y \text{ for all } x \in X\}$ for the *multiplier space* of X and Y. In the special cases of $Y = \{\ell_1, cs, bs\}$, we write $x^{\alpha} = x^{-1} * \ell_1$, $x^{\beta} = x^{-1} * cs$, $x^{\gamma} = x^{-1} * bs$ and $X^{\alpha} = M(X,\ell_1)$, $X^{\beta} = M(X,cs)$, $X^{\gamma} = M(X,bs)$ for the $\alpha-$dual, $\beta-$dual, $\gamma-$dual of X. By $A_n = (a_{nk})_{k=0}^{\infty}$ we denote the sequence in the $n-$th row of A, and we write $A_n(x) = \sum_{k=0}^{\infty} a_{nk}x_k$ $n = (0,1,...)$ and $A(x) = (A_n(x))_{n=0}^{\infty}$, provided $A_n \in x^{\beta}$ for all n.

Take the sequence spaces X and Y
If the spaces X and Y are $BK-$spaces, **then**
matrix transformations between X and Y are continuous
Choose the triangular matrix T and an infinite matrix A
do
$A \in (X : Y_T)$
while $TA(X : Y)$
Define the matrix B which is inverse of T
If the matrix B depending on a sequence $(a_k) \in \omega$, **then**
$\beta-$dual is defined by $X_T^{\beta} = \{a = (a_k) \in \omega : B^T \in (X : c)\}$ **and**
$\gamma-$dual is defined by $X_T^{\gamma} = \{a = (a_k) \in \omega : B^T \in (X : \ell_\infty)\}$

Table 3: Algorithm for dual spaces and matrix transformations

Now, we list the following useful conditions.

$$\sup_n \sum_k |a_{nk}| < \infty \tag{3.1}$$

$$\lim_{n\to\infty} a_{nk} - \alpha_k = 0 \tag{3.2}$$

$$\lim_{n\to\infty} \sum_k a_{nk} \quad exists \tag{3.3}$$

$$\lim_{n\to\infty} \sum_k |a_{nk}| = \sum_k \left| \lim_{n\to\infty} a_{nk} \right| \tag{3.4}$$

$$\lim_n a_{nk} = 0 \quad \text{for all k} \tag{3.5}$$

$$\sup_m \sum_k \left| \sum_{n=0}^{m} \right| < \infty \tag{3.6}$$

$$\sum_n a_{nk} \quad is\ convergent\ for\ all\ k \tag{3.7}$$

$$\sum_n \sum_k a_{nk} \quad is\ convergent \tag{3.8}$$

$$\lim_n a_{nk} \quad \text{exists for all k} \tag{3.9}$$

$$\lim_m \sum_k \left| \sum_{n=m}^{\infty} a_{nk} \right| = 0 \tag{3.10}$$

Lemma 3.1. *For the characterization of the class $(X : Y)$ with $X = \{c_0, c, \ell_\infty\}$ and $Y = \{\ell_\infty, c, cs, bs\}$, we can give the necessary and sufficient conditions from Table 4, where*

1. (3.1)	**2.** (3.1), (3.9)	**3.** (3.6)	**4.** (3.6), (3.7)
5. (3.1), (3.9), (3.3)	**6.** (3.6), (3.7), (3.8)	**7.** (3.9), (3.4)	**8.** (3.10)

To →	ℓ_∞	c	bs	cs
From ↓				
c_0	**1.**	**2.**	**3.**	**4.**
c	**1.**	**5.**	**3.**	**6.**
ℓ_∞	**1.**	**7.**	**3.**	**8.**

Table 4

For using in the proof of Theorem 3.2, we define the matrices $U = (u_{nk})$ and $V = (v_{nk})$ as below:

$$u_{nk} = \begin{cases} \frac{a_k}{k} - \frac{a_{k+1}}{k+1} & , \quad (k < n) \\ \frac{a_n}{n} & , \quad (k = n) \\ 0 & , \quad (k > n) \end{cases} \tag{3.11}$$

$$v_{nk} == \begin{cases} ka_k - (k+1)a_{k+1} & , \quad (k < n) \\ na_n & , \quad (k = n) \\ 0 & , \quad (k > n) \end{cases} \tag{3.12}$$

Theorem 3.2. *The $\beta-$ and $\gamma-$ duals of the new sequence spaces defined by*

$$[c_0(\Omega)]^\beta = \{a=(a_k)\in\omega : U\in(c_0:c)\}$$
$$[c(\Omega)]^\beta = \{a=(a_k)\in\omega : U\in(c:c)\}$$
$$[\ell_\infty(\Omega)]^\beta = \{a=(a_k)\in\omega : U\in(\ell_\infty:c)\}$$
$$[c_0(\Omega)]^\gamma = \{a=(a_k)\in\omega : U\in(c_0:\ell_\infty)\}$$
$$[c(\Omega)]^\gamma = \{a=(a_k)\in\omega : U\in(c:\ell_\infty)\}$$
$$[\ell_\infty(\Omega)]^\gamma = \{a=(a_k)\in\omega : U\in(\ell_\infty:\ell_\infty)\}$$

Proof. We will only show the $\beta-$ and $\gamma-$ duals of the new null convergent sequence spaces. Let $a=(a_k)\in\omega$. We begin the equality

$$\sum_{k=1}^{n} a_k x_k = \sum_{k=1}^{n} a_k k^{-1}(y_k - y_{k-1}) = \sum_{k=1}^{n-1}\left(\frac{a_k}{k}-\frac{a_{k+1}}{k+1}\right)y_k + \frac{a_n}{n}y_n = (Uy)_n \tag{3.13}$$

where $U=(u_{nk})$ is defined by (3.11). Using (3.13), we can see that $ax=(a_kx_k)\in cs$ or bs whenever $x=(x_k)\in c_0(\Omega)$ if and only if $Uy\in c$ or ℓ_∞ whenever $y=(y_k)\in c_0$. Then, from the algorithm in Table 3 , we obtain the result that $a=(a_k)\in(c_0(\Omega))^\beta$ or $a=(a_k)\in(c_0(\Omega))^\gamma$ if and only if $U\in(c_0:c)$ or $U\in(c_0:\ell_\infty)$, which is what we wished to prove. □

Theorem 3.3. *The $\beta-$ and $\gamma-$ duals of the new sequence spaces defined by*

$$[c_0(\Gamma)]^\beta = \{a=(a_k)\in\omega : V\in(c_0:c)\}$$
$$[c(\Gamma)]^\beta = \{a=(a_k)\in\omega : V\in(c:c)\}$$
$$[\ell_\infty(\Gamma)]^\beta = \{a=(a_k)\in\omega : V\in(\ell_\infty:c)\}$$
$$[c_0(\Gamma)]^\gamma = \{a=(a_k)\in\omega : V\in(c_0:\ell_\infty)\}$$
$$[c(\Gamma)]^\gamma = \{a=(a_k)\in\omega : V\in(c:\ell_\infty)\}$$
$$[\ell_\infty(\Gamma)]^\gamma = \{a=(a_k)\in\omega : V\in(\ell_\infty:\ell_\infty)\}$$

where $V=(v_{nk})$ is defined by (3.12).

4. Matrix mapping

Let X and Y be arbitrary subsets of ω. We shall show that, the characterizations of the classes (X,Y_T) and (X_T,Y) can be reduced to that of (X,Y), where T is a triangle.

It is well known that if $h_{c_0}(\Delta^{(m)})\cong c_0$, then the equivalence

$$x\in h_{c_0}(\Delta^{(m)}) \Leftrightarrow y\in c_0$$

holds. Then, the following theorems will be proved and given some corollaries which can be obtained to that of Theorems 4.1 and 4.2. Then, using the algorithm in Table 3, we have:

Theorem 4.1. *Consider the infinite matrices $A=(a_{nk})$ and $D=(d_{nk})$. These matrices get associated with each other in the following relations:*
These matr

$$d_{nk} = \frac{a_{nk}}{k} - \frac{a_{n,k+1}}{k+1} \tag{4.1}$$

for all $k,m,n\in\mathbb{N}$. Then, the following statements are true:
i. *$A\in(c_0(\Omega):Y)$ if and only if $\{a_{nk}\}_{k\in\mathbb{N}}\in[c_0(\Omega)]^\beta$ for all $n\in\mathbb{N}$ and $D\in(c_0:Y)$, where Y is any sequence space.*
ii. *$A\in(c(\Omega):Y)$ if and only if $\{a_{nk}\}_{k\in\mathbb{N}}\in[c(\Omega)]^\beta$ for all $n\in\mathbb{N}$ and $D\in(c:Y)$, where Y is any sequence space.*
iii. *$A\in(\ell_\infty(\Omega):Y)$ if and only if $\{a_{nk}\}_{k\in\mathbb{N}}\in[\ell_\infty(\Omega)]^\beta$ for all $n\in\mathbb{N}$ and $D\in(\ell_\infty:Y)$, where Y is any sequence space.*

Proof. We assume that the (4.1) holds between the entries of $A=(a_{nk})$ and $D=(d_{nk})$. Let us remember that from Theorem 2.1, the spaces $c_0(\Omega)$ and c_0 are linearly isomorphic. Firstly, we choose any $y=(y_k)\in c_0$ and consider $A\in(c_0(\Omega):Y)$. Then, we obtain that $D\Omega$ exists and $\{a_{nk}\}\in(c_0(\Omega))^\beta$ for all $k\in\mathbb{N}$. Therefore, the necessity of (4.1) yields and $\{d_{nk}\}\in c_0^\beta$ for all $k,n\in\mathbb{N}$. Hence, Dy exists for each $y\in c_0$. Thus, if we take $m\to\infty$ in the equality

$$\sum_{k=1}^{m} a_{nk}x_k = \sum_{k=1}^{m} a_{nk}\left(\frac{a_{nk}}{k}-\frac{a_{n,k+1}}{k+1}\right)y_k$$

for all $m,n\in\mathbb{N}$, then, we understand that $Dy=Ax$. So, we obtain that $D\in(c_0:Y)$.

Now, we consider that $\{a_{nk}\}_{k\in\mathbb{N}}\in(c_0(\Omega))^\beta$ for all $n\in\mathbb{N}$ and $D\in(c_0:Y)$. We take any $x=(x_k)\in c_0(\Omega)$. Then, we can see that Ax exists. Therefore, from the equality

$$\sum_k d_{nk} y_k = \sum_k a_{nk} x_k$$

for all $n \in \mathbb{N}$, we obtain that $Ax = Dy$. Therefore, this shows that $A \in (c_0(\Omega) : Y)$. □

Theorem 4.2. *Consider that the infinite matrices $A = (a_{nk})$ and $E = (e_{nk})$ with*

$$e_{nk} := \sum_{k=1}^{\infty} \sum_{j=1}^{n} a_{jk}. \tag{4.2}$$

Then, the following statements are true:
i. *$A = (a_{nk}) \in (X : c_0(\Omega))$ if and only if $E \in (X : c_0)$*
ii. *$A = (a_{nk}) \in (X : c(\Omega))$ if and only if $E \in (X : c)$*
iii. *$A = (a_{nk}) \in (X : \ell_\infty(\Omega))$ if and only if $E \in (X : \ell_\infty)$*

Proof. We take any $z = (z_k) \in X$. Using the (4.2), we have

$$\sum_{k=1}^{m} e_{nk} z_k = \sum_{k=1}^{m} \left(\sum_{k=1}^{\infty} \sum_{j=1}^{m} j b_{jk} \right) z_k \tag{4.3}$$

for all $m, n \in \mathbb{N}$. Then, for $m \to \infty$, equation (4.3) gives us that $(Ez)_n = \{\Omega(Az)\}_n$. Therefore, one can immediately observe from this that $Az \in c_0(\Omega)$ whenever $z \in X$ if and only if $Ez \in c_0$ whenever $z \in X$. Thus, the proof is completed. □

Theorem 4.3. *Consider the infinite matrices $A = (a_{nk})$ and $F = (f_{nk})$. These matrices get associated with each other in the following relations:*

$$f_{nk} = k a_{nk} - (k+1) a_{n,k+1} \tag{4.4}$$

for all $k, m, n \in \mathbb{N}$. Then, the following statements are true:
i. *$A \in (c_0(\Gamma) : Y)$ if and only if $\{a_{nk}\}_{k \in \mathbb{N}} \in [c_0(\Gamma)]^\beta$ for all $n \in \mathbb{N}$ and $F \in (c_0 : Y)$, where Y is any sequence space.*
ii. *$A \in (c(\Gamma) : Y)$ if and only if $\{a_{nk}\}_{k \in \mathbb{N}} \in [c(\Gamma)]^\beta$ for all $n \in \mathbb{N}$ and $F \in (c : Y)$, where Y is any sequence space.*
iii. *$A \in (\ell_\infty(\Gamma) : Y)$ if and only if $\{a_{nk}\}_{k \in \mathbb{N}} \in [\ell_\infty(\Gamma)]^\beta$ for all $n \in \mathbb{N}$ and $F \in (\ell_\infty : Y)$, where Y is any sequence space.*

Theorem 4.4. *Consider that the infinite matrices $A = (a_{nk})$ and $G = (g_{nk})$ with*

$$g_{nk} := \sum_{k=1}^{\infty} \sum_{j=1}^{n} j^{-1} a_{jk} \tag{4.5}$$

Then, the following statements are true:
i. *$A = (a_{nk}) \in (X : c_0(\Gamma))$ if and only if $G \in (X : c_0)$*
ii. *$A = (a_{nk}) \in (X : c(\Gamma))$ if and only if $G \in (X : c)$*
iii. *$A = (a_{nk}) \in (X : \ell_\infty(\Gamma))$ if and only if $G \in (X : \ell_\infty)$*

5. Examples

If we choose any sequence spaces X and Y in Theorem 4.1 and 4.2 in previous section, then, we can find several consequences in every choice. For example, if we take the space ℓ_∞ and the spaces which are isomorphic to ℓ_∞ instead of Y in Theorem 4.1, we obtain the following examples:

Example 5.1. *The Euler sequence space e^r_∞ is defined by $e^r_\infty = \{x \in \omega : \sup_{n \in \mathbb{N}} |\sum_{k=0}^{n} \binom{n}{k} (1-r)^{n-k} r^k x_k| < \infty\}$ ([2] and [3]). We consider the infinite matrix $A = (a_{nk})$ and define the matrix $H = (h_{nk})$ by*

$$h_{nk} = \sum_{j=0}^{n} \binom{n}{j} (1-r)^{n-j} r^j a_{jk} \qquad (k, n \in \mathbb{N}).$$

If we want to get necessary and sufficient conditions for the class $(c_0(\Omega) : e^r_\infty)$ in Theorem 4.1, then, we replace the entries of the matrix A by those of the matrix H.

Example 5.2. *Let $T_n = \sum_{k=0}^{n} t_k$ and $A = (a_{nk})$ be an infinite matrix. We define the matrix $P = (p_{nk})$ by*

$$p_{nk} = \frac{1}{T_n} \sum_{j=0}^{n} t_j a_{jk} \qquad (k, n \in \mathbb{N}).$$

Then, the necessary and sufficient conditions in order for A belongs to the class $(c_0(\Omega) : r^t_\infty)$ are obtained from in Theorem 4.1 by replacing the entries of the matrix A by those of the matrix P; where r^t_∞ is the space of all sequences whose R^t-transforms is in the space ℓ_∞ [7].

Example 5.3. *In the space r^t_∞, if we take $t = e$, then, this space become to the Cesaro sequence space of non-absolute type X_∞ [8]. As a special case, Example 5.2 includes the characterization of class $((c_0(\Omega) : r^t_\infty)$.*

Example 5.4. *The Taylor sequence space t_∞^r is defined by $t_\infty^r = \{x \in \omega : \sup_{n\in\mathbb{N}} |\sum_{k=n}^{\infty} \binom{k}{n}(1-r)^{n+1} r^{k-n} x_k| < \infty\}$ ([6]). We consider the infinite matrix $A = (a_{nk})$ and define the matrix $T = (t_{nk})$ by*

$$t_{nk} = \sum_{k=n}^{\infty} \binom{k}{n} (1-r)^{n+1} r^{k-n} a_{jk} \qquad (k,n \in \mathbb{N}).$$

If we want to get necessary and sufficient conditions for the class $(c_0(\Omega) : t_\infty^r)$ in Theorem 4.1, then, we replace the entries of the matrix A by those of the matrix T.

If we take the spaces c, cs and bs instead of X in Theorem 4.2, or Y in Theorem 4.1 we can write the following examples. Firstly, we give some conditions and following lemmas:

$$\lim_k a_{nk} = 0 \quad \text{for all n}, \tag{5.1}$$

$$\lim_{n\to\infty} \sum_k a_{nk} = 0, \tag{5.2}$$

$$\lim_{n\to\infty} \sum_k |a_{nk}| = 0, \tag{5.3}$$

$$\lim_{n\to\infty} \sum_k |a_{nk} - a_{n,k+1}| = 0, \tag{5.4}$$

$$\sup_n \sum_k \left|a_{nk} - a_{n,k+1}\right| < \infty \tag{5.5}$$

$$\lim_k \left(a_{nk} - a_{n,k+1}\right) \text{ exists for all k} \tag{5.6}$$

$$\lim_{n\to\infty} \sum_k |a_{nk} - a_{n,k+1}| = \sum_k \left|\lim_{n\to\infty} (a_{nk} - a_{n,k+1})\right| \tag{5.7}$$

$$\sup_n \left|\lim_k a_{nk}\right| < \infty \tag{5.8}$$

Lemma 5.5. *Consider that $X \in \{\ell_\infty, c, bs, cs\}$ and $Y \in \{c_0\}$. The necessary and sufficient conditions for $A \in (X : Y)$ can be read from the Table 5:*

9. (5.3)	**10.** (3.1), (3.5), (5.2)	**11.** (5.1), (5.4)	**12.** (3.5), (5.5)
13. (5.1), (5.6), (5.7)	**14.** (5.5), (3.9)	**15.** (5.1), (5.5)	**16.** (5.5), (5.8)

From →	ℓ_∞	c	bs	cs
To ↓				
c_0	**9.**	**10.**	**11.**	**12.**
c	**7.**	**5.**	**13.**	**14.**
ℓ_∞	**1.**	**1.**	**15.**	**16.**

Table 5

Example 5.6. *We choose $X \in \{c_0(\Omega), c(\Omega), \ell_\infty(\Omega)\}$ and $Y \in \{\ell_\infty, c, cs, bs, f\}$. The necessary and sufficient conditions for $A \in (X : Y)$ can be taken from the Table 6:*

1a. (3.1) holds with d_{nk} instead of a_{nk}.
2a. (3.1), (3.9) hold with d_{nk} instead of a_{nk}.
3a. (3.6) holds with d_{nk} instead of a_{nk}.
4a. (3.6), (3.7) hold with d_{nk} instead of a_{nk}.
5a. (3.1), (3.9), (3.3) hold with d_{nk} instead of a_{nk}.
6a. (3.6), (3.7), (3.8) hold with d_{nk} instead of a_{nk}.
7a. (3.9), (3.4) hold with d_{nk} instead of a_{nk}.
8a. (3.10) holds with d_{nk} instead of a_{nk}.

To →	ℓ_∞	c	bs	cs
From ↓				
$c_0(\Omega)$	**1a.**	**2a.**	**3a.**	**4a.**
$c(\Omega)$	**1a.**	**5a.**	**3a.**	**6a.**
$\ell_\infty(\Omega)$	**1a.**	**7a.**	**3a.**	**8a.**

Table 6

Example 5.7. *Consider that the $X \in \{\ell_\infty, c, bs, cs\}$ and $Y \in \{c_0(\Omega), c(\Omega), \ell_\infty(\Omega)\}$. The necessary and sufficient conditions for $A \in (X : Y)$ can be read from the Table 7:*

9a. *(5.3) holds with e_{nk} instead of a_{nk}.*
10a. *(3.1), (3.5), (5.2) hold with e_{nk} instead of a_{nk}.*
11a. *(5.1), (5.4) hold with e_{nk} instead of a_{nk}.*
12a. *(3.5), (5.5) hold with e_{nk} instead of a_{nk}.*
13a. *(5.1), (5.6), (5.7) hold with e_{nk} instead of a_{nk}.*
14a. *(5.5), (3.9) hold with e_{nk} instead of a_{nk}.*
15a. *(5.1), (5.5) hold with e_{nk} instead of a_{nk}.*
16a. *(5.5), (5.8) hold with e_{nk} instead of a_{nk}.*

From $\rightarrow$	ℓ_∞	c	bs	cs
To $\downarrow$				
$c_0(\Omega)$	**9a.**	**10a.**	**11a.**	**12a.**
$c(\Omega)$	**7a.**	**5a.**	**13a.**	**14a.**
$\ell_\infty(\Omega)$	**1a.**	**1a.**	**15a.**	**16a.**

Table 7

With the same idea of Example 5.6 and Example 5.7, we can write the examples related to the Γ matrix as table form. In examples which are writing with Γ matrix, we use the f_{nk} and g_{nk}.

6. Conclusion

We know that the most general linear operators between two sequence spaces is given by an infinite matrix. The theory of matrix transformations deals with establishing necessary and sufficient conditions on the entries of a matrix to map a sequence space X into a sequence space Y. This is a natural generalization of the problem to characterize all summability methods given by infinite matrices that preserve convergence.

In this work, we construct new sequence spaces by means of the matrix domain with two infinite matrices. We examine some properties such as isomorphism, $BK-$space, $AK-$ and $AB-$properties, monotone norm. Also, we give dual spaces and later the necessary and sufficient conditions on the matrix transformations of the classes $(X : Y)$. Afterward, in the last section, we obtain several examples related to new spaces.

In this paper, a flowchart showing the stages of the formation of a new sequence space is designed. Algorithms have been produced to construction of a new sequence space, base, separability, calculation of dual spaces and matrix characterizations.

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Separable solutions to a brood-parasite dynamics model

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Abstract

Common Cuckoo is a brood-parasite which lays its egg in the nest of other bird species and use them to raise its young. We present a Common Cuckoo and a host bird interaction deterministic model taking into account maternal care of offspring. The model consists of a coupled system of integro-partial differential equations subject to the conditions of the integral type. Number of equations in the system depends on a biologically possible maximal number of eggs of the same clutch laid by a host bird. Separable solutions of this model are studied.

1. Introduction

Brood parasites are organisms that use of host individuals either of the same or different species to raise the young of the brood-parasite. We consider the Common Cuckoo (*Cuculus canorus*), formerly European Cuckoo, and host birds interaction deterministic model. Cuckoo is a brood-parasite, which lays its eggs in the nests of other bird species, particularly of Dunnocks, Meadow Pipits, and Eurasian Reed Warblers. The cuckoo egg hatches earlier than eggs of the host bird. Cuckoo chick is much larger than its hosts [1]. It grows faster and monopolizes food supplied by the host parents [2]. Shortly after hatching it evicts all host eggs and chicks by rolling and pushing the other eggs and chicks out of the nest [2]. For the sake of simplicity we assume that it evicts all host chicks and eggs immediately after hatching and that the host bird takes care of only one cuckoo's chick living in the nest. If the hen cuckoo is out-of-phase with the host eggs, she will eat them all so that the hosts are forced to start another brood [2, 3].
In this paper, we present a common cuckoo and a host species interaction deterministic model described by a coupled system of integro-PDEs and prove the existence of its separable solutions. We take into account age of birds and a finite set of eggs in the nest and generalize a one-sex population model given in [4]. We assume that all individuals have pre-reproductive, reproductive, and post-reproductive age intervals. Individuals of reproductive age are divided into single and those who care of young offspring. Individuals of pre-reproductive age are divided into young (under maternal care) and juvenile classes. Juveniles can live without maternal care but cannot produce their offspring. It is assumed that after the death of mother all her young offspring die.
For the sake of simplicity, we consider (i) the joint parental care period which consists of the incubation and chick feeding periods and (ii) the same reproductive period for cuckoos and host birds. We also assume that the brood parasite lays his egg before incubation of clutch has started and do not take into account migration of cuckoos. To the best of our knowledge deterministic differential models have not been used yet for description of the interaction of cuckoos and host bird species.
The paper is organized as follows. In Section 2 we formulate the problem. In Section 3 we consider separable solutions of the model. Concluding remarks are given in Section 5.

2. Notation

$(0,T)$ and (T_1,T_3) $(T<T_1<T_3,\ T<T_3-T_1)$: the child care and reproductive age intervals, respectively, (the same for host birds and cuckoos),
$u(t,\tau_1)$: the age density of host birds aged τ_1 at time t who are of juvenile $(\tau_1\in(T,T_1))$, single $(\tau_1\in(T_1,T_3))$, or post-reproductive $(\tau_1>T_3)$ age,
$u_k(t,\tau_1,\tau_2)$: the age density of host birds aged τ_1 at time t who take care of their k, $1\le k\le n$, offspring aged τ_2 at the same time,
$v(t,\tau_1)$: the natural death rate of host birds aged $\tau_1>T$ at time t who are of juvenile or adult age,

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$\nu_k(t,\tau_1,\tau_2)$: the natural death rate of host birds aged τ_1 at time t who take care of their k offspring aged τ_2,
$\nu_{ks}(t,\tau_1,\tau_2)$: the natural death rate of $k-s$ host young offspring aged τ_2 at time t whose mother is aged τ_1 at the same time,
$\alpha_k(t,\tau_1)u(t,\tau_1)d\tau_1 dt$, $\alpha_k(t,\tau_1)<1$,: the average number of host birds of age from interval $[\tau_1,\tau_1+d\tau_1]$, $\tau_1\in(T_1,T_3)$, at time t who lay k eggs in their nest in time interval $[t,t+dt]$,
$u_c(t,\tau_1,\tau_2)$: the age density of host birds aged τ_1 at time t who take care of a cuckoo chick aged τ_2,
$\nu_c(t,\tau_1,\tau_2)$: the natural death rate of host birds aged τ_1 at time t who take care of a cuckoo chick aged τ_2,
$\nu_{c0}(t,\tau_1,\tau_2)$: the natural death rate of cuckoo chick aged τ_2 at time t whose host mother is aged τ_1 at the same time,
$\alpha_{ck}(t,\tau_1)\alpha_k(t,\tau_1)u(t,\tau_1)d\tau_1 dt$, $0<\alpha_{ck}(t,\tau_1)<1$,: the average number of nests formed of one Cuckoo's and k of host bird eggs laid in time interval $[t,t+dt]$ by host birds of age from interval $[\tau_1,\tau_1+d\tau_1]$,
$f(t,\tau_c)$: the age density of Cuckoos aged τ_c at time t who are of juvenile ($\tau_c\in(T,T_1)$), reproductive ($\tau_c\in(T_1,T_3)$), or post-reproductive ($\tau_c>T_3$) age,
$\nu_f(t,\tau_c)$: the natural death rate of Cuckoos aged τ_c at time t,
$u_0(\tau_1), u_{k0}(\tau_1,\tau_2), u_{c0}(\tau_1,\tau_2), f_0(\tau_c)$: the initial age distributions,
$T_2=T_1+T$: the minimal age of an individual finishing care of offspring of the first generation,
$T_4=T_3+T$: the maximal age of an individual finishing care of offspring of the last generation,
$\alpha=\sum_{k=1}^n\alpha_k$, $\tilde\nu_k=\nu_k+\sum_{s=0}^{k-1}\nu_{ks}$,
$Q=\{(\tau_1,\tau_2):\tau_1\in(T_1+\tau_2,T_3+\tau_2),\ \tau_2\in(0,T)\}$.

3. The model

In this section we present a deterministic model for co-evolution of an age-structured population of host birds and cuckoos taking into account a finite number of eggs in the nest. We assume that all young offspring become juveniles at age $\tau_1=T$ and all juveniles become adults at the age $\tau_1=T_1$. Let n be the biologically possible maximal number of eggs of prey laid in the nest. Denote

$$L_1u=\partial_t u+\partial_{\tau_1}u+\nu u, \tag{3.1}$$

$$L_2z=\partial_t z+\partial_{\tau_1}z+\partial_{\tau_2}z \quad \text{for } z=u_c,\,u_k. \tag{3.2}$$

The model is composed of the following coupled system of integro-differential equations:

$$\begin{cases} L_2u_c+\partial_{\tau_2}u_c+(\nu_c+\nu_{c0})u_c=0, \quad (\tau_1,\tau_2)\in Q, \quad t>0,\\ u_c\big|_{\tau_2=0}=\sum\limits_{k=1}^n\alpha_{ck}\alpha_k u, \quad \tau_1\in(T_1,T_3), \quad t\ge 0,\\ u_c\big|_{t=0}=u_{c0}, \quad (\tau_1,\tau_2)\in Q, \end{cases} \tag{3.3}$$

$$\begin{cases} L_2u_n+(\nu_n+\sum\limits_{s=0}^{n-1}\nu_{ns})u_n=0, \quad (\tau_1,\tau_2)\in Q, \quad t>0,\\ L_2u_k+\big(\nu_k+\sum\limits_{s=0}^{k-1}\nu_{ks}\big)u_k\\ \qquad =\sum\limits_{s=k+1}^{n}\nu_{sk}u_s, \quad 1\le k\le n-1, \quad (\tau_1,\tau_2)\in Q, \quad t>0,\\ u_k\big|_{t=0}=u_{k0}, \quad (\tau_1,\tau_2)\in Q, \quad k=1,\dots,n,\\ u_k\big|_{\tau_2=0}=\alpha_k u(1-\alpha_{ck}), \quad \tau_1\in(T_1,T_3), \quad t\ge0, \quad k=1,\dots,n, \end{cases} \tag{3.4}$$

$$L_1u=\begin{cases} 0, \quad \tau_1\in(T,T_1)\cup(T_4,\infty), \quad t>0,\\ -\alpha u+\int\limits_0^{\tau_1-T_1}\big(\sum\limits_{k=1}^n\nu_{k0}u_k+\nu_{c0}u_c\big)d\tau_2, \quad \tau_1\in(T_1,T_2), \quad t>0,\\ -\alpha u+\int\limits_0^{T}\big(\sum\limits_{k=1}^n\nu_{k0}u_k+\nu_{c0}u_c\big)d\tau_2\\ \qquad +\big(\sum\limits_{k=1}^n u_k+u_c\big)|_{\tau_2=T}, \quad \tau_1\in(T_2,T_3), \quad t>0,\\ \int\limits_{\tau_1-T_3}^{T}\big(\sum\limits_{k=1}^n\nu_{k0}u_k+\nu_{c0}u_c\big)d\tau_2\\ \qquad +\big(\sum\limits_{k=1}^n u_k+u_c\big)|_{\tau_2=T}, \quad \tau_1\in(T_3,T_4), \quad t>0,\\ u\big|_{\tau_1=T}=\int\limits_{T_2}^{T_4}\sum\limits_{k=1}^n ku_k\big|_{\tau_2=T}d\tau_1, \quad t\ge0,\\ u\big|_{t=0}=u_0, \quad \tau_1\in[T,\infty),\\ u\big|_{\tau_1=T_i-0}=u\big|_{\tau_1=T_i+0}, \quad i=1,2,3,4, \quad t\ge0, \end{cases} \tag{3.5}$$

$$\begin{cases} \partial_t f + \partial_{\tau_c} f = -\nu_f f, \ \tau_c > T, t > 0, \\ f\big|_{\tau_c = T} = \int\limits_{T_2}^{T_4} u_c\big|_{\tau_2 = T} d\tau_1, \quad t \geq 0, \\ f\big|_{t=0} = f_0, \quad \tau_c \in [T, \infty). \end{cases} \tag{3.6}$$

Here ∂_t and ∂_{τ_k} signify partial derivatives. We describe fraction α_{ck} by the function

$$\alpha_{ck}(t, \tau_1) = \frac{\int\limits_{T_1}^{T_3} \beta_k(t, \tau_1, \tau_c) f(t, \tau_c) d\tau_c}{\int\limits_{T_1}^{T_3} f(t, \tau_c) d\tau_c}. \tag{3.7}$$

The first term on the right-hand side in Eq. (3.5) is conditioned by individuals who produces offspring, the second and third terms are conditioned by individuals whose all young offspring die and who finish child care, respectively. The transition term $\sum\limits_{s=0}^{k-1} \nu_{ks} u_k$ on the left-hand side in Eq. (3.4) is conditioned by individuals aged τ_1 at time t who take care of k young offspring and whose at least one young offspring dies. Similarly, the term on the right-hand side in this equation is conditioned by individuals aged τ_1 at time t who take care of more than $k, 1 \leq k \leq n-1$, young offspring aged τ_2 whose number after the death of the other offspring is equal to k. As follows from the foregoing, the given functions $\nu, \nu_k, \nu_{ks}, \nu_c, \nu_{c0}, \nu_f, \alpha_k, \alpha_{ck}, u_0, u_{k0}, u_{c0}, f_0$ must be positive supported. Constants T, T_1, and T_3 are assumed to be given and positive. The assumptions $T < T_1$, $T < T_3 - T_1$ given in Section 2 are natural.

Densities of offspring of hosts and cuckoo we define by formulas

$$u(t, \tau_2) = \int\limits_{T_1+\tau_2}^{T_3+\tau_2} \sum_{k=1}^{n} k u_k(t, \tau_1, \tau_2) d\tau_1, \quad f(t, \tau_2) = \int\limits_{T_1+\tau_2}^{T_3+\tau_2} u_c(t, \tau_1, \tau_2) d\tau_1 \tag{3.8}$$

where $\tau_2 \in [0, T]$.

4. Separable solutions to problem (1)–(7)

In this section we restrict ourselves to the case where the vital rates ν, ν_c, ν_{c0}, ν_f, ν_k, ν_{ks}, α_k, α_{ck} and β_k do not depend on t. We seek solutions of the form

$$\begin{cases} u = Uv(\tau_1)\rho(t, \tau_1, \lambda), \ v(T) = 1, \\ u_k = Uv(\tau_1 - \tau_2) v_k(\tau_1, \tau_2) \rho(t, \tau_1, \lambda), \\ u_c = Uv(\tau_1 - \tau_2) v_c(\tau_1, \tau_2) \rho(t, \tau_1, \lambda), \\ f = Uw(\tau_c) \rho(t, \tau_c, \lambda), \end{cases} \tag{4.1}$$

$$\begin{cases} u_0 = Uv(\tau_1)\rho(0, \tau_1, \lambda), \\ u_{k0} = Uv(\tau_1 - \tau_2) v_k(\tau_1, \tau_2) \rho(0, \tau_1, \lambda) \\ u_{c0} = Uv(\tau_1 - \tau_2) v_c(\tau_1, \tau_2) \rho(0, \tau_1, \lambda) \\ f_0 = Uw(\tau_c) \rho(0, \tau_c, \lambda), \end{cases} \tag{4.2}$$

where $\rho(t, \tau_1, \lambda) = \exp\{\lambda(t - \tau_1 + T)\}$, $U > 0$ is an arbitrary constant while constant λ and functions v, v_k, v_c, and w are to be determined. Obviously, separable solutions are the steady-state solutions if $\lambda = 0$, die if $\lambda < 0$, and grow if $\lambda > 0$.

Theorem 4.1. *Let ν and ν_f, β_k, α_k, and functions ν_k, ν_{ks}, ν_c, ν_{c0} be positive in domains $[T, \infty)$, $[T_1, T_3] \times [T_1, T_3]$, $[T_1, T_3]$, and $\overline{Q}$, respectively, and let $\alpha < 1$ in $[T_1, T_3]$, $\beta_k < 1$ in $[T_1, T_3] \times [T_1, T_3]$.*

If $\beta_k \in C^{1,0}\big([T_1, T_3] \times [T_1, T_3]\big)$, ν_k, ν_{ks}, ν_c, and $\nu_{c0} \in C^0(\overline{Q}) \cap C^{10}(Q)$, $\alpha_k \in C^0([T_1, T_3] \cap C^1(T_1, T_3)$, ν and $\nu_f \in C^0[T, \infty)$, then system (1)–(7) has at least one class of positive separable solutions of type (4.1), (4.2).

If $\partial_{\tau_c} \beta_k = 0$ and $\beta_k \in C^0([T_1, T_3] \cap C^1((T_1, T_3))$, then system (1)–(7) has only one class of positive separable solutions of type (4.1), (4.2).

In both cases of β_k, v_c and $v_k \in C^0(\overline{Q}) \cap C^1(Q)$, $k = 1, \ldots, n$, $v \in C^0([T, \infty)) \cap C^1\big((T, \infty) \setminus \{T_1, T_2, T_3, T_4\}\big)$.

Proof. Inserting Eqs. (4.1), (4.2) into (1)–(7) we derive equations for v_c, v_k, w, v,

$$\begin{cases} \partial_{\tau_1} v_c + \partial_{\tau_2} v_c + (\nu_c + \nu_{c0}) v_c = 0 \quad \text{in } Q, \\ v_c(\tau_1, 0) = \sum\limits_{k=1}^{n} \alpha_k(\tau_1) q_k(\tau_1, \lambda), \quad \tau_1 \in (T_1, T_3), \end{cases} \tag{4.3}$$

$$\begin{cases} \partial_{\tau_1} v_n + \partial_{\tau_2} v_n + \tilde{\nu}_n v_n = 0 \quad \text{in } Q, \\ \partial_{\tau_1} v_k + \partial_{\tau_2} v_k + \tilde{\nu}_k v_k = \sum\limits_{s=k+1}^{n} \nu_{sk} v_s, \quad 1 \leq k \leq n-1 \text{ in } Q, \\ v_k(\tau_1, 0) = \alpha_k(1 - q_k(\tau_1, \lambda)), \quad k = 1, \ldots, n, \quad \tau_1 \in (T_1, T_3), \end{cases} \tag{4.4}$$

$$\begin{cases} w' = -\nu_f w \quad \text{in } (T,\infty) \\ w(T) = \int\limits_{T_1}^{T_3} v(x)\nu_c(x+T,T)\exp\{-\lambda x\}dx, \end{cases} \tag{4.5}$$

$$v' + \nu v = \begin{cases} 0 \quad \text{in} \quad (T,T_1)\cup(T_4,\infty), \quad v(T)=1, \\ -\alpha v + \int\limits_{T_1}^{\tau_1} K(\tau_1,\tau_1-x)v(x)dx \quad \text{in } (T_1,T_2), \\ -\alpha v + \int\limits_{\tau_1-T}^{\tau_1} K(\tau_1,\tau_1-x)v(x)dx + A(\tau_1)v(\tau_1-T) \\ \quad \text{in } (T_2,T_3), \\ \int\limits_{\tau_1-T}^{T_3} K(\tau_1,\tau_1-x)v(x)dx + A(\tau_1)v(\tau_1-T) \quad \text{in } (T_3,T_4), \\ v(T_i-0) = v(T_i+0), \quad i=1,2,3,4, \end{cases} \tag{4.6}$$

and the characteristic equation for λ,

$$\int\limits_{T_1}^{T_3} \exp\{-\lambda x\} \sum_{k=1}^{n} k\nu_k(x+T,T)v(x)dx = 1 \tag{4.7}$$

where

$q_k(\tau_1,\lambda) = \int\limits_{T_1}^{T_3} \beta_k(\tau_1,x)w(x)\exp\{-\lambda x\}dx\Big(\int\limits_{T_1}^{T_3} w(x)\exp\{-\lambda x\}dx\Big)^{-1}$,

$K(\tau_1,\tau_2,\lambda) = \sum\limits_{k=1}^{n} \nu_{k0}(\tau_1,\tau_2)\nu_k(\tau_1,\tau_2) + \nu_{c0}(\tau_1,\tau_2)\nu_c(\tau_1,\tau_2)$,

$A(\tau_1,\lambda) = \sum\limits_{k=1}^{n} \nu_k(\tau_1,T) + \nu_c(\tau_1,T)$.

Here and in what follows the prime indicates differentiation.
We integrate Eq. (4.5) obtaining

$$w(\tau_c) = w(T)\exp\Big\{-\int\limits_{T}^{\tau_c} \nu_f(\xi)d\xi\Big\}. \tag{4.8}$$

Therefore

$$q_k(\tau_1,\lambda) = \frac{\int_{T_1}^{T_3} \beta_k(\tau_1,x)\exp\{-\lambda x - \int_T^x \nu_f(\xi)d\xi\}\,dx}{\int_{T_1}^{T_3} \exp\{-\lambda x - \int_T^x \nu_f(\xi)d\xi\}\,dx}.$$

Then integrating Eqs. (4.3) and (4.4) we determine functions ν_c and ν_n,

$$\nu_c(\tau_1,\tau_2) = \sum_{k=1}^{n} \alpha_k(\tau_1-\tau_2)q_k(\tau_1-\tau_2,\lambda)\times\exp\{-\int\limits_0^{\tau_2}(\nu_c(x+\tau_1-\tau_2,x)+\nu_{c0}(x+\tau_1-\tau_2,x))\,dx\}, \tag{4.9}$$

$$\nu_n(\tau_1,\tau_2) = \alpha_n(\tau_1-\tau_2)(1-q_n(\tau_1-\tau_2,\lambda))\exp\Big\{-\int\limits_0^{\tau_2}\tilde{\nu}_n(x+\tau_1-\tau_2,x)\,dx\Big\} \tag{4.10}$$

and derive equations for ν_k, $k=1,...,n-1$,

$$\begin{aligned} \nu_k(\tau_1,\tau_2) &= \alpha_k(\tau_1-\tau_2)(1-q_k(\tau_1-\tau_2,\lambda)) \\ &\times\exp\{-\int\limits_0^{\tau_2}\tilde{\nu}_k(x+\tau_1-\tau_2,x)\,dx\} + \int\limits_0^{\tau_2}\sum_{s=k+1}^{n}(\nu_{sk}\nu_s)(y+\tau_1-\tau_2,y)\exp\{-\int\limits_y^{\tau_2}\tilde{\nu}_k(x+\tau_1-\tau_2,x)\,dx\}dy \end{aligned} \tag{4.11}$$

Equation (4.11) can be solved in the recurrent way starting with $k=n-1$ and using function (4.10). It is evident that ν_c and $\nu_k \in C^0(\overline{Q})\cap C^1(Q)$, $k=1,\ldots,n$.
Now we solve Eq. (4.6). From $(4.6)_1$ for $\tau_1\in[T,T_1]$ it follows that

$$v(\tau_1) = \exp\Big\{-\int_T^{\tau_1} \nu(\xi)\,d\xi\Big\}.$$

To determine v for $\tau_1 \in (T_1, T_2]$ we integrate Eq. (4.6)$_2$ together with the initial condition $v(T_1) = \exp\{-\int_T^{T_1} \nu(\xi)\,d\xi\}$ getting

$$v(\tau_1) = v(T_1)\exp\Big\{-\int_{T_1}^{\tau_1}(\nu(\xi)+\alpha(\xi))d\xi\Big\}v(\tau_1) + \int_{T_1}^{\tau_1}\exp\Big\{-\int_y^{\tau_1}(\nu(\xi)+\alpha(\xi))d\xi\Big\}dy\int_{T_1}^{y} K(y,y-x,\lambda)v(x)dx.$$

Then changing the order of integration we reduce it to the Volterra type equation

$$\begin{cases} v(\tau_1) = v(T_1)\exp\{-\int_{T_1}^{\tau_1}(\nu(\xi)+\alpha(\xi))d\xi\} \\ \quad +\int_{T_1}^{\tau_1} v(x)dx\int_x^{\tau_1} K(y,y-x,\lambda)\exp\{-\int_y^{\tau_1}(\nu(\xi)+\alpha(\xi))d\xi\}dy \end{cases} \tag{4.12}$$

which has a unique positive solution v for any finite λ.
To determine v in $(T_2, T_3]$ we have to solve Eq. (4.6)$_3$ with the initial value $v(T_2)$ determined by Eq. (4.12). Because of the retarded structure with delay T we consider this equation going with the step T along the axis τ_1. For $\tau_1 \in [T_2+sT, \min(T_2+(s+1)T, T_3))$, $s = 0,1,...,$ we rewrite it in the form

$$\begin{cases} v(\tau_1) = v(T_2+sT)\exp\{-\int_{T_2+sT}^{\tau_1}\big(\nu(\xi)+\alpha(\xi)\big)d\xi\} \\ \quad +\int_{T_2+sT}^{\tau_1}\exp\{-\int_y^{\tau_1}\big(\nu(\xi)+\alpha(\xi)\big)d\xi\}dy\int_{y-T}^{y} K(y,y-x,\lambda)v(x)\,dx \\ \quad +\int_{T_2+(s-1)T}^{\tau_1-T}\exp\{-\int_{x+T}^{\tau_1}\big(\nu(\xi)+\alpha(\xi)\big)d\xi\}A(x+T,\lambda)v(x)\,dx. \end{cases} \tag{4.13}$$

Since $\{(x,y) : x \in [y-T, y],\, y \in [T_2+sT, \tau_1]\} = D_1 \cup D_2 \cup D_3$,
where

$$\begin{aligned} D_1 &= \{(x,y) : x \in [y-T, \tau_1 - T],\, y \in [T_2+sT, \tau_1]\} \\ &= \{(x,y) : x \in [T_2+(s-1)T, \tau_1 - T],\, y \in [T_2+sT, x+T]\}, \\ D_2 &= \{(x,y); x \in [\tau_1 - T, T_2+sT],\, y \in [T_2+sT, \tau_1]\}, \\ D_3 &= \{(x,y) : x \in [T_2+sT, y], y \in [T_2+sT, \tau_1]\} \\ &= \{(x,y) : x \in [T_2+sT, \tau_1],\, y \in [x, \tau_1]\}, \end{aligned}$$

the second term in the right-hand side of Eq. (4.13) can be written as follows:

$$\begin{aligned} &\int_{T_2+sT}^{\tau_1}\exp\Big\{-\int_y^{\tau_1}(\nu(\xi)+\alpha(\xi))d\xi\Big\}dy\int_{y-T}^{y} K(y,y-x,\lambda)v(x)dx \\ &= \int_{T_2+(s-1)T}^{\tau_1-T} v(x)dx\int_{T_2+sT}^{x+T} K(y,y-x,\lambda)\exp\Big\{-\int_y^{\tau_1}(\nu(\xi)+\alpha(\xi))d\xi\Big\}dy \\ &+ \int_{\tau_1-T}^{T_2+sT} v(x)dx\int_{T_2+sT}^{\tau_1} K(y,y-x,\lambda)\exp\Big\{-\int_y^{\tau_1}(\nu(\xi)+\alpha(\xi))d\xi\Big\}dy \\ &+ \int_{T_2+sT}^{\tau_1} v(x)dx\int_x^{\tau_1} K(y,y-x,\lambda)\exp\Big\{-\int_y^{\tau_1}(\nu(\xi)+\alpha(\xi))d\xi\Big\}dy. \end{aligned}$$

Denote

$$\begin{aligned} g_s(\tau_1,\lambda) = {} & v(T_2+sT)\exp\Big\{-\int_{T_2+sT}^{\tau_1}(\nu(\xi)+\alpha(\xi))d\xi\Big\} + \int_{T_2+(s-1)T}^{\tau_1-T} A(x+T)v(x)\exp\Big\{-\int_{x+T}^{\tau_1}(\nu(\xi)+\alpha(\xi))\Big\}d\xi \\ &+ \int_{T_2+(s-1)T}^{\tau_1-T} v(x)dx\int_{T_2+sT}^{x+T} K(y,y-x,\lambda)\exp\Big\{-\int_y^{\tau_1}(\nu(\xi)+\alpha(\xi))d\xi\Big\}dy \\ &+ \int_{\tau_1-T}^{T_2+sT} v(x)dx\int_{T_2+sT}^{\tau_1} K(y,y-x,\lambda)\exp\Big\{-\int_y^{\tau_1}(\nu(\xi)+\alpha(\xi))d\xi\Big\}dy \end{aligned}$$

and rewrite Eq. (4.13) in the Volterra form

$$v(\tau_1) = g_s(\tau_1,\lambda) + \int\limits_{T_2+sT}^{\tau_1} v(x)dx \int\limits_{x}^{\tau_1} K(y,y-x,\lambda)\exp\Big\{-\int\limits_{y}^{\tau_1}(\nu(\xi)+\alpha(\xi))d\xi\Big\}dy \tag{4.14}$$

for $\tau_1 \in [T_2+sT, \min(T_2+(s+1)T, T_3)]$.

Starting with $s=0$ and using the recurrent way we first determine $g_s(\tau_1,\lambda)$ and then solve Volterra Eq. (4.14) getting $v \in C^0([T_2,T_3])$. It is evident that $v \in C^1(T_2+sT, \min(T_2+(s+1)T,T_3))$ for every fixed s. Direct calculation shows that v' is continuous at points $T_2+sT < T_3$ with $s>1$.

Then we solve Eq. $(4.6)_4$ for $\tau_1 \in (T_3,T_4]$ with known the right hand side to get

$$v(\tau_1) = v(T_3)\exp\Big\{-\int\limits_{T_3}^{\tau_1}\nu(\xi)d\xi\Big\} + \int\limits_{T_3}^{\tau_1}\exp\Big\{-\int\limits_{y}^{\tau_1}\nu(\xi)d\xi\Big\}dy\int\limits_{y-T}^{T_3}K(y,y-x,\lambda)v(x)dx$$
$$+\int\limits_{T_3-T}^{\tau_1-T}\exp\Big\{-\int\limits_{y+T}^{\tau_1}\nu(\xi)d\xi\Big\}A(y+T,\lambda)v(y)dy.$$

For $\tau_1 > T_4$ we solve Eq. $(4.6)_1$ to get $v(\tau_1) = v(T_4)\exp\{-\int_{T_4}^{\tau_1}\nu(\xi)d\xi\}$.
From Eqs. $(4.5)_2$ and (4.9) we get

$$w(T) = \int\limits_{T_1}^{T_3} v(x)\sum_{k=1}^{n}\alpha_k(x)q_k(x,\lambda)\exp\Big\{-\lambda x - \int\limits_{0}^{T}(\nu_c(\xi,\xi)+\nu_{c0}(\xi,\xi))d\xi\Big\}dx$$

where v is determined by Eqs. (4.12) and (4.14). It is evident that

$$v \in C^0([T,\infty)) \cap C^1\big((T,\infty)\setminus\{T_1,T_2,T_3,T_4\}\big).$$

At last, inserting v_k and v determined above into Eq. (4.7) we derive an equation for λ,

$$L(\lambda)=1,\, L(\lambda) := \int\limits_{T_1}^{T_3} e^{-\lambda x}\sum_{k=1}^{n} kv_k(x+T,T)v(x)dx. \tag{4.15}$$

If β_k is independent of τ_c, then q_k is independent of λ too. Hence, $q_k = \beta_k(\tau_1)$. Therefore, v, v_k, and v_c do not depend on λ as well. Because of the monotonicity in λ and since $L\to\infty$ as $\lambda\to-\infty$ and $L\to 0$ as $\lambda\to\infty$ Eq. (4.15) has a unique real root λ_0 such that $\lambda_0<0$, if $L(0)<1$ (in this case cuckoo and host bird populations die), $\lambda_0=0$, if $L(0)=1$ (both populations die), and $\lambda_0>0$, if $L(0)>1$ (both populations grow).

In the case where $\partial_{\tau_c}\beta_k \neq 0$, we have

$$0<\beta_{k*} = \min_{[T_1,T_3]\times[T_1,T_3]}\beta_k < q_k(\tau_1,\lambda) < \max_{[T_1,T_3]\times[T_1,T_3]}\beta_k = \beta_k^* < 1.$$

Let $v_c^*(\tau_1,\tau_2)$ and $v_{c*}(\tau_1,\tau_2)$ be functions defined by Eq. (4.9) with q_k replaced by β_k^* and β_{k*}, respectively. Let $v_k^*(\tau_1,\tau_2)$ and $v_{k*}(\tau_1,\tau_2)$, $k=1,2,\dots,n$, be functions defined by Eqs. (4.10) and (4.11) with q_k replaced by β_{k*} and β_k^*, respectively. Then $v_{c*}<v_c<v_c^*$ and $v_{k*}<v_k<v_k^*$. Hence,

$$K_*(\tau_1,\tau_2) := \sum_{k=1}^{n}\nu_{k0}(\tau_1,\tau_2)v_{k*}(\tau_1,\tau_2)+\nu_{c0}(\tau_1,\tau_2)v_{c*}(\tau_1,\tau_2) < K(\tau_1,\tau_2,\lambda)$$

$$< K^*(\tau_1,\tau_2) := \sum_{k=1}^{n}\nu_{k0}(\tau_1,\tau_2)v_k*(\tau_1,\tau_2)+\nu_{c0}(\tau_1,\tau_2)v_c*(\tau_1,\tau_2),$$

$$A_*(\tau_1) := \sum_{k=1}^{n} v_{k*}(\tau_1,T)+v_{c*}(\tau_1,T) < A(\tau_1,\lambda)$$

$$< A^*(\tau_1) := \sum_{k=1}^{n} v_k*(\tau_1,T)+v_c*(\tau_1,T).$$

Then we solve Eqs. (4.12) and (4.14) with $K(\tau_1,\tau_2,\lambda)$, $A(\tau_1,\lambda)$ replaced by $K_*(\tau_1,\tau_2)$, $A_*(\tau_1)$ and $K^*(\tau_1,\tau_2)$, $A^*(\tau_1)$ getting v_* and v^*, respectively, for $\tau_1\in[T_1,T_3]$. Obviously, $v_*<v<v^*$.

Therefore,

$$L_*(\lambda) := \int_{T_1}^{T_3} e^{-\lambda x} \sum_{k=1}^{n} k v_{k*}(x+T,T) v_*(x) dx < L(\lambda)$$

$$< L^*(\lambda) := \int_{t_1}^{T_3} e^{-\lambda x} \sum_{k=1}^{n} k v^{k*}(x+T,T) v^*(x) dx.$$

These equations show that Eq. (4.15) has at least one real root λ. Moreover, $\lambda > 0$, if $L_*(0) > 1$, and $\lambda < 0$, if $L^*(0) < 1$. The proof is complete.

Knowing v, v_c, v_k, $k = 1, \ldots, n$, we determine densities of cuckoo and host chicks of age $\tau_2 \leq T$ by formulas (3.8),

$$f(t,\tau_2) = U \int_{T_1}^{T_3} v(x) v_c(x+\tau_2,\tau_2) \exp\{\lambda(t-x+T-\tau_2)\} dx,$$

$$u(t,\tau_2) = U \int_{T_1}^{T_3} v(x) \sum_{k=1}^{n} k v_k(x+\tau_2,\tau_2) \exp\{\lambda(t-x+T-\tau_2)\} dx.$$

5. Conclusions

The rather generic phenomenological model for Common Cuckoo interaction with the other bird species is presented. The model is composed of a system of integro-partial differential equations. All individuals have pre-reproductive, reproductive, and post-reproductive age intervals. Individuals of reproductive age are divided into single and those who care of young offspring. Individuals of pre-reproductive age are divided into young (under maternal care) and juvenile classes. Juveniles can live without maternal care but cannot produce their offspring.

In the case of special initial distributions, the existence of separable solutions of type (4.1) is proved. The conditions for the convergence of separable solutions to a steady-state solution, populations death and growth are given. The solvability of the model for the initial distributions of a general type is an open problem.

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Modeling of fiber circumplacement around a hole using a streamline approach

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Abstract

The insertion of holes into laminates can be done by producing a fiber reinforced composite plate and, subsequently, drilling the borehole. Alternatively, we can bypass the fibers around the final hole before injecting the matrix material. In the first case, the spatial distribution of the axis of anisotropy, and the structural tensor concerned, are spatially constant. In the second case, i.e. the fiber circumplacement around the hole, a space-dependent anisotropy has to be considered. Instead of the common approach of defining region-wise constant fiber orientations, we propose a continuous formulation of fiber orientations using streamlines. To estimate the final stress and strain state for a unidirectional composite plate, three-dimensional finite element simulations are performed, where spatially constant transverse isotropy is compared to inhomogeneously distributed fiber orientation around the hole. It will turn out that the resulting stress states lead to both reduced stress amplitudes in loading direction as well as compressive strains in lateral direction. A detailed mathematical derivation of the basic equations accompanies the investigations.

1. Introduction

One possibility to produce laminates is to use one fiber direction. In this respect, woven fabrics are out the scope of the subsequent, first investigations. To compute a homogenized stress distribution, it is convenient to draw on linear elasticity for transverse isotropy for such kind of unidirectional laminates, see, for example, [1, 2], and, for a brief overview, [3]. These works go back to the fundamental contribution of [4]. In the field of small deformation, there exist a number of contributions, see, for example, [5, 6, 7, 8, 9]. There is a new area of applications of transverse isotropy, namely the field of biomechanics, since several soft tissues show this kind of anisotropic behavior, [10, 11, 12, 13, 14]. For particular descriptions, see [15, 16, 17]. Since there are only a very few analytical solutions (making sense for code verification purposes), numerical methods have to be chosen. Here, the finite element method is the most common approach. However, there might be also difficulties for low order elements. Regarding numerical difficulties, such as locking effects due to constraints in fiber direction under particular loading conditions, see [18, 19] and the literature cited therein. In our contribution fine meshes and quadratic shape functions are chosen to minimize numerical problems.

Fiber-reinforced plates must be connected to other structural elements, which is sometimes done using skews or rivets. Frequently, holes are drilled into the final plate so that some fibers are cut. To circumvent this, a suitable possibility is given by bypassing fibers around the hole in advance, i.e. during the production process.

Different methods have been developed for defining fiber orientation to investigate different issues of plates with a hole. Regarding stiffening laminates and to achieve variable stiffness in specimens, see [20, 21]. Detailed research is available to optimize the fiber orientation near a hole, path optimization of laminated composite structures, and to find the optimal fiber angle distribution, see [22, 23, 24, 25]. [26, 27] study problems in buckling, failure, and vibration in laminates reinforced by curvilinear fibers. For more studies on stress, strain, fracture, and the influence of the thickness distribution in a ply with variable fiber angles, see [28, 29, 30, 31, 32, 33, 34]. Commonly, the works treat fiber orientation defined with piece-wise functions, see [24], or by element-wise formulations, see [27, 23].

In this study, fiber orientation is presented by using streamline function in fluid mechanics to obtain continuously distributed fiber orientations. Thus, we are interested in modeling both approaches using finite elements, where for the bypass approach a fiber orientation model

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using streamline theory around a circular cylinder is proposed. Finally, the results of the computations with uniform fiber orientation and the bypass-approach are compared.

The article is structured as follows: first, the constitutive model of transverse isotropy for the small strain case is derived from the large strain theory by geometrical linearization. Afterwards, we propose the streamline model to describe how fiber directions lie around a circular hole. Finally, finite element simulations are provided, where the stress and strain states for small strain applications are compared.

The notation in use is defined in the following manner: geometrical vectors are symbolized by $\vec{a}$, second-order tensors $\mathbf{A}$ by bold-faced Roman letters, and calligraphic letters $\mathscr{A}$ define fourth order tensors. Furthermore, we introduce matrices by bold-faced Roman letters $\mathsf{\mathbf{A}}$.

2. Constitutive model of transverse isotropy

In the following, we motivate the model of transverse isotropy for linear elasticity and a small strain theory. Afterwards, a model for a fiber orientation around a hole is proposed, which is based on the theory of streamlines.

2.1. Basic equations

Frequently, constitutive equations for transverse isotropy of an elastic material are modeled using a strain-energy function depending on the Green strain tensor $\mathbf{E} = (1/2)(\mathbf{F}^T\mathbf{F} - \mathbf{I})$, or, alternatively, on the right Cauchy-Green tensor $\mathbf{C} = \mathbf{F}^T\mathbf{F}$, and a structural tensor $\mathbf{M}$ containing the information of the fiber orientation

$$\tilde{\mathbf{T}} = \rho_{\mathrm{R}} \frac{\partial \psi(\mathbf{E},\mathbf{M})}{\partial \mathbf{E}} = 2\rho_{\mathrm{R}} \frac{\partial \overline{\psi}(\mathbf{C},\mathbf{M})}{\partial \mathbf{C}}. \tag{1}$$

$\mathbf{F}(\vec{X},t) = \mathrm{Grad}\,\vec{\chi}_{\mathrm{R}}(\vec{X},t)$ represents the deformation gradient of the motion $\vec{x} = \vec{\chi}_{\mathrm{R}}(\vec{X},t)$, where the material point $\vec{X}$ occupies the spatial position $\vec{x}$ at time t. $\mathbf{I} = \delta_{ij}\vec{e}_i \otimes \vec{e}_j$ defines the second order identity tensor. In the case of transverse isotropy the structural tensor reads $\mathbf{M} = \vec{a} \otimes \vec{a}$, where $\vec{a}(\vec{X})$ defines the orientation of the fiber direction with the property $\|\vec{a}\| = 1$. $\tilde{\mathbf{T}}$ symbolizes the second Piola-Kirchhoff stress tensor, which is related to the (true) Cauchy stress tensor $\mathbf{T}$ by $\tilde{\mathbf{T}} = (\det\mathbf{F})\mathbf{F}^{-1}\mathbf{T}\mathbf{F}^{-T}$. ρ_{R} represents the density in reference configuration. The dependence of the strain energy function is not arbitrary but consists of particular number of invariants,

$$\tilde{\mathbf{T}} = \rho_{\mathrm{R}} \frac{\partial \hat{\psi}(\mathrm{I}_{\mathbf{E}},\mathrm{II}_{\mathbf{E}},\mathrm{III}_{\mathbf{E}},\mathrm{IV}_{\mathbf{E}},\mathrm{V}_{\mathbf{E}})}{\partial \mathbf{E}} = 2\rho_{\mathrm{R}} \frac{\partial \hat{\overline{\psi}}(\mathrm{I}_{\mathbf{C}},\mathrm{II}_{\mathbf{C}},\mathrm{III}_{\mathbf{C}},\mathrm{IV}_{\mathbf{C}},\mathrm{V}_{\mathbf{C}})}{\partial \mathbf{C}}, \tag{2}$$

which are defined by

$$\mathrm{I}_{\mathbf{E}} = \mathrm{tr}\,\mathbf{E},\ \mathrm{II}_{\mathbf{E}} = \mathrm{tr}\,\mathbf{E}^2,\ \mathrm{III}_{\mathbf{E}} = \mathrm{tr}\,\mathbf{E}^3,\ \mathrm{IV}_{\mathbf{E}} = \mathrm{tr}\,(\mathbf{E}\mathbf{M}) = \mathbf{E}\cdot\mathbf{M},\ \mathrm{V}_{\mathbf{E}} = \mathrm{tr}\,(\mathbf{E}^2\mathbf{M}) = \mathbf{E}^2\cdot\mathbf{M}, \tag{3}$$

$$\mathrm{I}_{\mathbf{C}} = \mathrm{tr}\,\mathbf{C},\ \mathrm{II}_{\mathbf{C}} = \mathrm{tr}\,\mathbf{C}^2,\ \mathrm{III}_{\mathbf{C}} = \mathrm{tr}\,\mathbf{C}^3,\ \mathrm{IV}_{\mathbf{C}} = \mathrm{tr}\,(\mathbf{C}\mathbf{M}) = \mathbf{C}\cdot\mathbf{M},\ \mathrm{V}_{\mathbf{C}} = \mathrm{tr}\,(\mathbf{C}^2\mathbf{M}) = \mathbf{C}^2\cdot\mathbf{M}, \tag{4}$$

see [4, 35]. The invariants of the Green strain and right Cauchy-Green tensor are related by

$$\mathrm{I}_{\mathbf{C}} = 2\mathrm{I}_{\mathbf{E}} + 3, \quad \mathrm{II}_{\mathbf{C}} = 4\mathrm{II}_{\mathbf{E}} + 4\mathrm{I}_{\mathbf{E}} + 3, \quad \mathrm{III}_{\mathbf{C}} = 8\mathrm{III}_{\mathbf{E}} + 8\mathrm{II}_{\mathbf{E}} + 6\mathrm{I}_{\mathbf{E}} + 3, \quad \mathrm{IV}_{\mathbf{C}} = 2\mathrm{IV}_{\mathbf{E}} + 1, \quad \mathrm{V}_{\mathbf{C}} = 4\mathrm{V}_{\mathbf{E}} + 4\mathrm{IV}_{\mathbf{E}} + 1$$

or vice versa

$$\mathrm{I}_{\mathbf{E}} = \frac{1}{2}(\mathrm{I}_{\mathbf{C}} - 3), \quad \mathrm{II}_{\mathbf{E}} = \frac{1}{4}(\mathrm{II}_{\mathbf{C}} - 2\mathrm{I}_{\mathbf{C}} + 3), \quad \mathrm{III}_{\mathbf{E}} = \frac{1}{8}(\mathrm{III}_{\mathbf{C}} - 2\mathrm{II}_{\mathbf{C}} + \mathrm{I}_{\mathbf{C}}), \quad \mathrm{IV}_{\mathbf{E}} = \frac{1}{2}(\mathrm{IV}_{\mathbf{C}} - 1), \quad \mathrm{V}_{\mathbf{E}} = \frac{1}{4}(\mathrm{V}_{\mathbf{C}} - 2\mathrm{IV}_{\mathbf{C}} + 1).$$

Since we are interested in formulating a constitutive model for small strains, we draw on the formulation using the Green strain tensor $\mathbf{E}$. Applying the chain rule on Eq.(2)$_1$ yields

$$\tilde{\mathbf{T}} = \rho_{\mathrm{R}} \left(\frac{\partial \hat{\psi}}{\partial \mathrm{I}_{\mathbf{E}}}\mathbf{I} + \frac{\partial \hat{\psi}}{\partial \mathrm{II}_{\mathbf{E}}}\mathbf{E} + \frac{\partial \hat{\psi}}{\partial \mathrm{III}_{\mathbf{E}}}\mathbf{E}^2 + \frac{\partial \hat{\psi}}{\partial \mathrm{IV}_{\mathbf{E}}}\mathbf{M} + \frac{\partial \hat{\psi}}{\partial \mathrm{V}_{\mathbf{E}}}(\mathbf{E}\mathbf{M} + \mathbf{M}\mathbf{E}) \right). \tag{5}$$

To obtain a linear elastic, small strain theory, which is fully appropriate in our application, several assumptions are introduced:

1. First, the Green strain tensor has to be linearized in view of the displacement gradient $\mathbf{H} = \mathrm{Grad}\,\vec{u}(\vec{X},t)$ with $\vec{u}(\vec{X},t) = \vec{\chi}_{\mathrm{R}}(\vec{X},t) - \vec{X}$,
$$\mathbf{E} = \frac{1}{2}(\mathbf{H} + \mathbf{H}^T + \mathbf{H}^T\mathbf{H}) = \frac{1}{2}(\mathbf{H} + \mathbf{H}^T) + \mathscr{O}(\|\mathbf{H}\|^2), \tag{6}$$
i.e. the quadratic term is omitted, see [36, 37].
2. Since we are interested in a theory of small displacements, it is assumed that there is no distinction between the spatial coordinates $\vec{x}$ and the coordinates in the reference configuration $\vec{X}$. Thus, the linearized Green strain tensor reads
$$\mathbf{E}_{\mathrm{L}}(\vec{x},t) = \frac{1}{2}(\mathrm{grad}\,\vec{u}(\vec{x},t) + \mathrm{grad}^T\,\vec{u}(\vec{x},t)) \approx \mathbf{E}, \tag{7}$$
implying that we have small displacements and small strains, i.e. $\vec{x} \approx \vec{X}$, $\vec{u}(\vec{X},t) = \vec{u}(\vec{x},t)$. There is no distinction of the configurations and all material properties are assigned to a spatial point $\vec{x}$. In the following, we omit the index $_{\mathrm{L}}$ for brevity.
3. If we are interested in a linear theory, i.e. the resulting stress state $\mathbf{T}$ depends linearly on the strain state $\mathbf{E}$, and the strain-energy function (5) must quadratically depend on the strain tensor $\mathbf{E}$. This is given by
$$\psi(\mathbf{E},\mathbf{M}) = \frac{\lambda}{2}\mathrm{I}_{\mathbf{E}}^2 + \mu_T\mathrm{II}_{\mathbf{E}} + \alpha\mathrm{I}_{\mathbf{E}}\mathrm{IV}_{\mathbf{E}} + 2(\mu_L - \mu_T)\mathrm{V}_{\mathbf{E}} + \frac{\beta}{2}\mathrm{IV}_{\mathbf{E}}^2. \tag{8}$$
There cannot be a dependence on the third invariant $\mathrm{III}_{\mathbf{E}}$, which automatically leads to a non-linear elastic material, see [4, 1, 2].

4. The stresses have the tendency to become equal for very small strains. Thus, we do not distinct between different stress measures, $\tilde{\mathbf{T}} \to \mathbf{T}$. Accordingly, we arrive at the final stress state applying Eq.(2)$_1$

$$\mathbf{T} = \rho_{\mathrm{R}}\left((\lambda \mathrm{I}_{\mathbf{E}} + \alpha \mathrm{IV}_{\mathbf{E}})\mathbf{I} + 2\mu_T \mathbf{E} + (\alpha \mathrm{I}_{\mathbf{E}} + \beta \mathrm{IV}_{\mathbf{E}})\mathbf{M} + 2(\mu_L - \mu_T)(\mathbf{EM} + \mathbf{ME})\right). \tag{9}$$

5. Frequently, the product of the density in the reference configuration with the material parameters is omitted so that the density is not visible in the final expression, i.e. we define

$$\lambda \leftarrow \rho_{\mathrm{R}}\lambda, \quad \mu_T \leftarrow \rho_{\mathrm{R}}\mu_T, \quad \mu_L \leftarrow \rho_{\mathrm{R}}\mu_L, \quad \alpha \leftarrow \rho_{\mathrm{R}}\alpha, \quad \beta \leftarrow \rho_{\mathrm{R}}\beta. \tag{10}$$

Using all these assumptions lead to a particular expression, where elasticity relation (9) can be expressed by a fourth order elasticity tensor $\mathscr{C}$. To show this, we need the relations

$$(\mathbf{B}\cdot\mathbf{C})\mathbf{A} = (\mathbf{A}\otimes\mathbf{B})\mathbf{C}, \quad \mathbf{ACB}^T = [\mathbf{A}\otimes\mathbf{B}]^{T_{23}}\mathbf{C}, \tag{11}$$

see, for example, [38, 39, 35], yielding

$$\mathbf{T} = \mathscr{C}\mathbf{E} \tag{12}$$

with

$$\mathscr{C} = \lambda \mathbf{I}\otimes\mathbf{I} + 2\mu_T \mathscr{I} + \alpha[\mathbf{I}\otimes\mathbf{M} + \mathbf{M}\otimes\mathbf{I}] + \beta \mathbf{M}\otimes\mathbf{M} + 2(\mu_L - \mu_T)[\mathbf{I}\otimes\mathbf{M} + \mathbf{M}\otimes\mathbf{I}]^{T_{23}}. \tag{13}$$

$\mathscr{I} = [\mathbf{I}\otimes\mathbf{I}]^{T_{23}} = \delta_{ik}\delta_{jl}\vec{e}_i\otimes\vec{e}_j\otimes\vec{e}_k\otimes\vec{e}_l$ is the fourth order identity tensor, $\mathbf{A} = \mathscr{I}\mathbf{A}$. The symbolic $\mathscr{A}^{T_{23}}$ implies the transposition of second and third index, i.e. for $\mathscr{A} = a_{ijkl}\vec{e}_i\otimes\vec{e}_j\otimes\vec{e}_k\otimes\vec{e}_l$ we obtain $\mathscr{A}^{T_{23}} = a_{ikjl}\vec{e}_i\otimes\vec{e}_j\otimes\vec{e}_k\otimes\vec{e}_l$. For $\vec{a} = \vec{e}_1$, the Voigt notation of Eq.(13) is compiled in Appendix B.

2.2. Model of fiber orientation

To obtain a function with continuously distributed fibers, which move around a circular hole in the vicinity of the hole, we lend ideas of streamlines in Fluid Mechanics. In [40] the streamlines are defined by

$$\psi = U_\infty \sin\theta\left(r - \frac{R^2}{r}\right) - k\frac{\log r}{R}. \tag{14}$$

We set $k = 0$, $a = \psi/U_\infty$, $x = r\cos\theta$, and $y = r\sin\theta$ to arrive at

$$g(x,y,a) = y^3 - ay^2 + (x^2 - R^2)y - ax^2 = 0 \tag{15}$$

representing a cubic polynomial in y. There are three solutions, two conjugate complex and one real. We are only interested in the real solution yielding

$$\hat{y}(x,a) = \frac{1}{6}\left(2^{2/3}\sqrt[3]{2a^3 + \sqrt{\left(2a^3 + 9aR^2 + 18ax^2\right)^2 - 4\left(a^2 + 3(R-x)(R+x)\right)^3} + 9aR^2 + 18ax^2} + \frac{2\left(a^2 + 3(R-x)(R+x)\right)}{\sqrt[3]{a^3 + \frac{1}{2}\sqrt{\left(2a^3 + 9aR^2 + 18ax^2\right)^2 - 4\left(a^2 + 3(R-x)(R+x)\right)^3} + \frac{9aR^2}{2} + 9ax^2}} + 2a\right). \tag{16}$$

In Fig. 1(b) the orientation lines are shown for different a. The arbitrary factor a is adapted in such a manner that it has a geometrical meaning, which is discussed in the following. The function (16) should have the value $\hat{y}(x^*, a^*) = y^* \rightsquigarrow a^*$. This constraint is fulfilled for

$$a^* = \frac{(x^{*2} - R^2)y^* + y^{*3}}{x^{*2} + y^{*2}}. \tag{17}$$

This parameter can be inserted into Eq.(15),

$$f(x,y) := g(x,y,a^*) \tag{18}$$

which can be evaluated, see Fig. 1(b), with $x^* = x$ and $y^* = y$. Thus, continuously distributed orientations are achieved. To obtain the tangent vector at the orientation function, we have to differentiate the position vector

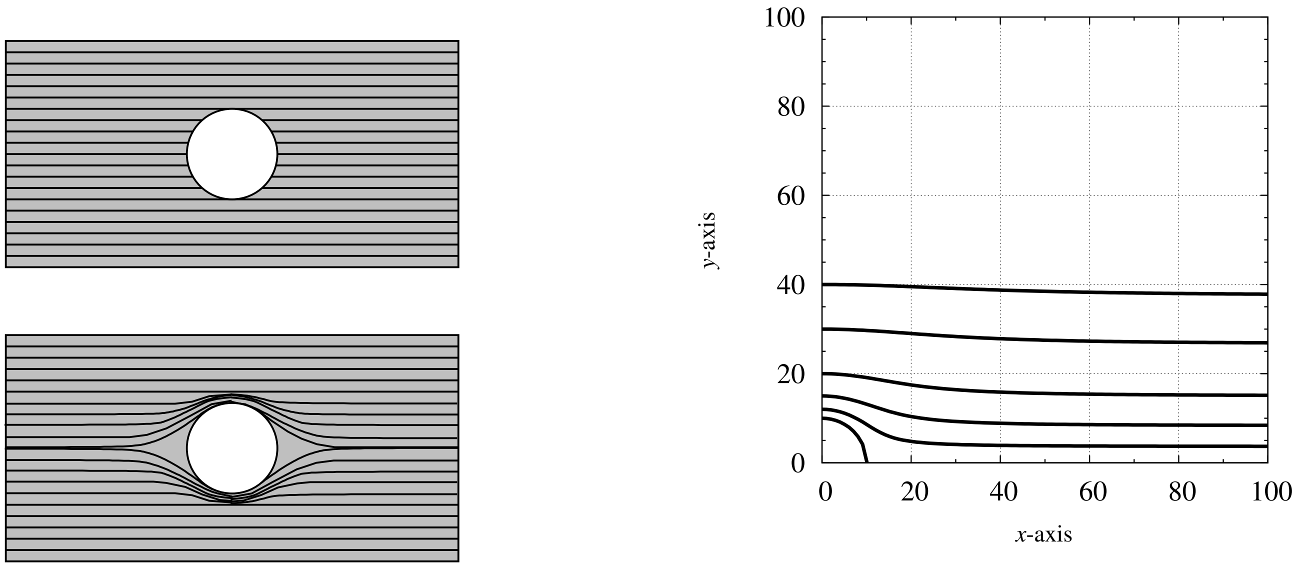
$$\vec{r} = x\vec{e}_x + \hat{y}(x)\vec{e}_y, \tag{19}$$

with respect to the arc-length, which can be chosen in the case of functions equivalent to x. Instead of applying Eq.(16), we apply the chain rule to Eq.(18),

$$f(x,\hat{y}(x)) = 0, \qquad f_{,x} + f_{,y}\hat{y}'(x) = 0 \quad \to \quad \hat{y}'(x) = -f_{,x}/f_{,y}, \tag{20}$$

which is a much simpler expression than using Eq.(16). Here, we draw on the abbreviations $f_{,x} := \partial f(x,y)/\partial x$ and $f_{,y} := \partial f(x,y)/\partial y$. The tangent vector at the orientation function reads

$$\vec{t}(x) = \frac{\partial\vec{r}}{\partial x} = \vec{e}_x + \hat{y}'(x)\vec{e}_y = \vec{e}_x - f_{,x}/f_{,y}\,\vec{e}_y. \tag{21}$$



(a) Basic sketch of uniformly distributed fibers, where borehole is applied after production process, and circumplacement of fibers around the hole during the production process

(b) Orientation lines using streamline functions

Figure 1: Orientation distribution of fibers

To obtain a unit vector at a point $x = x^*$ and $y = y^*$

$$\vec{a}(x^*, y^*) = \left. \frac{\vec{t}(x^*)}{|\vec{t}(x^*)|} \right|_{x=x^*, y=y^*}, \tag{22}$$

with

$$|\vec{t}(x)| = \sqrt{1 + f_{,x}^2 f_{,y}^{-2}} = \sqrt{f_{,y}^{-2}} \sqrt{f_{,x}^2 + f_{,y}^2}. \tag{23}$$

Thus, the final unit tangent vector reads

$$\vec{a} = \frac{1}{\sqrt{f_{,x}^2 + f_{,y}^2}} \left(\sqrt{f_{,y}^2} \vec{e}_x - f_{,x} \frac{\sqrt{f_{,y}^2}}{f_{,y}} \vec{e}_y \right), \tag{24}$$

which has to be evaluated at point $x = x^*$ and $y = y^*$. The superscript * can be omitted leading to the simple expressions

$$f_{,x} = 2x(y - a), \quad f_{,y} = x^2 - R^2 - 2ay + 3y^2 \tag{25}$$

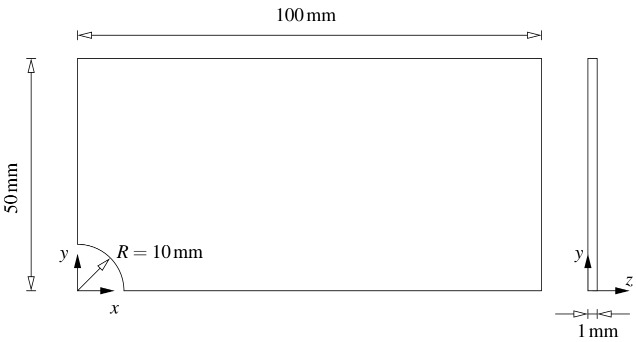
with

$$a = \frac{(x^2 - R^2)y + y^3}{x^2 + y^2}. \tag{26}$$

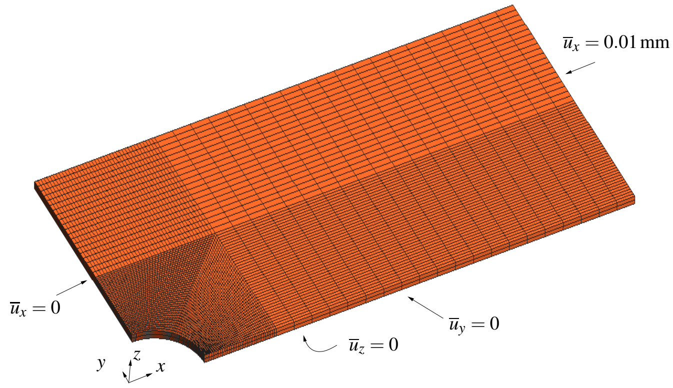
3. Finite element studies

In the following, we compare two computations of a plate with a hole having a radius of $R = 10\,\mathrm{mm}$. The plate is subjected to a displacement load of $u(100, y) = \bar{u}_x = 0.01\,\mathrm{mm}$. It is meshed using 20-noded, isoparametric hexahedral elements with $(3 \times 3 \times 3)$ Gauss-points. For the geometry and the mesh, see Fig. 2. Here, only one-eighth of the plate is discretized due to symmetry conditions, see Fig. 2(b). The material parameters are lend from [4], $\lambda = 5.64\,\mathrm{N\,mm^{-2}}$, $\mu_T = 2.46\,\mathrm{N\,mm^{-2}}$, $\alpha = -1.27\,\mathrm{N\,mm^{-2}}$, $\beta = 227.29\,\mathrm{N\,mm^{-2}}$, $\mu_L = 5.66\,\mathrm{N\,mm^{-2}}$. In the first computation, we choose the spatially constant fiber orientation $\vec{a} = \vec{e}_x$ (homogeneous distribution of fiber orientation, in the following called *uni-directional*). In the second example, the fiber orientation of Eq.(24) is used, which is represented by the tangent vectors to the coordinate lines shown in Fig. 1(b). This is called *bypass*. Since this computation has the disadvantage to generate a singularity at point $(x, y, z) = (R, 0, z)$ generating arbitrary stresses close to this point – transition of circle to horizontal line –, we consider an improvement, see Fig. 3. A polynomial of third order is used to distinct a region, where no fibers are (below this polynomial), and the region using fiber orientation of Eq.(24). This computation, where the stiffness in y-direction is essentially weaker, is called *bypass-reduced*. The polynomial starts at point $(x, y) = (3\sqrt{5}, 5)$ and ends at point $(x, y) = (15, 0)$. Its slopes are the same as of the circle on the left, and zero on the right. Inside the small region, we take only the isotropic part of the elasticity relation (13) – first two terms of the elasticity tensor.

In the following, we study the stress and strain states in two computations (unidirectional and bypassed-reduced). We evaluate the stresses σ_{xx} at $x = 0$ and $z = 1$ (vertical axis), $\sigma_{xx}(0, y, 1)$, see Fig. 4(a), and the stresses in vertical direction at the horizontal symmetry plane, $\sigma_{yy}(x, 0, 1)$, see Fig. 4(b). The plots are generated using GiD, where interpolation schemes transfer Gauss-point information to nodal



(a) Geometry of one-eighth of the specimen



(b) Geometry (mesh and boundary conditions)

Figure 2: Geometry, mesh, and boundary conditions

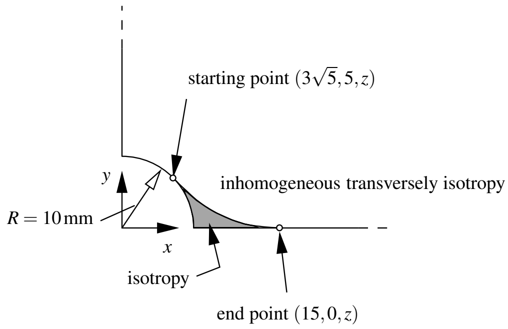


Figure 3: Region with purely isotropic material (bypass-reduced)

points. Obviously, the highest horizontal stresses are produced by a pure uniformly distributed fiber directions at point $(x,y) = (0,R)$. Very promising results are obtained for the "bypass-reduced"-simulation, where the stresses σ_{xx} do not show such large amplitudes in this region (more than three times smaller).

Similar results are obtained for the strains, see Fig. 5. However, we obtain larger compressive strains ε_{yy} for the bypass-reduced computation compared to the purely unidirectional reinforcement. However, negative strains are not so critical as tensile strains. Thus, promising results are obtained.

In Appendix C the entire stress and strain distributions are assembled.

4. Conclusions

In this paper, we investigate the stress and strain distribution of unidirectional and inhomogeneous fiber directions around a hole, which is of interest in manufacturing processes. The circumplacement of fibers is modeled using streamline function to obtain the inhomogeneous fiber direction, which can be evaluated in finite element simulations of transversely isotropic material. The comparison shows a promising results, namely, that the stresses are essentially reduced around the hole. However, compressive strains, vertical to the loading directions, increase, which can be seen as not so dramatic then tensile strains. Further investigations and comparisons to experimental data have to be followed.

A. Voigt-notation of transversal isotropy

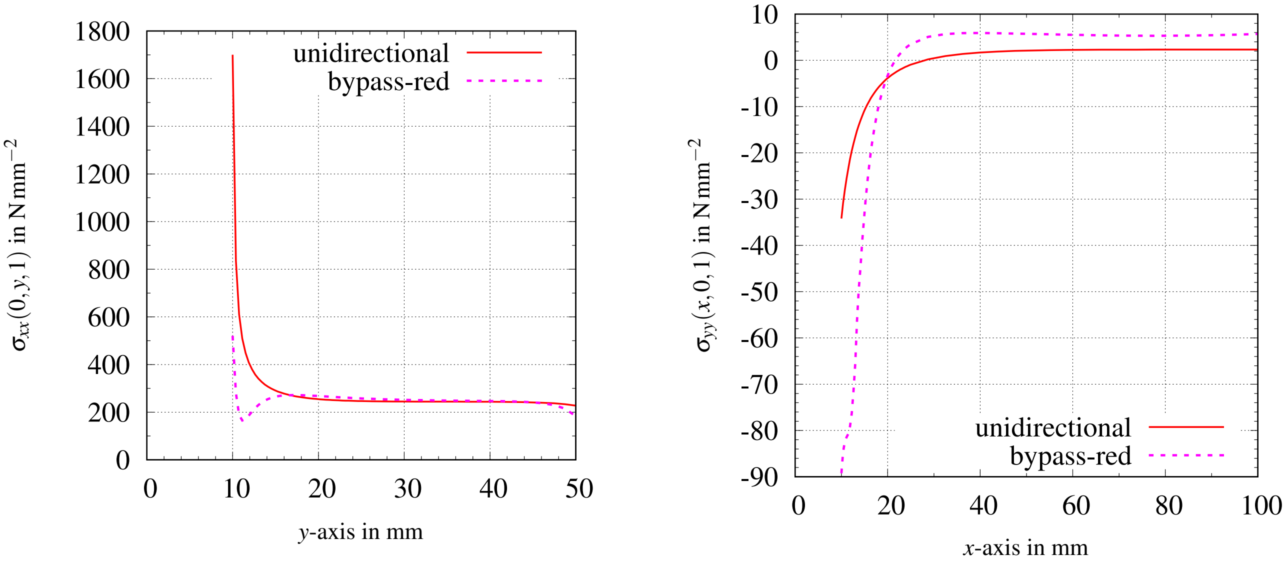
In the following, the tensorial notation is transferred into matrix notation so that no unnecessary computation should be performed. This is connected to the term *Voigt*-notation. Particularly, this is caused by the property of symmetry of the stress and strain tensor. Additionally, the scalar product in the principle of virtual displacements leads to a specific representation of the vector containing the independent strain tensor components leading to a symmetric elasticity matrix.

The (coefficients of the) components of the stress and strain tensor

$$\mathbf{T} = \begin{bmatrix} T_{11} & T_{12} & T_{31} \\ & T_{22} & T_{23} \\ \text{sym.} & & T_{33} \end{bmatrix} \vec{e}_i \otimes \vec{e}_j, \quad \mathbf{E} = \begin{bmatrix} E_{11} & E_{12} & E_{31} \\ & E_{22} & E_{23} \\ \text{sym.} & & E_{33} \end{bmatrix} \vec{e}_i \otimes \vec{e}_j$$

are assembled into column vectors

$$\mathbf{T}^T = \{T_{11}\; T_{22}\; T_{33}\; T_{12}\; T_{23}\; T_{31}\}, \quad \tilde{\mathbf{E}}^T = \{E_{11}\; E_{22}\; E_{33}\; E_{12}\; E_{23}\; E_{31}\}$$



(a) Normal stresses σ_{xx} in the vertical axis

(b) Normal stresses σ_{yy} in the horizontal axis

Figure 4: Stresses σ_{xx} and σ_{yy} at the vertical and the horizontal symmetry lines

To describe the influence of incorporating the symmetry properties into matrix formulation, we perform to steps: first, the tensorial quantities are transferred into (9×1) and (9×9)-matrices, respectively. In this context, we treat the product (12) leading to

$$\begin{Bmatrix} T_{11}\\ T_{22}\\ T_{33}\\ T_{12}\\ T_{23}\\ T_{31}\\ T_{13}\\ T_{21}\\ T_{32} \end{Bmatrix} = \begin{bmatrix} C_{1111} & C_{1122} & C_{1133} & C_{1112} & C_{1123} & C_{1131} & C_{1113} & C_{1121} & C_{1132}\\ C_{2211} & C_{2222} & C_{2233} & C_{2212} & C_{2223} & C_{2231} & C_{2213} & C_{2221} & C_{2232}\\ C_{3311} & C_{3322} & C_{3333} & C_{3312} & C_{3323} & C_{3331} & C_{3313} & C_{3321} & C_{3332}\\ C_{1211} & C_{1222} & C_{1233} & C_{1212} & C_{1223} & C_{1231} & C_{1213} & C_{1221} & C_{1232}\\ C_{2311} & C_{2322} & C_{2333} & C_{2312} & C_{2323} & C_{2331} & C_{2313} & C_{2321} & C_{2332}\\ C_{3111} & C_{3122} & C_{3133} & C_{3112} & C_{3123} & C_{3131} & C_{3113} & C_{3121} & C_{3132}\\ C_{1311} & C_{1322} & C_{1333} & C_{1312} & C_{1323} & C_{1331} & C_{1313} & C_{1321} & C_{1332}\\ C_{2111} & C_{2122} & C_{2133} & C_{2112} & C_{2123} & C_{2131} & C_{2113} & C_{2121} & C_{2132}\\ C_{3211} & C_{3222} & C_{3233} & C_{3212} & C_{3223} & C_{3231} & C_{3213} & C_{3221} & C_{3232} \end{bmatrix} \begin{Bmatrix} E_{11}\\ E_{22}\\ E_{33}\\ E_{12}\\ E_{23}\\ E_{31}\\ E_{13}\\ E_{21}\\ E_{32} \end{Bmatrix} \tag{27}$$

The symmetry of the stress tensor implies that the last three equations are the same as equations 4 to 6:

$$\begin{Bmatrix} T_{11}\\ T_{22}\\ T_{33}\\ T_{12}\\ T_{23}\\ T_{31} \end{Bmatrix} = \begin{bmatrix} C_{1111} & C_{1122} & C_{1133} & C_{1112} & C_{1123} & C_{1131} & C_{1113} & C_{1121} & C_{1132}\\ C_{2211} & C_{2222} & C_{2233} & C_{2212} & C_{2223} & C_{2231} & C_{2213} & C_{2221} & C_{2232}\\ C_{3311} & C_{3322} & C_{3333} & C_{3312} & C_{3323} & C_{3331} & C_{3313} & C_{3321} & C_{3332}\\ C_{1211} & C_{1222} & C_{1233} & C_{1212} & C_{1223} & C_{1231} & C_{1213} & C_{1221} & C_{1232}\\ C_{2311} & C_{2322} & C_{2333} & C_{2312} & C_{2323} & C_{2331} & C_{2313} & C_{2321} & C_{2332}\\ C_{3111} & C_{3122} & C_{3133} & C_{3112} & C_{3123} & C_{3131} & C_{3113} & C_{3121} & C_{3132} \end{bmatrix} \begin{Bmatrix} E_{11}\\ E_{22}\\ E_{33}\\ E_{12}\\ E_{23}\\ E_{31}\\ E_{13}\\ E_{21}\\ E_{32} \end{Bmatrix} \tag{28}$$

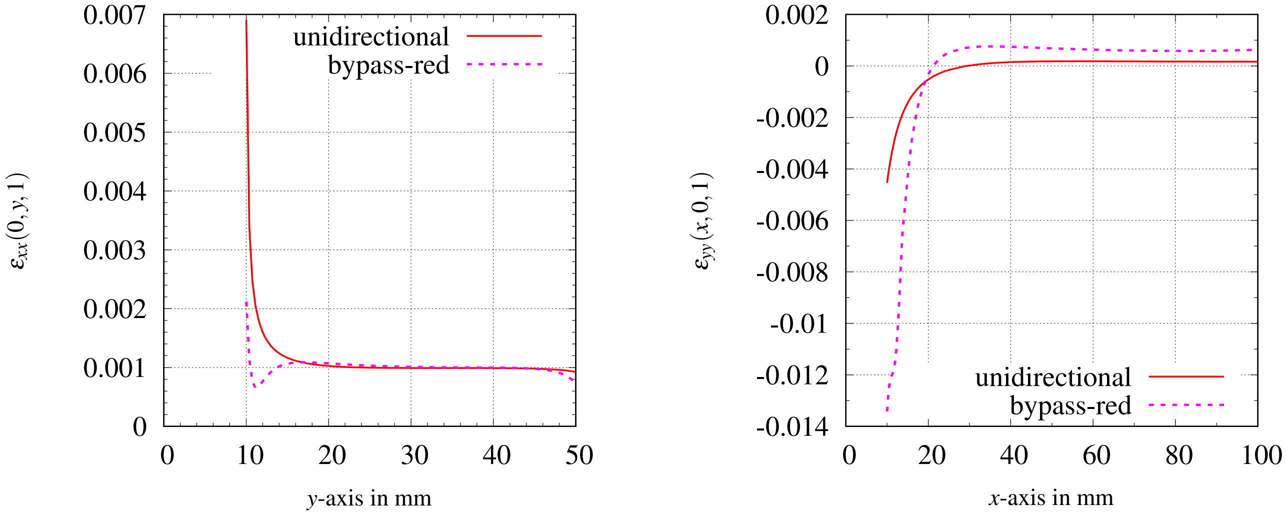
Now, we incorporate the symmetry of the strain tensor, $E_{12}=E_{21}$, $E_{23}=E_{32}$, $E_{31}=E_{13}$,

$$\begin{Bmatrix} T_{11}\\ T_{22}\\ T_{33}\\ T_{12}\\ T_{23}\\ T_{31} \end{Bmatrix} = \begin{bmatrix} C_{1111} & C_{1122} & C_{1133} & C_{1112}+C_{1121} & C_{1123}+C_{1132} & C_{1131}+C_{1113}\\ C_{2211} & C_{2222} & C_{2233} & C_{2212}+C_{2221} & C_{2223}+C_{2232} & C_{2231}+C_{2213}\\ C_{3311} & C_{3322} & C_{3333} & C_{3312}+C_{3321} & C_{3323}+C_{3332} & C_{3331}+C_{3313}\\ C_{1211} & C_{1222} & C_{1233} & C_{1212}+C_{1221} & C_{1223}+C_{1232} & C_{1231}+C_{1213}\\ C_{2311} & C_{2322} & C_{2333} & C_{2312}+C_{2321} & C_{2323}+C_{2332} & C_{2331}+C_{2313}\\ C_{3111} & C_{3122} & C_{3133} & C_{3112}+C_{3121} & C_{3123}+C_{3132} & C_{3131}+C_{3113} \end{bmatrix} \begin{Bmatrix} E_{11}\\ E_{22}\\ E_{33}\\ E_{12}\\ E_{23}\\ E_{31} \end{Bmatrix} \tag{29}$$

In other words, we obtain the representation $\mathbf{T}=\tilde{\mathbf{C}}\tilde{\mathbf{E}}$, with $\mathbf{T}\in\mathbb{R}^6$, $\tilde{\mathbf{E}}\in\mathbb{R}^6$ and $\tilde{\mathbf{C}}\in\mathbb{R}^{6\times 6}$. The arrangement of the coefficients of a fourth order tensor into the matrix $\tilde{\mathbf{C}}$ depends on the original calculation. For example, in the elasticity relation (13) we different products. First, we look at a product $\mathscr{C}=\mathbf{A}\otimes\mathbf{B}$, where $\mathbf{A}=\mathbf{A}^T$ and $\mathbf{B}=\mathbf{B}^T$ are symmetric tensors. The coefficients of $\mathscr{C}$ are given by $c_{ijkl}=a_{ij}b_{kl}$. In this case, the coefficient matrix (29) has the representation

$$\mathbf{C}=\begin{bmatrix} a_{11}b_{11} & a_{11}b_{22} & a_{11}b_{33} & 2a_{11}b_{12} & 2a_{11}b_{23} & 2a_{11}b_{31}\\ a_{22}b_{11} & a_{22}b_{22} & a_{22}b_{33} & 2a_{22}b_{12} & 2a_{22}b_{23} & 2a_{22}b_{31}\\ a_{33}b_{11} & a_{33}b_{22} & a_{33}b_{33} & 2a_{33}b_{12} & 2a_{33}b_{23} & 2a_{33}b_{31}\\ a_{12}b_{11} & a_{12}b_{22} & a_{12}b_{33} & 2a_{12}b_{12} & 2a_{12}b_{23} & 2a_{12}b_{31}\\ a_{23}b_{11} & a_{23}b_{22} & a_{23}b_{33} & 2a_{23}b_{12} & 2a_{23}b_{23} & 2a_{23}b_{31}\\ a_{31}b_{11} & a_{31}b_{22} & a_{31}b_{33} & 2a_{31}b_{12} & 2a_{31}b_{23} & 2a_{31}b_{31} \end{bmatrix}. \tag{30}$$

A fourth-order tensor having the transposition T_{23} requires a more detailed consideration. We consider the product $T_{ij}=C_{ikjl}E_{kl}$ yielding



(a) Strain ε_{xx} in the vertical axis

(b) Strain ε_{yy} in the vertical axis

Figure 5: Strains ε_{xx} and ε_{yy} at the vertical and the horizontal symmetry lines

the (9×9)-representation

$$\left\{\begin{array}{c} T_{11} \\ T_{22} \\ T_{33} \\ T_{12} \\ T_{23} \\ T_{31} \\ T_{13} \\ T_{21} \\ T_{32} \end{array}\right\} = \begin{bmatrix} C_{1111} & C_{1212} & C_{1313} & C_{1112} & C_{1213} & C_{1311} & C_{1113} & C_{1211} & C_{1312} \\ C_{2121} & C_{2222} & C_{2323} & C_{2122} & C_{2223} & C_{2321} & C_{2123} & C_{2221} & C_{2322} \\ C_{3131} & C_{3232} & C_{3333} & C_{3132} & C_{3233} & C_{3331} & C_{3133} & C_{3231} & C_{3332} \\ C_{1121} & C_{1222} & C_{1323} & C_{1122} & C_{1223} & C_{1321} & C_{1123} & C_{1221} & C_{1322} \\ C_{2131} & C_{2232} & C_{2333} & C_{2132} & C_{2233} & C_{2331} & C_{2133} & C_{2231} & C_{2332} \\ C_{3111} & C_{3212} & C_{3313} & C_{3112} & C_{3213} & C_{3311} & C_{3113} & C_{3211} & C_{3312} \\ C_{1131} & C_{1232} & C_{1333} & C_{1132} & C_{1233} & C_{1331} & C_{1133} & C_{1231} & C_{1332} \\ C_{2111} & C_{2212} & C_{2313} & C_{2112} & C_{2213} & C_{2311} & C_{2113} & C_{2211} & C_{2312} \\ C_{3121} & C_{3222} & C_{3323} & C_{3122} & C_{3223} & C_{3321} & C_{3123} & C_{3221} & C_{3322} \end{bmatrix} \left\{\begin{array}{c} E_{11} \\ E_{22} \\ E_{33} \\ E_{12} \\ E_{23} \\ E_{31} \\ E_{13} \\ E_{21} \\ E_{32} \end{array}\right\}. \tag{31}$$

Obviously, in each column the indices 2 and 3 are exchanged, compared to representation (27). If we have the symmetries $T_{12} = T_{21}$, $T_{23} = T_{32}$, and $T_{31} = T_{13}$, the last three rows contain the same information as in rows 4 to 6, and, accordingly, can be neglected. For the symmetries $E_{12} = E_{21}$, $E_{23} = E_{32}$, and $E_{31} = E_{13}$, we obtain

$$\begin{bmatrix} C_{1111} & C_{1212} & C_{1313} & C_{1112}+C_{1211} & C_{1213}+C_{1312} & C_{1311}+C_{1113} \\ C_{2121} & C_{2222} & C_{2323} & C_{2122}+C_{2221} & C_{2223}+C_{2322} & C_{2321}+C_{2123} \\ C_{3131} & C_{3232} & C_{3333} & C_{3132}+C_{3231} & C_{3233}+C_{3332} & C_{3331}+C_{3133} \\ C_{1121} & C_{1222} & C_{1323} & C_{1122}+C_{1221} & C_{1223}+C_{1322} & C_{1321}+C_{1123} \\ C_{2131} & C_{2232} & C_{2333} & C_{2132}+C_{2231} & C_{2233}+C_{2332} & C_{2331}+C_{2133} \\ C_{3111} & C_{3212} & C_{3313} & C_{3112}+C_{3211} & C_{3213}+C_{3312} & C_{3311}+C_{3113} \end{bmatrix}. \tag{32}$$

If we apply these ideas to the tensors $\mathbf{I}\otimes\mathbf{I}$, $\mathbf{M}\otimes\mathbf{M}$, $\mathbf{I}\otimes\mathbf{M}+\mathbf{M}\otimes\mathbf{I}$ and $[\mathbf{I}\otimes\mathbf{M}+\mathbf{M}\otimes\mathbf{I}]^{T_{23}}$, we obtain the following representations:

$$\mathbf{I}\otimes\mathbf{I} \longrightarrow \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \qquad \mathscr{I} = [\mathbf{I}\otimes\mathbf{I}]^{T_{23}} \longrightarrow \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \tag{33}$$

$$\mathscr{I} = [\mathbf{I}\otimes\mathbf{M}+\mathbf{M}\otimes\mathbf{I}]^{T_{23}} \longrightarrow \begin{bmatrix} 2m_{11} & 0 & 0 & 2m_{12} & 0 & 2m_{31} \\ 0 & 2m_{22} & 0 & 2m_{12} & 2m_{23} & 0 \\ 0 & 0 & 2m_{33} & 0 & 2m_{23} & 2m_{31} \\ m_{12} & m_{12} & 0 & m_{11}+m_{22} & m_{31} & m_{23} \\ 0 & m_{23} & m_{23} & m_{31} & m_{22}+m_{33} & m_{12} \\ m_{31} & 0 & m_{31} & m_{23} & m_{12} & m_{11}+m_{33} \end{bmatrix} \tag{34}$$

$$\mathbf{M}\otimes\mathbf{M} \longrightarrow \begin{bmatrix} m_{11}^2 & m_{11}m_{22} & m_{11}m_{33} & 2m_{11}m_{12} & 2m_{11}m_{23} & 2m_{11}m_{31} \\ m_{11}m_{22} & m_{22}^2 & m_{22}m_{33} & 2m_{12}m_{22} & 2m_{22}m_{23} & 2m_{22}m_{31} \\ m_{11}m_{33} & m_{22}m_{33}^2 & m_{33} & 2m_{12}m_{33} & 2m_{23}m_{33} & 2m_{31}m_{33} \\ m_{11}m_{12} & m_{12}m_{22} & m_{12}m_{33} & 2m_{12}^2 & 2m_{12}m_{23} & 2m_{12}m_{31} \\ m_{11}m_{23} & m_{22}m_{23} & m_{23}m_{33} & 2m_{12}m_{23} & 2m_{23}^2 & 2m_{23}m_{31} \\ m_{11}m_{31} & m_{22}m_{31} & m_{31}m_{33} & 2m_{12}m_{31} & 2m_{23}m_{31} & 2m_{31}^2 \end{bmatrix} \tag{35}$$

$$\mathbf{I}\otimes\mathbf{M}+\mathbf{M}\otimes\mathbf{I}\longrightarrow\begin{bmatrix} 2m_{11} & m_{11}+m_{22} & m_{11}+m_{33} & 2m_{12} & 2m_{23} & 2m_{31}\\ m_{11}+m_{22} & 2m_{22} & m_{22}+m_{33} & 2m_{12} & 2m_{23} & 2m_{31}\\ m_{11}+m_{33} & m_{22}+m_{33} & 2m_{33} & 2m_{12} & 2m_{23} & 2m_{31}\\ m_{12} & m_{12} & m_{12} & 0 & 0 & 0\\ m_{23} & m_{23} & m_{23} & 0 & 0 & 0\\ m_{31} & m_{31} & m_{31} & 0 & 0 & 0 \end{bmatrix} \tag{36}$$

In finite elements, we have the scalar product of the stress tensor with the strain tensor (or with the virtual strain tensor). Since we would like to reduce the amount of function evaluations, we consider this scalar product

$$\begin{aligned}\mathbf{T}\cdot\mathbf{E} &= \{T_{11}\ T_{22}\ T_{33}\ T_{12}\ T_{23}\ T_{31}\}\begin{Bmatrix}E_{11}\\E_{22}\\E_{33}\\E_{12}\\E_{23}\\E_{31}\end{Bmatrix}+\{T_{12}\ T_{23}\ T_{31}\}\begin{Bmatrix}E_{12}\\E_{23}\\E_{31}\end{Bmatrix}=\mathsf{T}^T\begin{Bmatrix}E_{11}\\E_{22}\\E_{33}\\2E_{12}\\2E_{23}\\2E_{31}\end{Bmatrix}\\ &=\mathsf{T}^T\mathsf{M}\tilde{\mathsf{E}}=\mathsf{T}^T\mathsf{E}\end{aligned} \tag{37}$$

with $\mathsf{M}=\mathrm{diag}(1,1,1,2,2,2)\in\mathbb{R}^{6\times6}$ and $\mathsf{E}^T=\{E_{11}\ E_{22}\ E_{33}\ 2E_{12}\ 2E_{23}\ 2E_{31}\}$, i.e.

$$\mathsf{E}=\mathsf{M}\tilde{\mathsf{E}}\qquad\Longrightarrow\qquad\tilde{\mathsf{E}}=\mathsf{M}^{-1}\mathsf{E}. \tag{38}$$

Commonly, the “shear angles” $\gamma_{12}=2E_{12}$, $\gamma_{23}=2E_{23}$, $\gamma_{31}=2E_{31}$ are introduced in Solid Mechanics implying that the product with 2 can be omitted. If we have the product $\mathbf{T}=\mathscr{C}\mathbf{E}$, i.e. $\mathsf{T}=\tilde{\mathsf{C}}\tilde{\mathsf{E}}=\mathsf{C}\mathsf{E}$, the scalar product (37) reads:

$$\mathscr{C}\mathbf{E}\cdot\mathbf{E}=\{\tilde{\mathsf{C}}\tilde{\mathsf{E}}\}^T\mathsf{E}=\left\{\tilde{\mathsf{C}}\mathsf{M}^{-1}\mathsf{E}\right\}^T\mathsf{E}=\{\mathsf{C}\mathsf{E}\}^T\mathsf{E}=\mathsf{E}^T\mathsf{C}\mathsf{E}. \tag{39}$$

In other words, the last three columns of the coefficient matrices $\tilde{\mathsf{C}}$ must be multiplied with a factor $1/2$,

$$\mathsf{C}=\tilde{\mathsf{C}}\mathsf{M}^{-1}, \tag{40}$$

yielding the side effect that all matrices (33) - (36) become symmetric.

B. Case $\vec{a}=\vec{e}_1$

Frequently, transversely isotropy is connected to the Voigt-notation, i.e. the (6×6)-representation. In this case the orientation of the fibers is connected to one spatial coordinate direction, $\vec{a}=\vec{e}_1$ leading to the structural tensor $\mathbf{M}=\vec{a}\otimes\vec{a}=\vec{e}_1\otimes\vec{e}_1$, i.e. there is only the component $m_{11}=1$. All others are zero. In this case the matrix representation degenerates to

$$\begin{Bmatrix}T_{11}\\T_{22}\\T_{33}\\T_{12}\\T_{23}\\T_{31}\end{Bmatrix}=\begin{bmatrix}\lambda+2\alpha+2\mu_T+4(\mu_L-\mu_T)+\beta & \lambda+\alpha & \lambda+\alpha & 0 & 0 & 0\\ \lambda+\alpha & \lambda+2\mu_T & \lambda & 0 & 0 & 0\\ \lambda+\alpha & \lambda & \lambda+2\mu_T & 0 & 0 & 0\\ 0 & 0 & 0 & \mu_L & 0 & 0\\ 0 & 0 & 0 & 0 & \mu_T & 0\\ 0 & 0 & 0 & 0 & 0 & \mu_L\end{bmatrix}\begin{Bmatrix}E_{11}\\E_{22}\\E_{33}\\\gamma_{12}\\\gamma_{23}\\\gamma_{31}\end{Bmatrix}. \tag{41}$$

C. Stress and strain distributions

In the following, we compile the whole stress and strain distributions in Figs. 6 and 7.

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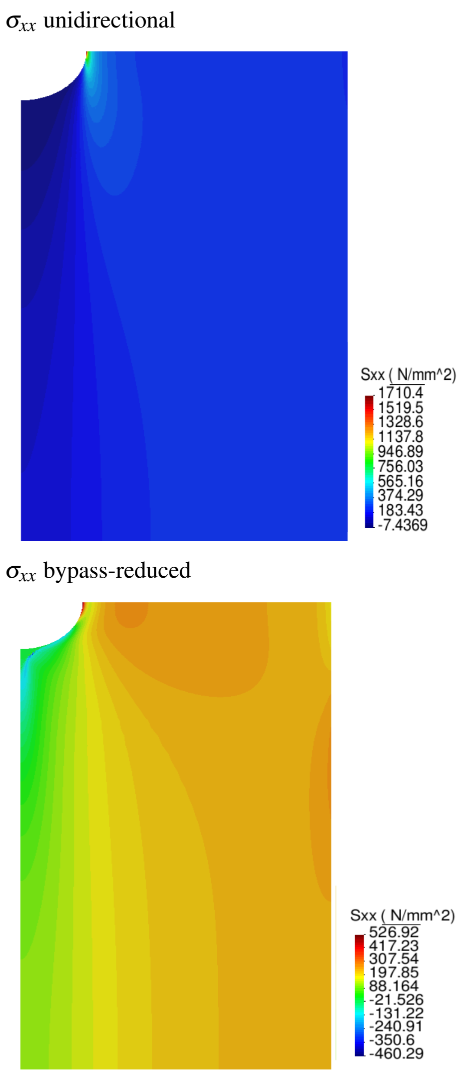
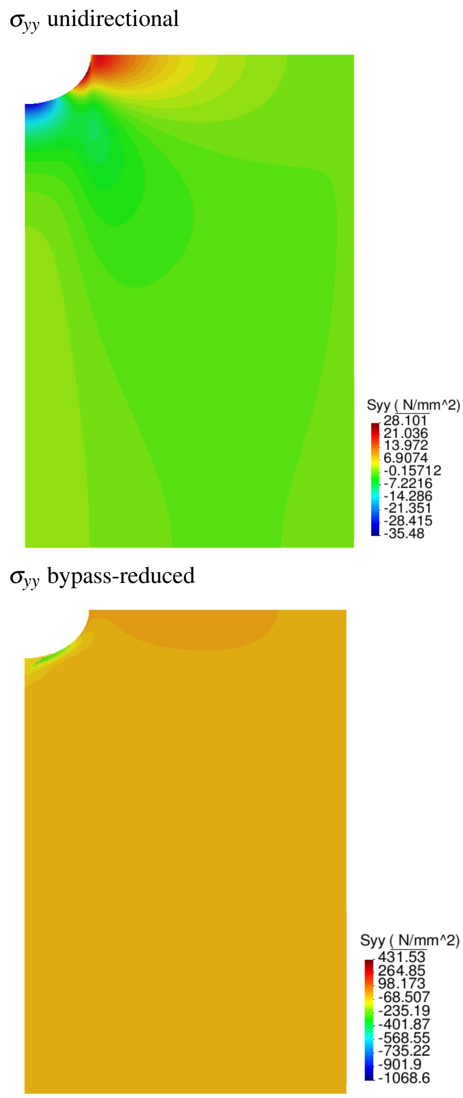
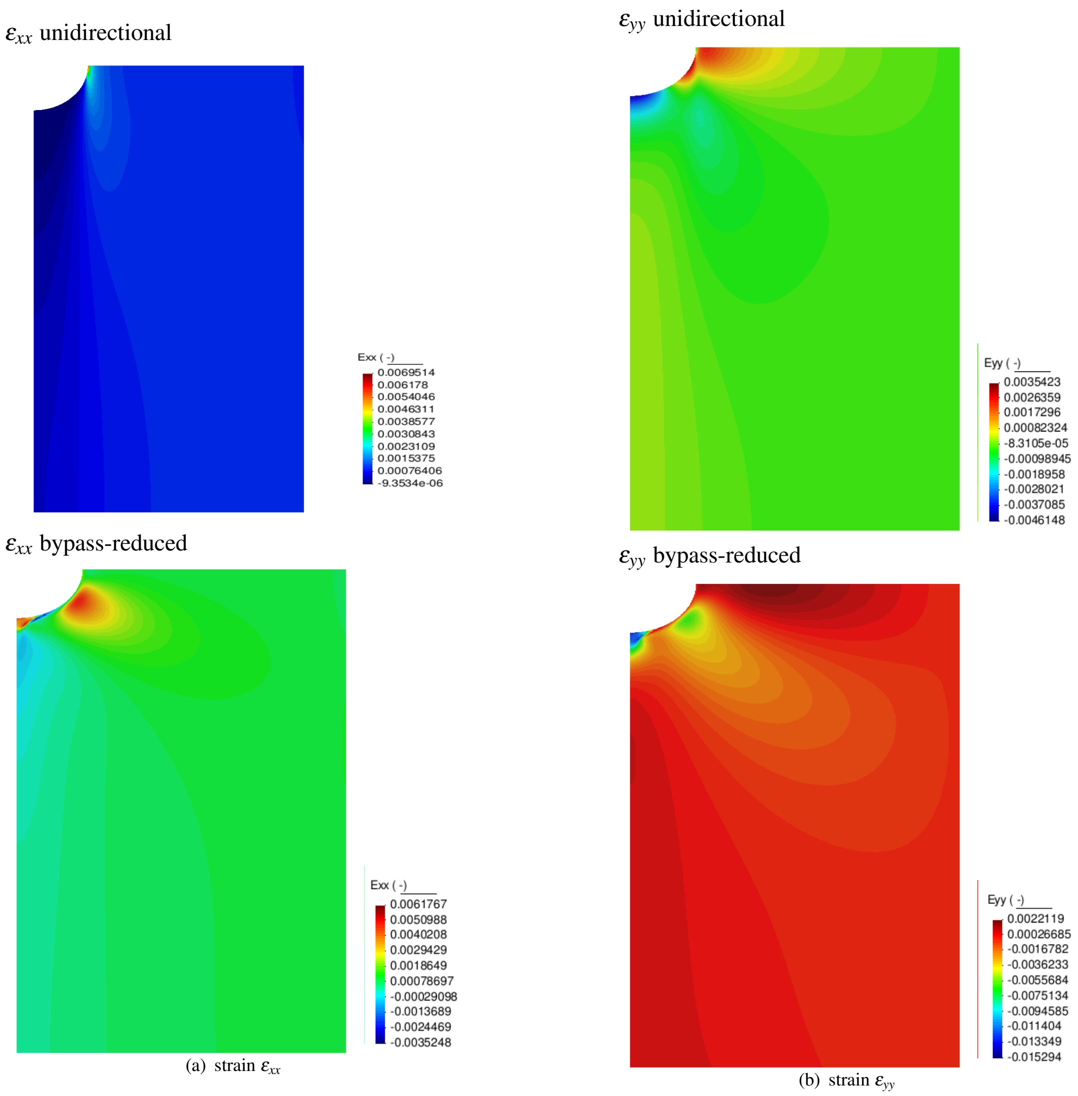


(a) σ_{xx} in $\mathrm{N\,mm^{-2}}$



(b) σ_{yy} in $\mathrm{N\,mm^{-2}}$

Figure 6: Stress distributions



(a) strain ε_{xx}

(b) strain ε_{yy}

Figure 7: Strain distributions

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An algorithm for constructing S-boxes for block symmetric encryption

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Abstract

This article presents an algorithm for the generation of S-boxes with the maximum algebraic immunity and high nonlinearity. The algorithm is founded method of the permutation of output element of S-box. On basis of the proposed method, $S(8\times 8)$-box created, with the algebraic immunity 3 (441) and nonlinearity 104. The algorithm given in this article can be used for oscillation of $S(8\times 8)$)-boxes with the increased resistance to algebraic, linear, differential and linear and differential methods of a cryptanalysis, for block symmetric algorithms of encryption.

1. Introduction

It is known that for determining the reliability of (cryptographic stability) encryption algorithms is required to assess their well-known modern methods of cryptanalysis. This shows that the emergence of a new method of cryptanalysis or development of existing methods of cryptanalysis can affect the cryptographic stability of the encryption algorithms used in practice. Today, algebraic method of cryptanalysis based on solving systems of equations over finite fields, is a modern and rapidly developing methods of cryptanalysis for block symmetric encryption algorithms [4]. As a result of research, experts, it has been proposed option ”algebraic immunity” encryption algorithms that allows you to determine the stability (instability) it to the algebraic methods of cryptanalysis. Therefore, the use of encryption algorithms convert with high algebraic immunity, will serve as a basis for ensuring the reliability of its methods to algebraic cryptanalysis. After the introduction of the parameter has become an urgent task for research aimed at creating change, with the maximum algebraic immunity. In developing the new block symmetric encryption algorithms take into account the use of these transformations, with the maximum algebraic immunity, that is, its cryptographic stability to methods of algebraic cryptanalysis. For example, in algorithms of standard STB 34.101.31-2011, GOST R 34.11-2012 and GOST R 34.12-2015 used S-boxes with the maximum algebraic immunity. This article describes the algorithms for generating S-boxes, the maximum algebraic immunity and high degree of nonlinearity.

2. Generation of S-boxes

It should be noted that as a part of round function the modern block symmetric algorithms of enciphering two following main transformations are used: substitution (S-box) and permutation (P-box) [3, 5]. The main the purpose of the S-box is “hashing of bits” and their use as the main non-linear transformation in round function. P-box to serve “a dispelling of bits” and is the linear transformation.
Each S-box transformation is defined over some finite field. S-box represent a $S(n\times m)$ wherein the input bit length (n) and output bit length (m).
Below is a sample $S(4\times 4)$-box.

$$S=\left\{\begin{array}{llllllllllllllll} x=\{0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15\} \\ \quad\updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow \\ y=\{15 & 14 & 13 & 11 & 6 & 12 & 9 & 2 & 5 & 10 & 4 & 8 & 0 & 1 & 3 & 7\} \end{array}\right\} \tag{2.1}$$

where: x - the input sequence in $S(4\times 4)$-box, y - output sequence with x respectively. For example: $S(6)=9$, $S(14)=3$.

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Determination of [1, 6]. Let the following system of Boolean equations satisfies $S(n\times m)$-box:

$$G=\begin{cases} g_1(x_1,x_1,...,x_n,y_1,y_1,...,y_m)=0;\\ g_2(x_1,x_1,...,x_n,y_1,y_1,...,y_m)=0;\\ \dots\\ g_r(x_1,x_1,...,x_n,y_1,y_1,...,y_m)=0.\end{cases} \tag{2.2}$$

Minimum degree algebraic equation (deg(g)) in the system (2.2) is called an algebraic immunity $S(n\times m)$-box ($AI(S)$). That is, it can be formally written as follows:

$$AI(S)=\min\{\deg(g)|g\in G\} \tag{2.3}$$

From this it follows that the high value of the parameter AI to S-box provides a high degree algebraic equation system.
After introducing the concept of the AI, it has become an urgent task of evaluating the maximum possible value of this parameter for the S-box of fixed length and a minimum number (N_{TS}) of possible equations in the system. As a result of investigations for solving this problem has been created in the following table indicating the maximum value of the AI and the minimum value of the N_{TS}, depending on the size of $S(n\times n)$-box.

n	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**	**13**	**14**
AI	2	2	2	2	3	3	3	3	4	4	4	4
N_{TS}	14	21	24	15	342	441	476	327	7061	8855	9710	7774

Table 1: Optimal values of AI and N_{TS} for $S(n\times n)$-box

In general it should be noted that *if the values AI S-box is less than the possible maximum value or value N_{TS} is greater than the minimum possible value, the S-box does not provide maximum resistance to algebraic techniques of cryptanalysis*.

It is known that the linear and differential methods of a cryptanalysis are also the modern methods of a cryptanalysis, for block symmetric algorithms of encryption. For ensuring resistance of an algorithm to the linear cryptanalysis, it is required to use S-boxes with the maximal value of nonlinearity ($N(S)$), and for ensuring resistance of algorithms to a differential cryptanalysis, it is required to use S-boxes in which the maximal value (δ) in a matrix of differences is less. Therefore, in algorithms of encryption it is necessary to use those S-boxes in which not only AI value is maximal, but also $N(S)$ value is also maximal, and value δ minimum.
There are several methods of creation of the S-boxes having the maximal value of nonlinearity [2]. However, because of the made experiments it became of known that $AI(S)$ and $N(S)$ value for S-boxes would not be at the same time maximum (that is maximality of these values is mutually excluded). This condition demands to solve the following problem:
At what maximal values of the N(S) parameter value of the AI and N_{TS} parameters will be optimum?
Solution of this task nonlinearity demands to construct several S-boxes having some high (that is, smaller maximal) nonlinearity and to make the corresponding experiments with them. Because of the carried-out analysis, the following statement are more efficient approach for creation of S-boxes with high values of nonlinearity.
Statement [7]. Let, the following equalities for S_1(nxm) and S_2(nxm) of boxes are carried out.

$$\begin{cases} S_2(p_1)=S_1(p_2),\ \ p_1\neq p_2;\\ S_2(p_2)=S_1(p_1);\\ S_2(x)=S_1(x),\ \ x\neq p_1,p_2.\end{cases} \tag{2.4}$$

Then truly following expression.

$$N(S_1)-2\le N(S_2)\le N(S_1)+2 \tag{2.5}$$

It means, according to statements, as a result of permutation among themselves of two elements of the S-box of its value of nonlinearity either decreases on 2 or increases on 2 or does not change. At the same time the statement also follows from this statement the following: *if as a result of permutation between itself two different elements S_1-box having general degree nonlinearity N(S_1)=a, is present probability of the creation S_2-box with the common degree nonlinearity equal N(S_2)=a-2, that as a result of permutation different 4 elements S_1-box created S_3-box can have importance with the general degree nonlinearity equal N(S_3)=a-4*. This statement will be a basis for oscillation of S-boxes with different values of nonlinearity. That is, increasing quantity of mutually rearranged elements of the S-box it is possible to reduce value of nonlinearity sequentially. Put into practice experiments with S(8x8)-box it was revealed that at $N(S)$=104 values, δ=8 the AI and N_{TS} parameters can have optimum degree.

Generally, the algorithm of creation of the S-boxes having such properties has the following sequence of steps:

Input: Certain $S(8\times 8)_{\ max}$ – box having maximal (that is: $N(S)$=112) nonlinearity.
Output: $S(8\times 8)$ – box satisfying to values: $N(S)$=104, δ=8, $AI(S)$=3 and N_{TS}=441.
1. $S(8\times 8)$=$S(8\times 8)_{\ max}$.
2. Permutation mutually 39 elements of the S(8x8)-box.
3. Determine value of the *N(S)* and δ parameters of the $S(8\times 8)$-box, created in 2 step.
4. If $N(S)$<104 or δ>8 that return to 1 step.
5. Define values of the $AI(S)$ and N_{TS} parameters of the $S(8\times 8)$-box, created 2 step.
6. If AI (S) $\neq$3 or N_{TS} $\neq$441 that return to 1 step.

7. Announce $S(8 \times 8)$-box as output dates.
8. End.

Below the example of model $S(8 \times 8)$-box created by means of the algorithm developed by the software is given (the output elements of the $S(8 \times 8)$-box):
$S(8 \times 8)_{\ example}$ = {173, 175, 17, 133, 114, 99, 57, 231, 126, 42, 247, 209, 230, 68, 181, 109, 248, 236, 115, 48, 188, 125, 18, 120, 53, 105, 4, 239, 32, 121, 76, 246, 6, 155, 13, 221, 254, 180, 226, 224, 36, 143, 196, 219, 78, 146, 227, 31, 96, 118, 92, 22, 249, 217, 49, 79, 67, 138, 198, 251, 93, 215, 60, 24, 69, 88, 50, 154, 253, 140, 206, 123, 184, 81, 160, 229, 98, 159, 139, 113, 233, 223, 238, 204, 153, 237, 107, 234, 225, 242, 14, 7, 183, 178, 72, 128, 203, 94, 124, 191, 84, 170, 205, 116, 29, 190, 150, 131, 103, 207, 97, 164, 51, 194, 65, 21, 37, 106, 58, 145, 212, 213, 172, 101, 100, 168, 163, 136, 9, 55, 86, 102, 195, 199, 15, 80, 132, 127, 61, 83, 176, 20, 122, 241, 38, 255, 82, 161, 171, 19, 89, 148, 220, 110, 8, 43, 3, 85, 66, 56, 142, 250, 40, 2, 59, 162, 134, 240, 182, 228, 141, 129, 211, 185, 179, 74, 11, 34, 62, 210, 193, 167, 197, 33, 156, 108, 30, 117, 95, 214, 187, 245, 35, 26, 27, 0, 252, 104, 202, 44, 208, 158, 147, 64, 157, 52, 192, 77, 5, 25, 152, 41, 12, 232, 87, 149, 119, 216, 165, 46, 75, 235, 169, 135, 222, 200, 39, 70, 91, 174, 112, 166, 54, 189, 243, 177, 218, 28, 10, 137, 144, 244, 16, 130, 45, 90, 73, 23, 201, 111, 47, 71, 151, 1, 63, 186}.
It is known that today in many algorithms of encryption $S(8 \times 8)$-box is used. For comparison of the S(8x8)$_{example}$-box with some $S(8 \times 8)$-boxes, created the table of assessment (Table 2).

Encrypting algorithm	**N(S)**	δ	**AI (N$_{TS}$)**	**IGS**
AES	112	4	2 (39)	0,886285
Camellia	112	4	2 (39)	0,886285
SQUARE	112	4	2 (39)	0,886285
UzDSt 1105:2009	112	4	2 (39)	0,886285
STB 34.101.31-2011	102	8	3 (441)	0,962426
GOST P 34.12-2015	100	8	3 (441)	0,956473
S(8x8)$_{example}$	**104**	**8**	**3(441)**	**0,968378**

Table 2: Comparative properties in algorithms of encryption $S(8 \times 8)$-boxes.

Values of the IGS parameter (the **I**ndex of the **G**eneral **S**tability, 0≤IGS≤1) specified in this table it is calculated by means of (2.6) formula, considering an indicator of resistance to the linear, differential and algebraic cryptanalysis of $S(8 \times 8)$-box.

$$\mathrm{IGS} = \frac{\frac{\mathrm{N}(S)}{112} + \frac{\mathrm{AI}}{3} + \frac{258-\delta}{256}}{3} \tag{2.6}$$

Follows from this expression that for some $S(8 \times 8)$-box there correspond the $N(S)$=112, δ=2 and AI=3 parameters, IGS values of this $S(8 \times 8)$-box it will be maximal, that is IGS=1. Besides, the IGS value of the $S(8 \times 8)_{\ example}$-box is higher than other $S(8 \times 8)$-boxes given in the table.
In Figure 2.1 the comparative schedule on IGS value of the specified S-boxes is represented.

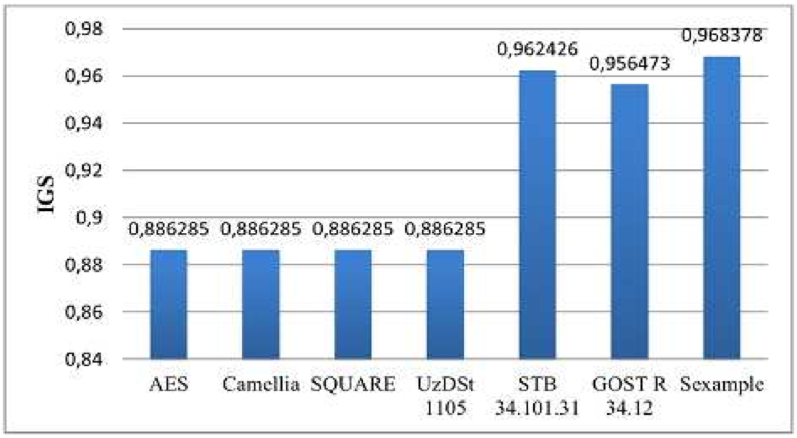

Figure 2.1: The comparative schedule on IGS value of the specified S-blocks.

3. Conclusions

The proposed method is based method of permutation of output element of S-box. It allows to find S-box with desired properties. Such S-box can be used in modern symmetric algorithms that demand high level of robustness against various types of attacks.

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On ψ-Hilfer fractional differential equation with complex order

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Abstract

The objectives of this paper is to investigate some adequate results for the existence of solution to a ψ-Hilfer fractional derivatives (HFDEs) involving complex order. Appropriate conditions for the existence of at least one solution are developed by using Schauder fixed point theorem (SFPT) to the consider problem. Moreover, we also investigate the Ulam-Hyers stability for the proposed problem.

1. Introduction

Fractional calculus deals with the study of fractional order integral and derivative operators over real or complex domains and some of their applications are in the area of fluid flow, control theory of dynamical systems, diffusive transport akin to diffusion, electrical networks, probability and statistics, viscoelasticity, electrochemistry of corrosion, dynamical processes in self-similar and porous structures, optics and rheology etc. There has been significant development in fractional differential equations in recent year (see [1]-[6])

The generalization of Riemann-Liouville and Caputo fractional derivatives was introduced by R. Hilfer [1] in 1999. A significant development and interest has been shown by many researchers. Vanterler et al. interpolated HFD and ψ-fractional derivative is called as ψ-HFD [7]. This fractional derivative is different from the other classical fractional derivative because the kernel is in terms of function. The study on ψ-HFD with classical properties and interpolation of many fractional derivatives.

Alternatively, the stability problem of functional equations initiated form a question of Ulam, created in 1940, relating to the stability of group homomorphism. In the next year, Hyers gave a partial affirmative respond to the question of Ulam in the background of Banach spaces that was the opening momentous breakthrough and a step towards more solutions in this area. In view of the fact that a large number of papers have been published in connections with various generalizations appeared devoted to the data dependence in the theory of fractional differential equations [8]-[11].

Inspired by the above discussion, we introduce complex order to ψ-HFD and we establish the existence, uniqueness and stability of solutions. Consider the differential equations with ψ-HFD with complex order of the form

$$\mathfrak{D}^{\theta_1,\theta_2;\psi}\mathfrak{h}(t) = \mathfrak{g}(t,\mathfrak{h}(t)),\ t \in J := (a,b], \tag{1.1}$$

$$\mathfrak{I}^{1-\theta;\psi}\mathfrak{h}(t)|_{t=a} = \mathfrak{h}_a, \theta = \theta_1 + \theta_2 - \theta_1\theta_2, \tag{1.2}$$

where $D^{\theta_1,\theta_2;\psi}(\theta_1,\theta_2 \in \mathbb{C})$ is ψ-HFD of order $\theta_1 = \alpha + i\beta$ and type $\theta_2 = \gamma + i\eta$. Here, $0 < \Re(\theta_1) < 1$ and $0 \le \Re(\theta_2) \le 1$, with α, β, γ and η are constants. Consider a Banach space R and $\mathfrak{g} : J \times R \to R$ be a continuous function.

The paper is organised as follows. In Section 2, we give some basic definitions and results concerning with the ψ-HFD. In Section 3, we present existence results based on SFPT and further stability result is also discussed. Finally, an example is included to check the theoretical results.

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2. Preliminaries

For the ease of the readers, we discuss some basic definitions and lemmas. The ideas are adopted from [12, 13]. Next, consider the following spaces, let $C(J)$ a space of continuous functions from J into R with the norm

$$\|x\|_C = \max\{|x(t)| : t \in J\}.$$

The weighted space $C_{1-\xi,\psi}(J)$ of functions $\mathfrak{g}$ on J is defined by

$$C_{1-\xi,\psi}(J) = \left\{\mathfrak{g} : J \to R : (\psi(t)-\psi(a))^{1-\xi}\,\mathfrak{g}(t) \in C(J)\right\}, 0 \le \xi (= \Re(\theta)) < 1,$$

with the norm

$$\|\mathfrak{g}\|_{C_{1-\xi,\psi}} = \left\|(\psi(t)-\psi(a))^{1-\xi}\,\mathfrak{g}(t)\right\|_{C[a,b]} = \max_{t\in J}\left|(\psi(t)-\psi(a))^{1-\xi}\,\mathfrak{g}(t)\right|.$$

Definition 2.1. *The ψ-Riemann Liouville (RL) fractional integral of order $\theta \in \mathbb{C}, (\Re(\theta) > 0)$ of a function $\mathfrak{g}$ is defined by,*

$$\mathfrak{I}^{\theta;\psi}\mathfrak{g}(t) = \frac{1}{\Gamma(\theta)}\int_0^t \psi'(s)\,(\psi(t)-\psi(s))^{\theta-1}\,\mathfrak{g}(s)ds, \quad t \ge 0.$$

Definition 2.2. *The ψ-RL fractional derivative of order $\theta \in \mathbb{C}, (\Re(\theta) > 0)$ of a function $\mathfrak{g}$ is defined by,*

$$\mathfrak{D}^{\theta;\psi}\mathfrak{g}(t) = \frac{1}{\Gamma(n-\theta)}\left(\frac{1}{\psi'(t)}\frac{d}{dt}\right)^n \int_0^t \psi'(s)\,(\psi(t)-\psi(s))^{n-\theta-1}\,\mathfrak{g}(s)ds, \quad t \ge 0,$$

where $n = [\Re(\theta)] + 1$.

Definition 2.3. *The ψ-Caputo fractional derivative of order $\theta \in \mathbb{C}, (\Re(\theta) > 0)$ of function $\mathfrak{g}$ is defined by,*

$$\mathfrak{D}^{\theta;\psi}\mathfrak{g}(t) = \mathfrak{I}^{n-\theta;\psi}\left(\frac{1}{\psi'(t)}\frac{d}{dt}\right)^n \mathfrak{g}(t) \quad t \ge 0,$$

where $n = [\Re(\theta)] + 1$.

Definition 2.4. *The ψ-HFD of order $0 < \theta_1 < 1$ and $0 \le \theta_2 \le 1$ of function $\mathfrak{g}(t)$ is defined by*

$$\mathfrak{D}^{\theta_1,\theta_2;\psi}\mathfrak{g}(t) = \mathfrak{I}^{\theta_2(1-\theta_1);\psi}\left(\frac{1}{\psi'(t)}\frac{d}{dt}\right)\mathfrak{I}^{(1-\theta_2)(1-\theta_1);\psi}\mathfrak{g}(t). \tag{2.1}$$

The ψ-HFD as above defined, can be written in the following

$$\mathfrak{D}^{\theta_1,\theta_2;\psi}\mathfrak{g}(t) = \mathfrak{I}^{\theta-\theta_1;\psi}\mathfrak{D}^{\theta;\psi}\mathfrak{g}(t).$$

Remark 2.5. *(a) If $\theta_2 = 0 (\gamma = 0,\ \eta = 0)$, then $\mathfrak{D}^{\theta_1,\theta_2;\psi} = \mathfrak{D}^{\theta_1,0;\psi}$ is called the RL fractional derivative of complex order.*
(b) If $\theta_2 = 1 (\gamma = 1,\ \eta = 0)$, then $\mathfrak{D}^{\theta_1,\theta_2;\psi} = \mathfrak{I}^{1-\theta_1;\psi}\mathfrak{D}^{1;\psi}$ is called the Caputo fractional derivative of complex order.

Definition 2.6. *The Stirling asymptotic formula of gamma function for $z \in \mathbb{C}$ is following*

$$\Gamma(z) = (2\pi)^{1/2} z^{z-\frac{1}{2}} e^{-z}\left[1 + O\left(\frac{1}{z}\right)\right] \quad (|\arg(z)| < \pi; |z| \to \infty),$$

and its result for $|\Gamma(a+ib)|$, $(a, b \in R)$ is

$$|\Gamma(a+ib)| = (2\pi)^{1/2}\,|b|^{a-\frac{1}{2}}\,e^{-a-\frac{\pi|b|}{2}}\left[1 + O\left(\frac{1}{z}\right)\right] \quad (b \to \infty).$$

Here, we shall give the definitions of Ulam-Hyers(U-H) stability and Ulam-Hyers-Rassias(U-H-R) stability for ψ-HFDEs of complex order. Let $\varepsilon > 0$ be a positive real number and $\varphi : J \to R^+$ be a continuous function. We consider the following inequalities:

$$\left|\mathfrak{D}^{\theta_1,\theta_2;\psi}\mathfrak{v}(t) - \mathfrak{g}(t,\mathfrak{v}(t))\right| \le \varepsilon, \quad t \in J, \tag{2.2}$$

$$\left|\mathfrak{D}^{\theta_1,\theta_2;\psi}\mathfrak{v}(t) - \mathfrak{g}(t,\mathfrak{v}(t))\right| \le \varepsilon\varphi(t), \quad t \in J, \tag{2.3}$$

$$\left|\mathfrak{D}^{\theta_1,\theta_2;\psi}\mathfrak{v}(t) - \mathfrak{g}(t,\mathfrak{v}(t))\right| \le \varphi(t), \quad t \in J. \tag{2.4}$$

Definition 2.7. *Eq. (1.1) is U-H stable if there exists a real number $C_f > 0$ such that for each $\varepsilon > 0$ and for each solution $\mathfrak{v} \in C_{1-\xi,\psi}(J)$ of the inequality (2.2) there exists a solution $\mathfrak{h} \in C_{1-\xi,\psi}(J)$ of Eq. (1.1) with*

$$|\mathfrak{v}(t) - \mathfrak{h}(t)| \le C_f\,\varepsilon, \quad t \in J.$$

Definition 2.8. *Eq. (1.1) is generalized U-H stable if there exist $\varphi \in C_{1-\xi,\psi}(J)$, $\varphi_f(0)=0$ such that for each solution $\mathfrak{v} \in C_{1-\xi,\psi}(J)$ of the inequality (2.2) there exists a solution $\mathfrak{h} \in C_{1-\xi,\psi}(J)$ of Eq. (1.1) with*

$$|\mathfrak{v}(t)-\mathfrak{h}(t)| \leq \varphi_f\, \varepsilon, \quad t \in J.$$

Definition 2.9. *Eq. (1.1) is U-H-R stable with respect to $\varphi \in C_{1-\xi,\psi}(J)$ if there exists a real number $C_{f,\varphi}>0$ such that for each $\varepsilon>0$ and for each solution $\mathfrak{v} \in C_{1-\xi,\psi}(J)$ of the inequality (2.3) there exists a solution $\mathfrak{h} \in C_{1-\xi,\psi}(J)$ of Eq. (1.1) with*

$$|\mathfrak{v}(t)-\mathfrak{h}(t)| \leq C_{f,\varphi}\, \varepsilon\varphi(t), \quad t \in J.$$

Definition 2.10. *Eq. (1.1) is generalized U-H-R stable with respect to $\varphi \in C_{1-\xi,\psi}(J)$ if there exists a real number $C_{f,\varphi}>0$ such that for each solution $\mathfrak{v} \in C_{1-\xi,\psi}(J)$ of the inequality (2.4) there exists a solution $\mathfrak{h} \in C_{1-\xi,\psi}(J)$ of Eq. (1.1) with*

$$|\mathfrak{v}(t)-\mathfrak{h}(t)| \leq C_{f,\varphi}\varphi(t), \quad t \in J.$$

Remark 2.11. *A function $\mathfrak{v} \in C_{1-\xi,\psi}(J)$ is a solution of the inequality*

$$\left|\mathfrak{D}^{\theta_1,\theta_2;\psi}\mathfrak{v}(t)-\mathfrak{g}(t,\mathfrak{v}(t))\right| \leq \varepsilon, \ \ t \in J,$$

iff there exist a function $g \in C_{1-\xi,\psi}(J)$ such that

(i) $|g(t)| \leq \varepsilon, t \in J.$
(ii) $\mathfrak{D}^{\theta_1,\theta_2;\psi}\mathfrak{v}(t)=\mathfrak{g}(t,\mathfrak{v}(t))+g(t), t \in J.$
(iii) If $\mathfrak{v}$ is solution of the inequality (2.2), then z is a solution of the following integral inequality

$$\left|\mathfrak{v}(t)-\frac{\mathfrak{v}_a}{\Gamma(\theta)}(\psi(t)-\psi(a))^{\theta-1}-\frac{1}{\Gamma(\theta_1)}\int_a^t \psi'(s)(\psi(t)-\psi(s))^{\theta_1-1}\mathfrak{g}(s,\mathfrak{v}(s))ds\right| \leq \frac{(\psi(b)-\psi(a))^{\alpha}}{\alpha|\Gamma(\theta_1)|}\varepsilon.$$

Lemma 2.12. *Suppose $\alpha(=\Re(\theta))>0$, $a(t)$ is a nonnegative function locally integrable on $a \leq t < b$ (some $b \leq \infty$), and let $g(t)$ be a nonnegative, nondecreasing continuous function defined on $a \leq t < b$, such that $g(t) \leq K$ for some constant K. Further let $\mathfrak{h}(t)$ be a nonnegative locally integrable on $a \leq t < b$ function with*

$$|\mathfrak{h}(t)| \leq a(t)+g(t)\int_a^t \psi'(s)(\psi(t)-\psi(s))^{\alpha-1}\mathfrak{h}(s)ds, \ \ t \in J$$

with some $\alpha>0$. Then

$$|\mathfrak{h}(t)| \leq a(t)+\int_a^t\left[\sum_{n=1}^{\infty}\frac{(g(t)\Gamma(\alpha))^n}{\Gamma(n\alpha)}\psi'(s)(\psi(t)-\psi(s))^{n\alpha-1}\right]a(s)ds, \ \ a \leq t < b.$$

Theorem 2.13. *(SFPT) Let E be a Banach space and Q be a nonempty bounded convex and closed subset of E and $N: Q \to Q$ is compact, and continuous map. Then N has at least one fixed point in Q.*

Lemma 2.14. *A function $\mathfrak{h}$ is the solution of*

$$\begin{cases}\mathfrak{D}^{\theta_1,\theta_2;\psi}\mathfrak{h}(t)=\mathfrak{g}(t), \ \ t \in J,\\ \mathfrak{I}^{1-\theta;\psi}\mathfrak{h}(t)|_{t=a}=\mathfrak{h}_a, \ \ \theta=\theta_1+\theta_2-\theta_1\theta_2,\end{cases} \tag{2.5}$$

equivalent to the solution of integral equation:

$$\mathfrak{h}(t)=\frac{\mathfrak{h}_a}{\Gamma(\theta)}(\psi(t)-\psi(a))^{\theta-1}+\frac{1}{\Gamma(\theta_1)}\int_a^t \psi'(s)(\psi(t)-\psi(s))^{\theta_1-1}\mathfrak{g}(s)ds. \tag{2.6}$$

3. Main results

Consider the following assumptions in order to solve the problem (1.1)-(1.2).

(H1) Let $\mathfrak{g}: J \times R \to R$ be continuous. For $\mathfrak{h},\mathfrak{v} \in R$, there exists a positive constant $L>0$ such that

$$|\mathfrak{g}(t,\mathfrak{h})-\mathfrak{g}(t,\mathfrak{v})| \leq L|\mathfrak{h}-\mathfrak{v}|, \quad t \in J.$$

(H2) The constant

$$\rho=\frac{L}{|\Gamma(\theta_1)|}(\psi(b)-\psi(a))^{\alpha}B(\xi,\alpha)<1.$$

(H3) Let $\mathfrak{g}: J \times R \to R$ be continuous. For $\mathfrak{h} \in R$, there exists $M \geq 0$ and $N>0$ such that

$$|\mathfrak{g}(t,\mathfrak{h})| \leq M|\mathfrak{h}|+N.$$

(H4) Suppose that there exists $\lambda_{\varphi}>0$ such that

$$\mathfrak{I}^{\theta_1;\psi}\varphi(t) \leq \lambda_{\varphi}\varphi(t).$$

Theorem 3.1. *If assumptions (H1) and (H2) are satisfied. Then, the Eq. (1.1)-(1.2) has a unique solution.*

Proof. Consider the operator $N : C_{1-\xi;\psi}(J) \to C_{1-\xi;\psi}(J)$ given by

$$(N\mathfrak{h})(t) = \frac{\mathfrak{h}_a}{\Gamma(\theta)}\left(\psi(t)-\psi(a)\right)^{\theta-1} + \frac{1}{\Gamma(\theta_1)}\int_a^t \psi^{'}(s)\left(\psi(t)-\psi(s)\right)^{\theta_1-1}\mathfrak{g}(s,\mathfrak{h}(s))ds. \tag{3.1}$$

Define a ball $B_r = \left\{\mathfrak{h} \in C_{1-\xi;\psi}(J) : \|\mathfrak{h}\| \leq r\right\}$. First, we show $N(B_r) \subset B_r$, for $\mathfrak{h} \in B_r$

$$\begin{aligned}
|(N\mathfrak{h})(t)| &= \left|\frac{\mathfrak{h}_a}{\Gamma(\theta)}\left(\psi(t)-\psi(a)\right)^{\theta-1} + \frac{1}{\Gamma(\theta_1)}\int_a^t \psi^{'}(s)\left(\psi(t)-\psi(s)\right)^{\theta_1-1}\mathfrak{g}(s,\mathfrak{h}(s))ds\right| \\
&\leq \frac{|\mathfrak{h}_a|}{|\Gamma(\theta)|}\left|\left(\psi(t)-\psi(a)\right)^{\theta-1}\right| + \frac{1}{|\Gamma(\theta_1)|}\int_a^t \psi^{'}(s)\left|\left(\psi(t)-\psi(s)\right)^{\theta_1-1}\right||\mathfrak{g}(s,\mathfrak{h}(s))|ds \\
&\leq \frac{|\mathfrak{h}_a|}{|\Gamma(\theta)|}\left(\psi(t)-\psi(a)\right)^{\xi-1} + \frac{1}{|\Gamma(\theta_1)|}\int_a^t \psi^{'}(s)\left(\psi(t)-\psi(s)\right)^{\alpha-1}|\mathfrak{g}(s,\mathfrak{h}(s))-\mathfrak{g}(s,0)|ds \\
&\quad + \frac{1}{|\Gamma(\theta_1)|}\int_a^t \psi^{'}(s)\left(\psi(t)-\psi(s)\right)^{\alpha-1}|\mathfrak{g}(s,0)|ds.
\end{aligned}$$

Thus,

$$\begin{aligned}
\|(N\mathfrak{h})(t)\|_{C_{1-\xi;\psi}} &\leq \frac{|\mathfrak{h}_a|}{|\Gamma(\theta)|} + \frac{\left(\psi(t)-\psi(a)\right)^{1-\xi}}{|\Gamma(\theta_1)|}\int_a^t \psi^{'}(s)\left(\psi(t)-\psi(s)\right)^{\alpha-1}L|\mathfrak{h}(s)|ds \\
&\quad + \frac{\left(\psi(t)-\psi(a)\right)^{1-\xi}}{|\Gamma(\theta_1)|}\int_a^t \psi^{'}(s)\left(\psi(t)-\psi(s)\right)^{\alpha-1}|\tilde{\mathfrak{g}}(s)|ds \\
&\leq \frac{|\mathfrak{h}_a|}{|\Gamma(\theta)|} + \frac{\left(\psi(b)-\psi(a)\right)^{\alpha}}{\alpha|\Gamma(\theta_1)|}B(\xi,\alpha)\left(L\|\mathfrak{h}\|_{C_{1-\xi;\psi}} + \|\tilde{\mathfrak{g}}\|_{C_{1-\xi;\psi}}\right) \\
&:= r.
\end{aligned}$$

Let $\mathfrak{h},\mathfrak{v} \in C_{1-\xi;\psi}(J)$ and for $t \in J$, we have

$$\begin{aligned}
&\left|\left((N\mathfrak{h})(t)-(N\mathfrak{v})(t)\right)\left(\psi(t)-\psi(a)\right)^{1-\xi}\right| \\
&\leq \frac{\left(\psi(t)-\psi(a)\right)^{1-\xi}}{|\Gamma(\theta_1)|}\int_a^t \psi^{'}(s)\left(\psi(t)-\psi(s)\right)^{\alpha-1}L|\mathfrak{h}(s)-\mathfrak{v}(s)|ds \\
&\leq \frac{L\left(\psi(t)-\psi(a)\right)^{1-\xi}}{|\Gamma(\theta_1)|}\left(\psi(t)-\psi(a)\right)^{\alpha+\xi-1}B(\xi,\alpha)\|\mathfrak{h}-\mathfrak{v}\|_{C_{1-\xi;\psi}} \\
&\leq \frac{L}{|\Gamma(\theta_1)|}\left(\psi(b)-\psi(a)\right)^{\alpha}B(\xi,\alpha)\|\mathfrak{h}-\mathfrak{v}\|_{C_{1-\xi;\psi}} \\
&\leq \|\mathfrak{h}-\mathfrak{v}\|_{C_{1-\xi;\psi}}.
\end{aligned}$$

□

Theorem 3.2. *Assume that [H3] is satisfied. Then, Eq.(1.1)-(1.2) has at least one solution.*

Proof. Consider the operator N, we check $N(B_r) \subset B_r$. For $\mathfrak{h} \in C_{1-\xi;\psi}(J)$ and $\|\mathfrak{h}\|_{C_{1-\xi;\psi}} < r^{'}$. By using assumption [H3], we can obtain

$$\begin{aligned}
|(N\mathfrak{h})(t)| &= \left|\frac{\mathfrak{h}_a}{\Gamma(\theta)}\left(\psi(t)-\psi(a)\right)^{\theta-1} + \frac{1}{\Gamma(\theta_1)}\int_a^t \psi^{'}(s)\left(\psi(t)-\psi(s)\right)^{\theta_1-1}\mathfrak{g}(s,\mathfrak{h}(s))ds\right| \\
&\leq \frac{|\mathfrak{h}_a|}{|\Gamma(\theta)|}\left|\left(\psi(t)-\psi(a)\right)^{\theta-1}\right| + \frac{1}{|\Gamma(\theta_1)|}\int_a^t \psi^{'}(s)\left|\left(\psi(t)-\psi(s)\right)^{\theta_1-1}\right||\mathfrak{g}(s,\mathfrak{h}(s))|ds \\
&\leq \frac{|\mathfrak{h}_a|}{|\Gamma(\theta)|}\left(\psi(t)-\psi(a)\right)^{\xi-1} + \frac{1}{|\Gamma(\theta_1)|}\int_a^t \psi^{'}(s)\left(\psi(t)-\psi(s)\right)^{\alpha-1}(M|\mathfrak{h}|+N)ds \\
\|(N\mathfrak{h})(t)\|_{C_{1-\xi;\psi}} &\leq \frac{|\mathfrak{h}_a|}{|\Gamma(\theta)|} + \frac{\left(\psi(t)-\psi(a)\right)^{1-\xi}}{|\Gamma(\theta_1)|}\int_a^t \psi^{'}(s)\left(\psi(t)-\psi(s)\right)^{\alpha-1}M|\mathfrak{h}|ds \\
&\quad + \frac{\left(\psi(t)-\psi(a)\right)^{1-\xi}}{|\Gamma(\theta_1)|}\int_a^t \psi^{'}(s)\left(\psi(t)-\psi(s)\right)^{\alpha-1}Nds \\
&\leq \frac{|\mathfrak{h}_a|}{|\Gamma(\theta)|} + M\frac{\left(\psi(b)-\psi(a)\right)^{\alpha}}{|\Gamma(\theta_1)|}B(\xi,\alpha)\|\mathfrak{h}\|_{C_{1-\xi;\psi}} + N\frac{\left(\psi(b)-\psi(a)\right)^{\alpha-\xi+1}}{\alpha|\Gamma(\theta_1)|} \\
&:= r^{'}.
\end{aligned}$$

Now we show that $N : B_r \to B_r$ is continuous. Let $\mathfrak{h}_n$ be a sequence such that $\mathfrak{h}_n \to \mathfrak{h}$ in B_r. Then for each $t \in J$, we have

$$\begin{aligned}
&\left|(N\mathfrak{h}_n(t) - N\mathfrak{h}(t))(\psi(t)-\psi(a))^{1-\xi}\right| \\
&\le \frac{(\psi(t)-\psi(a))^{1-\xi}}{|\Gamma(\theta_1)|}\int_0^t \psi'(s)\left|(\psi(t)-\psi(s))^{\theta_1-1}\right| |\mathfrak{g}(t,\mathfrak{h}_n(t)) - \mathfrak{g}(t,\mathfrak{h}(t))|\,ds \\
&\le \frac{(\psi(t)-\psi(a))^{1-\xi}}{|\Gamma(\theta_1)|}\int_0^t \psi'(s)(\psi(t)-\psi(s))^{\alpha-1} |\mathfrak{g}(t,\mathfrak{h}_n(t)) - \mathfrak{g}(t,\mathfrak{h}(t))|\,ds \\
&\le \frac{(\psi(t)-\psi(a))^{1-\xi}}{|\Gamma(\theta_1)|}(\psi(t)-\psi(a))^{\alpha+\xi-1} B(\xi,\alpha)\,\|\mathfrak{g}(\cdot,\mathfrak{h}_n(\cdot)) - \mathfrak{g}(\cdot,\mathfrak{h}(\cdot))\|_{C_{1-\xi,\psi}} \\
&\le \frac{1}{|\Gamma(\theta_1)|}(\psi(b)-\psi(a))^{\alpha} B(\xi,\alpha)\,\|\mathfrak{g}(\cdot,\mathfrak{h}_n(\cdot)) - \mathfrak{g}(\cdot,\mathfrak{h}(\cdot))\|_{C_{1-\xi,\psi}}.
\end{aligned}$$

Since $\mathfrak{g}$ is continuous, then by the Lebesgue dominated convergence theorem which implies

$$\|(N\mathfrak{h}_n)(t) - (N\mathfrak{h})(t)\|_{C_{1-\xi,\psi}} \to 0 \ \text{ as } \ n \to \infty.$$

Thus $N(B_r)$ is uniformly bounded. It is clear that $N(B_r) \subset B_r$ is bounded. Next we show that $N(B_r)$ is equicontinuous. Let $t_1, t_2 \in J$, such that $t_1 < t_2$, we get

$$\begin{aligned}
&\left|(\psi(t_2)-\psi(a))^{1-\xi}(N\mathfrak{h})(t_2) - (\psi(t_1)-\psi(a))^{1-\xi}(N\mathfrak{h})(t_1)\right| \\
&= \left|\frac{(\psi(t_2)-\psi(a))^{1-\xi}}{\Gamma(\theta_1)}\int_a^{t_2}\psi'(s)(\psi(t_2)-\psi(s))^{\theta_1-1}\mathfrak{g}(s,\mathfrak{h}(s))ds\right. \\
&\qquad \left. + \frac{(\psi(t_1)-\psi(a))^{1-\xi}}{\Gamma(\theta_1)}\int_a^{t_1}\psi'(s)(\psi(t_1)-\psi(s))^{\theta_1-1}\mathfrak{g}(s,\mathfrak{h}(s))ds\right| \\
&\le \frac{\|\mathfrak{g}\|_{C_{1-\xi,\psi}}}{|\Gamma(\theta_1)|}B(\xi,\alpha)\left|(\psi(t_2)-\psi(a))^{\alpha} + (\psi(t_1)-\psi(a))^{\alpha}\right|.
\end{aligned}$$

Thus from Steps 1 to 3 with the Arzelä-Ascoli theorem, the operator N is continuous and compact. From Theorem 2.13 the operator N has a fixed point $\mathfrak{h}$ which is a solution of the problem Eq.(2.5). □

Theorem 3.3. *The assumptions [H1] and [H4] hold. Then Eq.(1.1)-(1.2) is generalised U-H-R stable.*

Proof. Let $\mathfrak{v}$ be solution of 2.4 and by Theorem 3.1 there $\mathfrak{h}$ is unique solution of the problem

$$\begin{aligned}
&\mathfrak{D}^{\theta_1,\theta_2;\psi}\mathfrak{h}(t) = \mathfrak{g}(t,\mathfrak{h}(t)), \\
&\mathfrak{I}^{1-\theta;\psi}\mathfrak{h}(t)|_{t=a} = \mathfrak{I}^{1-\theta;\psi}\mathfrak{v}(t)|_{t=a} = \mathfrak{h}_a.
\end{aligned}$$

Then we have

$$\mathfrak{h}(t) = \frac{\mathfrak{v}_a}{\Gamma(\theta)}(\psi(t)-\psi(a))^{\theta-1} + \frac{1}{\Gamma(\theta_1)}\int_a^t \psi'(s)(\psi(t)-\psi(s))^{\theta_1-1}\mathfrak{g}(s,\mathfrak{h}(s))ds.$$

By differentiating inequality (2.4), we have

$$\begin{aligned}
&\left|\mathfrak{v}(t) - \frac{\mathfrak{v}_a}{\Gamma(\theta)}(\psi(t)-\psi(a))^{\theta-1} - \frac{1}{\Gamma(\theta_1)}\int_a^t \psi'(s)(\psi(t)-\psi(s))^{\theta_1-1}\mathfrak{g}(s,\mathfrak{v}(s))ds\right| \\
&\le \left|\frac{1}{\Gamma(\theta_1)}\int_a^t \psi'(s)(\psi(t)-\psi(s))^{\theta_1-1}\varphi(s)ds\right| \\
&\le \lambda_\varphi \varphi(t).
\end{aligned}$$

Hence it follows that,

$$\begin{aligned}
&|\mathfrak{v}(t) - \mathfrak{h}(t)| \\
&\le \left|\mathfrak{v}(t) - \frac{\mathfrak{v}_a}{\Gamma(\theta)}(\psi(t)-\psi(a))^{\theta-1} - \frac{1}{\Gamma(\theta_1)}\int_a^t \psi'(s)(\psi(t)-\psi(s))^{\theta_1-1}\mathfrak{g}(s,\mathfrak{h}(s))ds\right| \\
&\le \left|\mathfrak{v}(t) - \frac{\mathfrak{v}_a}{\Gamma(\theta)}(\psi(t)-\psi(a))^{\theta-1} - \frac{1}{\Gamma(\theta_1)}\int_a^t \psi'(s)(\psi(t)-\psi(s))^{\theta_1-1}\mathfrak{g}(s,\mathfrak{v}(s))ds\right| \\
&+ \int_a^t \psi'(s)(\psi(t)-\psi(s))^{\alpha-1}|\mathfrak{g}(s,\mathfrak{v}(s)) - \mathfrak{g}(s,\mathfrak{h}(s))|\,ds \\
&\le \lambda_\varphi \varphi(t) + \frac{L(\psi(b)-\psi(a))^{\alpha}}{\alpha|\Gamma(\theta_1)|}|\mathfrak{v}(t) - \mathfrak{h}(t)|
\end{aligned}$$

By Lemma 2.12, there exists a constant $K^* > 0$ independent of $\lambda_\varphi \varphi(t)$ such that

$$|\mathfrak{v}(t) - \mathfrak{h}(t)| \le K^* \varphi(t) := C_{f,\varphi}\varphi(t).$$

Thus, Eq.(1.1)-(1.2) is generalized U-H-R stable. □

4. An example

In this section, here we consider the following Cauchy problem in order to verify our results.

$$\begin{cases} \mathfrak{D}^{\theta_1,\theta_2;\psi}\mathfrak{h}(t) = \frac{1}{20}(\psi(t)-\psi(a))\cos(t)\mathfrak{h}(t),\ t\in J:=(a,b], \\ \mathfrak{I}^{1-\theta;\psi}\mathfrak{h}(t)|_{t=a} = \mathfrak{h}_a, \theta=\theta_1+\theta_2-\theta_1\theta_2, \end{cases} \tag{4.1}$$

By taking $\psi(t)=\ln t,\ a=1,\ b=e,\ \theta_1=\frac{1}{2}+\frac{1}{3}i,\ \theta_2=\frac{1}{3}+\frac{1}{2}i$, then we get a particular case of the proposed problem (4.1) using the Hadamard fractional derivative.

$$\mathfrak{D}^{\theta_1,\theta_2;\ln t}\mathfrak{h}(t) = \frac{1}{20}\ln t^{1/2}\cos(t)\mathfrak{h}(t),\ t\in(1,e], \tag{4.2}$$

$$\mathfrak{I}^{1-\theta;\ln t}\mathfrak{h}(1) = 1. \tag{4.3}$$

Here the function $\mathfrak{g}$ is continuous. Then, for all $\mathfrak{h},\mathfrak{v}\in R$, and $t\in(1,e]$, we have

$$|\mathfrak{g}(t,\mathfrak{h})-\mathfrak{g}(t,\mathfrak{v})| \le \frac{1}{20}|\mathfrak{h}-\mathfrak{v}|$$

Thus condition (H2) is satisfied with $L=\frac{1}{20}$. Then, for $\lambda_\varphi = \frac{2}{\sqrt{\pi}}\varphi(t) = \ln t^{1/2}$, condition (H4) is satisfied. Hence, by Theorem 3.1 and Theorem 3.3, the problem has a unique solution and it is U-H-R stability.

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Positive solutions for nonlinear fractional differential equation with nonlocal boundary conditions

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Abstract

In this paper, we study the boundary value problem of a class of fractional differential equations involving the Riemann-Liouville fractional derivative with nonlocal integral boundary conditions. To establish the existence results for the given problems, we use the properties of the Green's function and the monotone iteration technique, one shows the existence of positive solutions and constructs two successively iterative sequences to approximate the solutions. The results are illustrated with an example.

1. Introduction

In this paper, we are interested in the existence of solutions for the nonlinear fractional differential equation

$$D_{0^+}^{\alpha}u(t)+f(t,u(t))=0, \quad t\in(0,1), \tag{1.1}$$

subject to the boundary conditions

$$u^{(i)}(0)=0,\ 0\le i\le 2,\ D_{0^+}^{\beta}u(1)=\lambda I_{0^+}^{\beta}u(\eta), \tag{1.2}$$

where $D_{0^+}^{\alpha}$, $D_{0^+}^{\beta}$ are the standard Riemann-Liouville fractional derivative of order $\alpha\in(3,4]$, $\beta\in(0,2]$, I_{0+}^{β} is the standard Riemann-Liouville fractional integral of order $\beta\in(0,2]$ and $0\le\frac{\lambda\Gamma(\alpha-\beta)\eta^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}<1$.

The first definition of fractional derivative was introduced at the end of the nineteenth century by Liouville and Riemann, but the concept of non-integer derivative and integral, as a generalization of the traditional integer order differential and integral calculus, was mentioned already in 1695 by Leibniz and L'Hospital. In fact, fractional derivatives provide an excellent tool for the description of memory and hereditary properties of various materials and processes. The mathematical modelling of systems and processes in the fields of physics, chemistry, aerodynamics, electrodynamics of complex medium, polymer rheology, Bode's analysis of feedback amplifiers, capacitor theory, electrical circuits, electro-analytical chemistry, biology, control theory, fitting of experimental data, involves derivatives (or q-derivatives) of fractional order see for example [5, 6]. For more details we refer the reader to [2, 11] and the references cited therein.
Many mathematicians show strong interest in fractional differential equations and many wonderful results have been obtained. The techniques of nonlinear analysis, as the main method to deal with the problems of nonlinear fractional differential equations, plays an essential role in the research of this field, such as establishing the existence and the uniqueness or the multiplicity of solutions to nonlinear fractional differential equations boundary value problems, see [4, 7, 9, 10, 11, 12, 15, 16, 17, 18, 19] and the references therein.
In [3], the authors studied the boundary value problems of the fractional order differential equation:

$$\begin{cases} D_{0^+}^{\alpha}u(t)+f(t,u(t))=0, \quad t\in(0,1), \\ u(0)=0,\ D_{0^+}^{\beta}u(1)=aD_{0^+}^{\beta}u(\eta), \end{cases}$$

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where $1<\alpha\leq 2$, $0<\eta<1$, $0<a,\,\beta<1$, $f\in C\left([0,1]\times\mathbb{R}^2,\mathbb{R}\right)$ and $D_{0^+}^{\alpha}$, $D_{0^+}^{\beta}$ are the standard Riemann-Liouville fractional derivative of order α, β respectively. They obtained the multiple positive solutions by the Leray-Schauder nonlinear alternative and the fixed point theorem on cones.

The monotone iteration scheme is an interesting and effective way to investigate the existence of solutions to nonlinear fractional problem (see for example [8, 13, 14]). Inspired and motivated by the works mentioned above, we focus on the existence of positive solutions for the nonlocal boundary value problem $(1.1)-(1.2)$ by using the fixed point theorem for increasing operators on the order intervals, we also establish two iterative sequences to approximate the solutions. The paper is organized as follows. In Section 2, we recall some preliminary facts that we need in the sequel, for more details; see [1]. The existence of the positive solutions to the problem $(1.1)-(1.2)$, is proved and two successively iterative sequences to approximate the solutions are constructed and we give an example to illustrate our results in Section 3.

2. Preliminaries

In this section, we recall some definitions and facts which will be used in the later analysis.

Definition 2.1. *Let E be a real Banach space. A nonempty closed set $K\subset E$ is said to be a cone provided that*
(i) $c_1u+c_2v\in K$ for all $c_1\geq 0$, $c_2\geq 0$, and
(ii) $u\in K$, $-u\in K$ implies $u=0$.
Every cone K induces an ordering in E given by $u\leq v$ if and only if $v-u\in K$.

Definition 2.2. *The Riemann-Liouville fractional integral of order $\alpha>0$ of a function $u:(0,\infty)\to\mathbb{R}$ is given by*

$$I_{0^+}^{\alpha}u(t)=\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}u(s)\,ds,\quad t>0,$$

where $\Gamma(\cdot)$ is the Euler Gamma function, provided that the right side is pointwise defined on $(0,\infty)$.

Definition 2.3. *[1]. The Riemann-Liouville fractional derivative order $\alpha>0$ of a continuous function $u:(0,\infty)\to\mathbb{R}$ is defined by*

$$D_{0^+}^{\alpha}u(t)=\frac{1}{\Gamma(n-\alpha)}\frac{d^n}{dt^n}\int_0^t(t-s)^{n-\alpha-1}u(s)\,ds,\quad t>0,$$

where $n=\lceil\alpha\rceil+1$, $\lceil\alpha\rceil$ denotes the integer part of number α, provided that the right side is pointwise defined on $(0,\infty)$.

Lemma 2.4. *[1] (i) If $u\in L^p(0,1)$, $1\leq p\leq+\infty$, $\beta>\alpha>0$, then $D_{0^+}^{\alpha}I_{0^+}^{\beta}u(t)=I_{0^+}^{\beta-\alpha}u(t)$, $D_{0^+}^{\alpha}I_{0^+}^{\alpha}u(t)=u(t)$, $I_{0^+}^{\alpha}I_{0^+}^{\beta}u(t)=I_{0^+}^{\alpha+\beta}u(t)$.*
(ii) If $\beta>\alpha>0$, then $D^{\alpha}t^{\beta-1}=\frac{\Gamma(\beta)t^{\beta-\alpha-1}}{\Gamma(\beta-\alpha)}$.
(iii) If $\alpha>0$ and $\gamma\in(-1,+\infty)$, then $I_{0^+}^{\alpha}t^{\gamma}=\frac{\Gamma(\gamma+1)}{\Gamma(\alpha+\gamma+1)}t^{\alpha+\gamma}$.

Lemma 2.5. *[1] Let $\alpha>0$ and for any $y\in L^1(0,1)$. Then, the general solution of the fractional differential equation $D_{0^+}^{\alpha}u(t)+y(t)=0$, $0<t<1$ is given by*

$$u(t)=-\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}y(s)\,ds+c_1t^{\alpha-1}+c_2t^{\alpha-2}+\cdots+c_nt^{\alpha-n},$$

where $c_0,c_1,\ldots,c_{n-1}$ are real constants and $n=\lceil\alpha\rceil+1$.

Lemma 2.6. *Let $y\in C[0,1]$. Then the solution of the fractional boundary value problem*

$$\begin{cases} D_{0^+}^{\alpha}u(t)+y(t)=0,\\ u^{(i)}(0)=0,\ 0\leq i\leq 2,\\ D_{0^+}^{\beta}u(1)=\lambda I_{0^+}^{\beta}u(\eta), \end{cases}\tag{2.1}$$

is given by

$$u(t)=\int_0^1 G(t,s)y(s)\,ds,\tag{2.2}$$

where

$$G(t,s)=\begin{cases} \frac{-P\Gamma(\alpha-\beta)\Gamma(\alpha+\beta)(t-s)^{\alpha-1}+\Delta}{P\Gamma(\alpha)\Gamma(\alpha-\beta)\Gamma(\alpha+\beta)}, & 0\leq s\leq t\leq 1,\ s\leq\eta,\\ \frac{\Delta}{P\Gamma(\alpha)\Gamma(\alpha-\beta)\Gamma(\alpha+\beta)}, & 0\leq t\leq s\leq\eta\leq 1,\\ \frac{-P\Gamma(\alpha-\beta)\Gamma(\alpha+\beta)(t-s)^{\alpha-1}+\Lambda}{P\Gamma(\alpha)\Gamma(\alpha-\beta)\Gamma(\alpha+\beta)}, & 0\leq\eta\leq s\leq t\leq 1,\\ \frac{\Gamma(\alpha)\Gamma(\alpha+\beta)(1-s)^{\alpha-\beta-1}t^{\alpha-1}}{P\Gamma(\alpha)\Gamma(\alpha-\beta)\Gamma(\alpha+\beta)}, & 0\leq t\leq s\leq 1,\ s\geq\eta, \end{cases}\tag{2.3}$$

where $\Delta=t^{\alpha-1}\left[\Gamma(\alpha)\Gamma(\alpha+\beta)(1-s)^{\alpha-\beta-1}-\lambda\Gamma(\alpha)\Gamma(\alpha-\beta)(\eta-s)^{\alpha+\beta-1}\right]$,
$\Lambda=\Gamma(\alpha+\beta)\Gamma(\alpha)(1-s)^{\alpha-\beta-1}t^{\alpha-1}$.

and $P=\frac{\Gamma(\alpha)}{\Gamma(\alpha-\beta)}-\frac{\lambda\Gamma(\alpha)}{\Gamma(\alpha+\beta)}\eta^{\alpha+\beta-1}$.

Proof. In view of Lemma 2.5, the general solution for the above equation is

$$u(t) = -\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} y(s)\,ds + c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + c_3 t^{\alpha-3} + C_4 t^{\alpha-4},$$

where $c_1, c_2, c_3, c_4 \in \mathbb{R}$.
The boundary conditions $u(0) = u'(0) = u''(0) = 0$, implies that $c_2 = c_3 = c_4 = 0$. Thus

$$u(t) = -\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} y(s)\,ds + c_1 t^{\alpha-1}. \tag{2.4}$$

By (2.4) and Lemma 2.4, we get

$$D_{0^+}^{\beta} u(t) = \frac{1}{\Gamma(\alpha-\beta)}\left[c_1 \Gamma(\alpha) t^{\alpha-\beta-1} - \int_0^t (t-s)^{\alpha-\beta-1} y(s)\,ds\right].$$

In view of boundary condition $D_{0^+}^{\beta} u(1) = \lambda I_{0^+}^{\beta} u(\eta)$, we conclude that

$$c_1 = \frac{1}{P}\left[\frac{1}{\Gamma(\alpha-\beta)}\int_0^1 (1-s)^{\alpha-\beta-1} y(s)\,ds - \frac{\lambda}{\Gamma(\alpha+\beta)}\int_0^{\eta} (\eta-s)^{\alpha+\beta-1} y(s)\,ds\right].$$

Therefore, the unique solution of the problem (2.1) is given by

$$u(t) = \frac{t^{\alpha-1}}{P\Gamma(\alpha-\beta)}\int_0^1 (1-s)^{\alpha-\beta-1} y(s)\,ds - \frac{\lambda t^{\alpha-1}}{P\Gamma(\alpha+\beta)}\int_0^{\eta} (\eta-s)^{\alpha+\beta-1} y(s)\,ds$$

$$-\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} y(s)\,ds.$$

For $t \leq \eta$, one has

$$u(t) = \frac{t^{\alpha-1}}{P\Gamma(\alpha-\beta)}\left[\int_0^t (1-s)^{\alpha-\beta-1} y(s)\,ds + \int_t^{\eta} (1-s)^{\alpha-\beta-1} y(s)\,ds + \int_{\eta}^1 (1-s)^{\alpha-\beta-1} y(s)\,ds\right]$$

$$-\frac{\lambda t^{\alpha-1}}{P\Gamma(\alpha+\beta)}\left[\int_0^t (\eta-s)^{\alpha+\beta-1} y(s)\,ds + \int_t^{\eta} (\eta-s)^{\alpha+\beta-1} y(s)\,ds\right] - \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} y(s)\,ds,$$

$$= \int_0^t \frac{-P\Gamma(\alpha-\beta)\Gamma(\alpha+\beta)(t-s)^{\alpha-1} + \Delta}{P\Gamma(\alpha)\Gamma(\alpha+\beta)\Gamma(\alpha-\beta)} y(s)\,ds + \int_t^{\eta} \frac{\Delta}{P\Gamma(\alpha)\Gamma(\alpha+\beta)\Gamma(\alpha-\beta)} y(s)\,ds$$

$$+\int_{\eta}^1 \frac{\Gamma(\alpha)\Gamma(\alpha+\beta)(1-s)^{\alpha-\beta-1} t^{\alpha-1}}{P\Gamma(\alpha)\Gamma(\alpha+\beta)\Gamma(\alpha-\beta)} y(s)\,ds,$$

$$= \int_0^1 G(t,s) y(s)\,ds. \qquad .$$

For $t \geq \eta$, one has

$$u(t) = \int_0^{\eta} \frac{-P\Gamma(\alpha-\beta)\Gamma(\alpha+\beta)(t-s)^{\alpha-1} + \Delta}{P\Gamma(\alpha)\Gamma(\alpha+1)\Gamma(\alpha-\beta)} y(s)\,ds$$

$$+\int_{\eta}^t \frac{-P\Gamma(\alpha-\beta)\Gamma(\alpha+\beta)(t-s)^{\alpha-1} + \Gamma(\alpha)\Gamma(\alpha+\beta)(1-s)^{\alpha-\beta-1} t^{\alpha-1}}{P\Gamma(\alpha)\Gamma(\alpha+\beta)\Gamma(\alpha-\beta)} y(s)\,ds$$

$$+\int_t^1 \frac{\Gamma(\alpha)\Gamma(\alpha+\beta)(1-s)^{\alpha-\beta-1} t^{\alpha-1}}{P\Gamma(\alpha)\Gamma(\alpha+\beta)\Gamma(\alpha-\beta)} y(s)\,ds, \qquad ..$$

$$= \int_0^1 G(t,s) y(s)\,ds. \qquad .$$

The proof is complete. □

We need some properties of function $G(t,s)$ to establish the existence of positive solutions.

Lemma 2.7. *The Green's function $G(t,s)$ has the following properties:*
(i) *The function $G(t,s)$ is continuous and $G(t,s)>0$ for all $t,\ s\in(0,1)$.*
(ii) *For all $t,\ s\in(0,1)$, we have*

$$t^{\alpha-1}w_2(s)\le G(t,s)\le t^{\alpha-1}w_1(s), \tag{2.5}$$

where

$$w_1(s)=\frac{\lambda\eta^{\alpha+\beta-1}\left[(1-s)^{\alpha-\beta-1}-(1-s)^{\alpha+\beta-1}\right]}{P\Gamma(\alpha+\beta)},$$

and

$$w_2(s)=\frac{(1-s)^{\alpha-\beta-1}}{P\Gamma(\alpha-\beta)}.$$

Proof. It is easy to prove (i). Now, we prove (ii), assume that $0\le\frac{\lambda\Gamma(\alpha-\beta)\eta^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}<1$, then for $0\le s\le t\le 1,\ s\le\eta$, we get

$$\begin{aligned}P\Gamma(\alpha)\Gamma(\alpha+\beta)\Gamma(\alpha-\beta)G(t,s)&=-P\Gamma(\alpha-\beta)\Gamma(\alpha+\beta)(t-s)^{\alpha-\beta-1}\\&\quad+\Gamma(\alpha)\Gamma(\alpha+\beta)(1-s)^{\alpha-\beta-1}t^{\alpha-1}-\lambda\Gamma(\alpha)\Gamma(\alpha-\beta)(\eta-s)^{\alpha+\beta-1}t^{\alpha-1},\\&=\lambda\Gamma(\alpha)\Gamma(\alpha-\beta)\left\{\eta^{\alpha+\beta-1}(t-s)^{\alpha-1}-(\eta-s)^{\alpha+\beta-1}t^{\alpha-1}\right\}\\&\quad+\Gamma(\alpha)\Gamma(\alpha+\beta)\left\{-(t-s)^{\alpha-1}+(1-s)^{\alpha-\beta-1}t^{\alpha-1}\right\},\\&\ge\lambda\eta^{\alpha+\beta-1}\Gamma(\alpha)\Gamma(\alpha-\beta)t^{\alpha-1}\left[(1-s)^{\alpha-\beta-1}-(1-s)^{\alpha+\beta-1}\right],\end{aligned}$$

and

$$\begin{aligned}P\Gamma(\alpha)\Gamma(\alpha+\beta)\Gamma(\alpha-\beta)G(t,s)&=\Gamma(\alpha)\Gamma(\alpha+\beta)(1-s)^{\alpha-\beta-1}t^{\alpha-1}-\lambda\Gamma(\alpha)\Gamma(\alpha-\beta)(\eta-s)^{\alpha+\beta-1}t^{\alpha-1},\\&\le\Gamma(\alpha)\Gamma(\alpha+\beta)(1-s)^{\alpha-\beta-1}t^{\alpha-1}.\end{aligned}$$

For $0\le\eta\le s\le t\le 1$, we have

$$\begin{aligned}P\Gamma(\alpha)\Gamma(\alpha+\beta)\Gamma(\alpha-\beta)G(t,s)&=-P\Gamma(\alpha-\beta)\Gamma(\alpha+\beta)(t-s)^{\alpha-1}+\Gamma(\alpha)\Gamma(\alpha+\beta)(1-s)^{\alpha-\beta-1}t^{\alpha-1},\\&=-\Gamma(\alpha)\Gamma(\alpha+\beta)(t-s)^{\alpha-1}+\lambda\Gamma(\alpha)\Gamma(\alpha-\beta)\eta^{\alpha+\beta-1}(t-s)^{\alpha-1}\\&\quad+\Gamma(\alpha)\Gamma(\alpha+\beta)(1-s)^{\alpha-\beta-1}t^{\alpha-1},\\&\ge\lambda\Gamma(\alpha)\Gamma(\alpha-\beta)\eta^{\alpha+\beta-1}(t-s)^{\alpha-1}-\Gamma(\alpha)\Gamma(\alpha-\beta)\eta^{\alpha+\beta-1}(t-s)^{\alpha-1}\\&\quad+\Gamma(\alpha)\Gamma(\alpha-\beta)\eta^{\alpha+\beta-1}(1-s)^{\alpha-\beta-1}t^{\alpha-1},\\&\ge\lambda\Gamma(\alpha)\Gamma(\alpha-\beta)\eta^{\alpha-\beta-1}(1-s)^{\alpha-\beta-1}t^{\alpha-1}\\&\ge\lambda\Gamma(\alpha)\Gamma(\alpha-\beta)\eta^{\alpha+\beta-1}t^{\alpha-1}\left[(1-s)^{\alpha-\beta-1}-(1-s)^{\alpha+\beta-1}\right],\end{aligned}$$

and

$$\begin{aligned}P\Gamma(\alpha)\Gamma(\alpha+\beta)\Gamma(\alpha-\beta)G(t,s)&=-P\Gamma(\alpha-\beta)\Gamma(\alpha+\beta)(t-s)^{\alpha-1}+\Gamma(\alpha)\Gamma(\alpha+\beta)(1-s)^{\alpha-\beta-1}t^{\alpha-1},\\&=-\Gamma(\alpha)\Gamma(\alpha+\beta)(t-s)^{\alpha-1}t^{\alpha-1}+\lambda\Gamma(\alpha)\Gamma(\alpha-\beta)\eta^{\alpha+\beta-1}(t-s)^{\alpha-1}\\&\quad+\Gamma(\alpha)\Gamma(\alpha+\beta)(1-s)^{\alpha-\beta-1}t^{\alpha-1},\\&\le\Gamma(\alpha)\Gamma(\alpha+\beta)(1-s)^{\alpha-\beta-1}t^{\alpha-1}.\end{aligned}$$

For $0\le t\le s\le 1,\ s\ge\eta$, we get

$$\begin{aligned}P\Gamma(\alpha)\Gamma(\alpha+\beta)\Gamma(\alpha-\beta)G(t,s)&=\Gamma(\alpha)\Gamma(\alpha+\beta)(1-s)^{\alpha-\beta-1}t^{\alpha-1},\\&\ge\lambda\Gamma(\alpha)\Gamma(\alpha-\beta)\eta^{\alpha+\beta-1}t^{\alpha-1}\left[(1-s)^{\alpha-\beta-1}-(1-s)^{\alpha+\beta-1}\right].\end{aligned}$$

$\square$

3. Existence results

We shall consider the Banach space $E = C[0,1]$ equipped with the norm $\|u\| = \max_{0\le t\le 1} |u(t)|$ and let a closed cone $K \subset E$ by $K = \{u \in E : u \geq 0\}$ where 0 is the the zero function. Then K is normal.

Set $K_a = \{u \in E : \|u\| \leq a\}$. Define the operator $T : K_a \to E$ as

$$(Tu)(t) = \int_0^1 G(t,s) f(s,u(s))\, ds, \quad t \in [0,1], \tag{3.1}$$

where $G(t,s)$ is given by (2.3). It is not hard to see that fixed points of operator T coincide with the solutions to the problem (1.1) – (1.2).

Lemma 3.1. *[9] Let E be a Banach space ordered by a normal cone $K \subset E$. Assume that $T : [x_1,x_2] \to E$ is completely continuous operator such that $x_1 \leq Tx_2$, $x_2 \geq Tx_2$. Then T has a minimal fixed point x_* and a maximal fixed point x^* such that $x_1 \leq x_* \leq x^* \leq x_2$. Moreover, $x_* = \lim_{n\to\infty} T^n x_1$, $\quad x^* = T^n x_2$, where $\{T^n x_1\}_{n=1}^{\infty}$ is an increasing sequence and $\{T^n x_2\}_{n=1}^{\infty}$ is a decreasing sequence.*

First, for the existence results of problem (1.1) – (1.2), we need the following assumptions.
(A_1) $f : [0,1] \times [0,a] \to [0,\infty)$ is continuous and $f(t,0) \neq 0$,
(A_2) There exists a nonnegative function $q \in C[0,1] \subseteq L^1[0,1]$ such that $|f(t,u)| \leq q(t)$, $(t,u) \in [0,1] \times [0,a]$,
(A_3) $f(t,\underline{u}) \leq f(t,\overline{u})$, $t \in [0,1]$, $0 \leq \underline{u} \leq \overline{u} \leq a$.

Lemma 3.2. *Assume that $(A_1) - (A_3)$ hold. Then the operator T defined in (3.1) is a completely continuous increasing operator.*

Proof. Firstly, the operator T is continuous in view of the continuity of functions $f(t,u(t))$ and $G(t,s)$. Secondly, we will show that $T(K_a)$ is bounded. Let

$$L = \int_0^1 q(t)\, dt < \infty.$$

Then, for any $u \in K_a$, we have

$$\|(Tu)(t)\| = \max_{t\in[0,1]} \int_0^1 G(t,s) |f(s,u(s))|\, ds \leq \frac{L}{P\Gamma(\alpha-\beta)}, \quad t \in [0,1].$$

For each $u \in K_a$, one can show that

$$\begin{aligned}
\left|(Tu)'(t)\right| &= \left| \frac{(\alpha-1)t^{\alpha-2}}{P\Gamma(\alpha-\beta)} \int_0^1 (1-s)^{\alpha-\beta-1} f(s,u(s))\, ds \right. \\
&\quad \left. - \frac{\lambda(\alpha-1)t^{\alpha-2}}{P\Gamma(\alpha+\beta)} \int_0^\eta (\eta-s)^{\alpha+\beta-1} f(s,u(s))\, ds - \frac{\alpha-1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-2} f(s,u(s))\, ds \right|, \\
&= \left| \frac{(\alpha-1)t^{\alpha-2}}{P\Gamma(\alpha-\beta)} \int_0^1 (1-s)^{\alpha+\beta-1} f(s,u(s))\, ds \right. \\
&\quad \left. - \frac{\lambda(\alpha-1)t^{\alpha-2}}{P\Gamma(\alpha+\beta)} \int_0^\eta (\eta-s)^{\alpha+\beta-1} f(s,u(s))\, ds - \frac{1}{\Gamma(\alpha-1)} \int_0^t (t-s)^{\alpha-2} f(s,u(s))\, ds \right|, \\
&\leq \frac{(\alpha-1)t^{\alpha-2}}{P\Gamma(\alpha-\beta)} \int_0^1 (1-s)^{\alpha+\beta-1} |f(s,u(s))|\, ds \\
&\quad + \frac{\lambda(\alpha-1)t^{\alpha-2}}{P\Gamma(\alpha+\beta)} \int_0^\eta (\eta-s)^{\alpha+\beta-1} |f(s,u(s))|\, ds + \frac{1}{\Gamma(\alpha-1)} \int_0^t (t-s)^{\alpha-2} |f(s,u(s))|\, ds, \\
&\leq \frac{(\alpha-1)L}{P\Gamma(\alpha-\beta)} + \frac{\lambda(\alpha-1)L}{P\Gamma(\alpha+\beta)} + \frac{L}{\Gamma(\alpha-1)} = \bar{L}.
\end{aligned}$$

Therefore, for any t_1, $t_2 \in [t_1,t_2] \leq \overline{L}(t_2-t_1)$. with $t_1 < t_2$, we have

$$|(Tu)(t_1) - (Tu)(t_2)| \leq \int_{t_1}^{t_2} \left|(Tu)'(s)\right| ds \leq \overline{L}(t_2-t_1) \to 0 \, as \, t_2 \to t_1,$$

The Arzela-Ascoli theorem implies that the operator $T : K_a \to E$ is completely continuous. The assumption (A_3) provides that the operator $T : K_a \to E$ is an increasing operator. The proof is completed.

$\square$

Theorem 3.3. *Assume that* $(A_1)-(A_3)$ *hold, and*

$$\int_0^1 w_1(s) f(s,0)\, ds \geq 0, \qquad \int_0^1 w_2(s) f\left(a, as^{\alpha-1}\right) \leq a, \quad s \in [0,1]. \tag{3.2}$$

Then the problem $(1.1)-(1.2)$ *has two positive solutions* u^*, v^* *satisfying* $0 < u^* \leq v^* \leq a$. *Moreover, there exist a non-decreasing iterative sequence* $\{u_n\}_{n=0}^{\infty}$ *with*

$$\lim_{n\to\infty} u_n = u^*, \quad u_0 = 0, \quad u_{n+1} = Tu_n, \quad n = 0,1,2,\ldots, \tag{3.3}$$

and a non-decreasing iterative sequence $\{v_n\}_{n=0}^{\infty}$ *with*

$$\lim_{n\to\infty} v_n = u^*, \quad v_0 = at^{\alpha-1}, \quad v_{n+1} = Tv_n, \quad n = 0,1,2,\ldots. \tag{3.4}$$

Proof. We only need to prove that $Tu_0 \geq u_0$ and $Tv_0 \leq u_0$.

$$(Tu_0)(t) = \int_0^1 G(t,s) f(s,u_0)\, ds = \int_0^1 G(t,s) f(s,0)\, ds,$$

$$\geq t^{\alpha-1} \int_0^1 w_1(s) f(s,0)\, ds \geq 0 = u_0, \quad t \in [0,1], \tag{3.5}$$

this implies $u_1 \geq u_0$, wich combined with (A_3) gives

$$u_2 = (Tu_1)(t) = \int_0^1 G(t,s) f(s,u_1(t))\, ds \geq u_1, \quad t \in [0,1]$$

Similarly, we have

$$\begin{aligned} v_1 = Tv_0 &= \int_0^1 G(t,s) f(s,v_0)\, ds, \\ &\leq t^{\alpha-1} \int_0^1 w_2(s) f\left(s, at^{\alpha-1}\right) ds, \\ &\leq at^{\alpha-1} = v_0, \quad t \in [0,1] \end{aligned} \tag{3.6}$$

Then, by $(3.5)-(3.6)$ and induction, the iterative sequences $\{u_n\}$, $\{v_n\}$ satisfy

$$u_0(t) \leq u_1(t) \leq \ldots \leq u_n(t) \leq \ldots \leq v_n(t) \leq \ldots \leq v_1(t) \leq v_0(t), \ \forall t \in [0,1].$$

By induction, one can prove that $u_{n+1} \geq u_n$ and $v_{n+1} \leq v_n$.
Lemma 3.1 shows that the operator T has a minimal fixed point u^* and a maximal fixed point v^* satisfying $0 \leq u^* \leq v^* \leq a$. From (A_1) we find that the zero function is not the solution to the problem $(1.1)-(1.2)$. Thus $0 < u^* \leq v^* \leq a$. The proof is complete. □

We construct an example to illustrate the applicability of the results presented.

Example 3.4. *Consider the following boundary value problem*

$$D_{0^+}^{\frac{7}{2}} u(t) + \frac{1}{\sqrt{t}}\left(t + u^{\frac{1}{3}}(t) \tanh(u(t)) + u^{\frac{1}{3}}(t)\right) = 0,\ t \in (0,1),$$

$$u(0) = u'(0) = u''(0) = 0,\ D^{\frac{5}{2}} u(1) = \frac{1}{2} I_{0^+}^{\frac{5}{2}} u\left(\frac{1}{2}\right),$$

where $\alpha = \frac{7}{2}$, $\beta = \frac{5}{2}$, $\lambda = \frac{1}{2}$, $\eta = \frac{1}{2}$ *and* $f(t,u(t)) = \frac{1}{\sqrt{t}}\left(t + u^{\frac{1}{3}}(t) \tanh(u(t)) + u^{\frac{1}{3}}(t)\right)$.
We take $a = 10$. *By simple calculation we have*
$P = 3,3229182, \quad f(t,0) = \frac{t}{\sqrt{t}}$ *and* $f\left(t, 10t^{\frac{5}{2}}\right) = \frac{1}{\sqrt{t}}\left(t + (10)^{\frac{1}{3}} t^{\frac{5}{6}} + t^{\frac{5}{6}}\right)$.
A simple calculation leads to

$$\int_0^1 w_1(s) f(s,0)\, ds \simeq 0,0000239 \geq 0,$$

and

$$\int_0^1 w_2(s) f\left(s, 10s^{\frac{5}{2}}\right) ds \simeq 0,9124781 \leq 10.$$

Hence, all the assumptions of Theorem 3.3 are satisfied. Which implies that the boundary value $(1.1)-(2.1)$ *has two nontrivial solutions* u^*, v^* *with* $0 \leq u^* \leq v^* \leq 10$, *and the two monotone iterative sequences* $\{u_n\}_{n=1}^{\infty}$ *can be taken as*

$$u_0 = 0, \quad u_{n+1} = Tu_n, \quad v_0 = 10t^{\alpha-1}, \quad v_{n+1} = Tv_n, \quad n = 0,1,2,\ldots.$$

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Some remarks regarding the $(p,q)-$Fibonacci and Lucas octonion polynomials

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Abstract

We investigate the $(p,q)-$Fibonacci and Lucas octonion polynomials. The main purpose of this paper is using of some properties of the $(p,q)-$Fibonacci and Lucas polynomials. Also for present some results involving these octonion polynomials, we obtain some interesting computational formulas.

1. Introduction

Fibonacci, Lucas, Pell and the other special numbers are the special case of the second order linear recurrence $R=\{R_i\}_{i=0}^{\infty}$ if the recurrence relation for $i\geq 2,\ R_i=PR_{i-1}-QR_{i-2}$ holds for its terms, where P and Q are integers such that $D=P^2-4Q\neq 0$ (to exclude a degenerate case) and R_0,R_1 are fixed integers. Define the sequences

$$\begin{aligned} U_n &= PU_{n-1}-QU_{n-2} \\ V_n &= PV_{n-1}-QV_{n-2} \end{aligned} \tag{1.1}$$

for $n\geq 2$. The characteristic equation of them is $x^2-Px+Q=0$ and hence the roots of it are $\alpha=\frac{P+\sqrt{D}}{2}$ and $\beta=\frac{P-\sqrt{D}}{2}$. So by Binet's formula, $U_n=\frac{\alpha^n-\beta^n}{\alpha-\beta}$ and $V_n=\alpha^n+\beta^n$. Further the generating function for U_n and V_n is

$$\sum_{n=0}^{\infty}U_nx^n=\frac{x}{1-Px+Qx^2} \quad \text{and} \quad \sum_{n=0}^{\infty}V_nx^n=\frac{2-Px}{1-Px+Qx^2}$$

[8, 9].
Polynomials can be defined by Fibonacci-like recursion relations are called Fibonacci polynomials. More mathematicians were involved in the study of Fibonacci polynomials. Let $p(x)$ and $q(x)$ be polynomials with real coefficients. The $(p,q)-$Fibonacci polynomials are defined by the recurrence relation

$$F_{p,q,n+1}(x)=p(x)F_{p,q,n}(x)+q(x)F_{p,q,n-1}(x) \tag{1.2}$$

with the initial conditions $F_{p,q,0}(x)=0,\ F_{p,q,1}(x)=1$. Also for the $p(x)$ and $q(x)$ polynomials with real coefficients the $(p,q)-$Lucas polynomials are defined by the recurrence relation

$$L_{p,q,n+1}(x)=p(x)L_{p,q,n}(x)+q(x)L_{p,q,n-1}(x)$$

with the initial conditions $L_{p,q,0}(x)=2,\ L_{p,q,1}(x)=p(x)$. Let $\alpha_1(x)=\frac{p(x)+\sqrt{p^2(x)+4q(x)}}{2}$ and $\alpha_2(x)=\frac{p(x)-\sqrt{p^2(x)+4q(x)}}{2}$ denote the roots of the characteristic equation

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$$\alpha^2 - p(x)\alpha - q(x) = 0$$

on the recurrence relation of (1.2). Binet formulas for the $(p,q)-$Fibonacci polynomials and $(p,q)-$Lucas polynomials are

$$F_{p,q,n}(x) = \frac{\alpha_1^n(x) - \alpha_2^n(x)}{\alpha_1(x) - \alpha_2(x)} \text{ and } L_{p,q,n}(x) = \alpha_1^n(x) + \alpha_2^n(x).$$

[10]
Note that

$$\begin{aligned} \alpha_1(x) + \alpha_2(x) &= p(x) \\ \alpha_1(x) - \alpha_2(x) &= \sqrt{p^2(x) + 4q(x)} \\ \alpha_1(x).\alpha_2(x) &= -q(x) \\ \frac{\alpha_1(x)}{\alpha_2(x)} &= \frac{-\alpha_1^2(x)}{q(x)}, q(x) \neq 0 \\ \frac{\alpha_2(x)}{\alpha_1(x)} &= \frac{-\alpha_2^2(x)}{q(x)}, \ q(x) \neq 0. \end{aligned} \tag{1.3}$$

In [5], they introduce $(p,q)-$Fibonacci quaternion polynomials that generalize $h(x)-$Fibonacci quaternion polynomials. Division algebras are defined on real numbers $\mathbb{R}$, complex numbers $\mathbb{C}$, quaternions $\mathbf{H}$, and octonions $\mathbb{Q}$. There are different types of sequences of quaternions like Fibonacci Quaternions, Split Fibonacci Quaternions and Complex Fibonacci Quaternions [1].
The octonions in Clifford algebra are a normed division algebra with eight dimensions over the real numbers larger than the quaternions. The field $\mathbb{Q} \cong \mathbb{C}^4$ of octonions

$$\alpha = \sum_{s=0}^{7} \alpha_s e_s, \quad \alpha_i \in \mathbb{R}(i = 0, 1, \cdots, 7)$$

is an eight-dimensional non-commutative and non-associative $\mathbb{R}-$field generated by eight base elements $e_0, e_1, \cdots, e_6$ and e_7 which satisfy the non-commutative and non-associative multiplication rules are listed in below Table.

$\times$	e_0	e_1	e_2	e_3	e_4	e_5	e_6	e_7
e_0	e_0	e_1	e_2	e_3	e_4	e_5	e_6	e_7
e_1	e_1	$-e_0$	e_3	$-e_2$	e_5	$-e_4$	$-e_7$	e_6
e_2	e_2	$-e_3$	$-e_0$	e_1	e_6	e_7	$-e_4$	$-e_5$
e_3	e_3	e_2	$-e_1$	$-e_0$	e_7	$-e_6$	e_5	$-e_4$
e_4	e_4	$-e_5$	$-e_6$	$-e_7$	$-e_0$	e_1	e_2	e_3
e_5	e_5	e_4	$-e_7$	e_6	$-e_1$	$-e_0$	$-e_3$	e_2
e_6	e_6	e_7	e_4	$-e_5$	$-e_2$	e_3	$-e_0$	$-e_1$
e_7	e_7	$-e_6$	e_5	e_4	$-e_3$	$-e_2$	e_1	$-e_0$

The multiplication table for the basis of $\mathbb{Q}$

For $n \geq 0$, the Fibonacci octonion numbers that are given for the $n-th$ classic Fibonacci F_n number are defined by the following recurrence relations:

$$\mathbb{Q}_n = \sum_{s=0}^{7} F_{n+s} e_s.$$

Besides $h(x)$ $-$Fibonacci octonion polynomials can be defined by [6] that generalized both Catalan's Fibonacci octonion polynomials $\Psi_n(x)$ and Byrd's Fibonacci octonion polynomials and also $k-$ Fibonacci octonion numbers. Moreover in [2] they derived the Binet formula and generating function of $h(x)$ $-$Fibonacci octonion polynomial sequence.
Let $h(x)$ be a polynomial with real coefficients. The $h(x)-$Fibonacci octonion polynomials $\{O_{h,n}(x)\}_{n=0}^{\infty}$ are defined by the recurrence relation

$$O_{h,n}(x) = \sum_{s=0}^{7} F_{h,n+s}(x) e_s$$

where $F_{h,n}(x)$ is the $n-$th $h(x)-$Fibonacci polynomial in [2].

2. Main theorems of the $(p,q)-$Fibonacci and Lucas octonion polynomials

In the main section, we introduce the $(p,q)-$Fibonacci and Lucas octonion polynomials and formulate the Binet-style formula, the generating function and some identities of the $(p,q)-$Fibonacci octonion and Lucas octonion polynomial sequence. In [7], the authors obtained similar results for the $(p,q)-$Fibonacci and Lucas quaternion polynomials.
For $n \geq 0$ the Fibonacci octonion numbers that are given for the n-th classic Fibonacci F_n number are defined in [4]. Also $(p,q)-$Fibonacci octonions are investigated by [3].
So $(p,q)-$Fibonacci octonion polynomials $OF_{p,q,n}(x)$ are defined by the recurrence relation

$$OF_{p,q,n}(x) = \sum_{k=0}^{7} F_{p,q,n+k}(x)e_k$$

where $F_{p,q,n+k}(x)$ is the $(n+k)-th$ $(p,q)-$Fibonacci polynomial.
The initial conditions of this sequence are given by

$$\begin{aligned} OF_{p,q,0}(x) = \sum_{k=0}^{7} F_{p,q,k}(x)e_k &= e_1 + p(x)e_2 + (p^2(x)+q(x))e_3 + (p^3(x)+2p(x)q(x))e_4 \\ &+ (p^4(x)+3p^2(x)q(x)+q^2(x))e_5 + (p^5(x)+4p^3(x)q(x)+3p(x)q^2(x))e_6 \\ &+ (p^6(x)+5p^4(x)q(x)+6p^2(x)q^2(x)+q^3(x))e_7 \end{aligned}$$

and

$$\begin{aligned} OF_{p,q,1}(x) = \sum_{k=0}^{7} F_{p,q,1+k}(x)e_k &= e_0 + p(x)e_1 + (p^2(x)+q(x))e_2 + (p^3(x)+2p(x)q(x))e_3 \\ &+ (p^4(x)+3p^2(x)q(x)+q^2(x))e_4 + (p^5(x)+4p^3(x)q(x)+3p(x)q^2(x))e_5 \\ &+ (p^6(x)+5p^4(x)q(x)+6p^2(x)q^2(x)+q^3(x))e_6 \\ &+ (p^7(x)+6p^5(x)q(x)+10p^3(x)q^2(x)+4p(x)q^3(x))e_7. \end{aligned}$$

Also $OF_{p,q,n}(x)$ is written by a recurrence relation of order two;

$$\begin{aligned} OF_{p,q,n+1}(x) &= \sum_{k=0}^{7} F_{p,q,n+1+k}(x)e_k \\ &= \sum_{k=0}^{7} (p(x)F_{p,q,n+k}(x) + q(x)F_{p,q,n-1+k}(x))e_k \\ &= p(x)\sum_{k=0}^{7} F_{p,q,n+k}(x)e_k + q(x)\sum_{k=0}^{7} F_{p,q,n-1+k}(x)e_k \end{aligned}$$

and thus,

$$OF_{p,q,n+1}(x) = p(x)OF_{p,q,n}(x) + q(x)OF_{p,q,n-1}(x).$$

For the $n-th$ $(p,q)-$Lucas octonion polynomials $OL_{p,q,n}(x) = \sum_{k=0}^{7} L_{p,q,n+k}(x)e_k$,where $L_{p,q,n+k}$ is the $(n+k)-th$ $(p,q)-$Lucas polynomial. For $n \geq 1$

$$OL_{p,q,n+1}(x) = p(x)OL_{p,q,n}(x) + q(x)OL_{p,q,n-1}(x)$$

with the initial conditions.

Theorem 2.1. *The generating functions for the $(p,q)-$Fibonacci octonion polynomials $OF_{p,q,n}(x)$ and the $(p,q)-$Lucas octonion polynomials $OL_{p,q,n}(x)$ are*

$$g_{OF}(t) = \frac{OF_{p,q,0}(x) + [OF_{p,q,1}(x) - p(x)OF_{p,q,0}(x)]t}{1 - p(x)t - q(x)t^2}$$

and

$$g_{OL}(t) = \frac{OL_{p,q,0}(x) + [OL_{p,q,1}(x) - p(x)OL_{p,q,0}(x)]t}{1 - p(x)t - q(x)t^2}.$$

respectively.

Proof. The generating function $g_{OF}(t)$ for $OF_{p,q,n}(x)$ is to be of the form

$$\sum_{n=0}^{\infty} OF_{p,q,n}(x)t^n = OF_{p,q,0}(x) + OF_{p,q,1}(x)t + OF_{p,q,2}(x)t^2 + \cdots + OF_{p,q,n}(x)t^n + \cdots. \tag{2.1}$$

The formal power series expansions of $g_{OF}(t), -p(x)tg_{OF}(t)$ and $-q(x)t^2 g_{OF}(t)$ are

$$\begin{aligned}
g_{OF}(t) &= \sum_{n=0}^{\infty} OF_{p,q,n}(x)t^n = OF_{p,q,0}(x) + OF_{p,q,1}(x)t + OF_{p,q,2}(x)t^2 \\
&\quad + \cdots + OF_{p,q,n}(x)t^n + \cdots \\
-p(x)tg_{OF}(t) &= -p(x)OF_{p,q,0}(x)t - p(x)OF_{p,q,1}(x)t^2 - p(x)OF_{p,q,2}(x)t^3 \\
&\quad - \cdots - p(x)OF_{p,q,n}(x)t^{n+1} - \cdots \\
-q(x)t^2 g_{OF}(t) &= -q(x)OF_{p,q,0}(x)t^2 - q(x)OF_{p,q,1}(x)t^3 - q(x)OF_{p,q,2}(x)t^4 \\
&\quad - \cdots - q(x)OF_{p,q,n}(x)t^{n+2} - \cdots
\end{aligned}$$

respectively. So the expansion for $g_{OF}(t) - g_{OF}(t)p(x)t - g_{OF}(t)q(x)t^2$ is

$$\begin{aligned}
g_{OF}(t)[1 - p(x)t - q(x)t^2] &= OF_{p,q,0}(x) + OF_{p,q,1}(x)t - p(x)OF_{p,q,0}(x)t \\
&\quad + [OF_{p,q,2}(x) - p(x)OF_{p,q,1}(x) - q(x)OF_{p,q,0}(x)]t^2 \\
&\quad + [OF_{p,q,3}(x) - p(x)OF_{p,q,2}(x) - q(x)OF_{p,q,1}(x)]t^3 \\
&\quad + \ldots + [OF_{p,q,n}(x) - p(x)OF_{p,q,n-1}(x) - q(x)OF_{p,q,n-2}(x)]t^n \\
&\quad + \ldots \\
&= OF_{p,q,0}(x) + [OF_{p,q,1}(x) - p(x)OF_{p,q,0}(x)]t.
\end{aligned}$$

Hence $OF_{p,q,0}(x) + [OF_{p,q,1}(x) - p(x)OF_{p,q,0}(x)]t$ is a finite series, so we can rewrite $[1 - p(x)t - q(x)t^2]g_{OF}(t) = OF_{p,q,0}(x) + [OF_{p,q,1}(x) - p(x)OF_{p,q,0}(x)]t$ and hence

$$g_{OF}(t) = \frac{OF_{p,q,0}(x) + [OF_{p,q,1}(x) - p(x)OF_{p,q,0}(x)]t}{1 - p(x)t - q(x)t^2} \tag{2.2}$$

as we claimed.
Similarly, it can be also proved that $g_{OL}(t) = \frac{OL_{p,q,0}(x) + [OL_{p,q,1}(x) - p(x)OL_{p,q,0}(x)]t}{1 - p(x)t - q(x)t^2}$. □

Lemma 2.2. *For the generating function given in Theorem 2.1, we have*

$$\begin{aligned}
g_{OF}(t) &= \frac{1}{\alpha_1(x) - \alpha_2(x)} \left(\frac{OF_{p,q,1}(x) - \alpha_2(x)OF_{p,q,0}(x)}{1 - \alpha_1(x)t} - \frac{OF_{p,q,1}(x) - \alpha_1(x)OF_{p,q,0}(x)}{1 - \alpha_2(x)t} \right) \\
g_{OL}(t) &= \frac{1}{\alpha_1(x) - \alpha_2(x)} \left(\frac{OL_{p,q,1}(x) - \alpha_2(x)OL_{p,q,0}(x)}{1 - \alpha_1(x)t} - \frac{OL_{p,q,1}(x) - \alpha_1(x)OL_{p,q,0}(x)}{1 - \alpha_2(x)t} \right).
\end{aligned}$$

Proof. Using the expression of $g_{OF}(t)$ in Teorem 2.1 and (1.3) ,we found

$$\frac{OF_{p,q,0}(x) + [OF_{p,q,1}(x) - p(x)OF_{p,q,0}(x)]t}{1 - p(x)t - q(x)t^2} = \frac{OF_{p,q,0}(x) + [OF_{p,q,1}(x) - p(x)OF_{p,q,0}(x)]t}{(1 - \alpha_1(x)t)(1 - \alpha_2(x)t)}$$

$$= \left(\frac{OF_{p,q,0}(x) + [OF_{p,q,1}(x) - (\alpha_1(x) + \alpha_2(x))OF_{p,q,0}(x)]t}{(1 - \alpha_1(x)t)(1 - \alpha_2(x)t)} \right) \times \left(\frac{\alpha_1(x) - \alpha_2(x)}{\alpha_1(x) - \alpha_2(x)} \right)$$

$$= \frac{\left\{ \begin{array}{c} \alpha_1(x)OF_{p,q,0}(x) + \alpha_1(x)OF_{p,q,1}(x)t - \alpha_1^2(x)OF_{p,q,0}(x)t \\ -\alpha_1(x)\alpha_2(x)OF_{p,q,0}(x)t - \alpha_2(x)OF_{p,q,0}(x) - \alpha_2(x)OF_{p,q,1}(x)t \\ +\alpha_1(x)\alpha_2(x)OF_{p,q,0}(x)t + \alpha_2^2(x)OF_{p,q,0}(x)t + OF_{p,q,1}(x) - OF_{p,q,1}(x) \end{array} \right\}}{(\alpha_1(x) - \alpha_2(x))(1 - \alpha_1(x)t)(1 - \alpha_2(x)t)}$$

$$= \frac{\left\{ \begin{array}{c} OF_{p,q,1}(x)(1 - \alpha_2(x)t) + \alpha_2(x)OF_{p,q,0}(x)(-1 + \alpha_2(x)t) \\ +OF_{p,q,1}(x)(-1 + \alpha_1(x)t) + \alpha_1(x)OF_{p,q,0}(x)(1 - \alpha_1(x)t) \end{array} \right\}}{(\alpha_1(x) - \alpha_2(x))(1 - \alpha_1(x)t)(1 - \alpha_2(x)t)}$$

$$= \frac{\left\{ \begin{array}{c} (1 - \alpha_2(x)t)(OF_{p,q,1}(x) - \alpha_2(x)OF_{p,q,0}(x)) \\ -(1 - \alpha_1(x)t)(OF_{p,q,1}(x) - \alpha_1(x)OF_{p,q,0}(x)) \end{array} \right\}}{(\alpha_1(x) - \alpha_2(x))(1 - \alpha_1(x)t)(1 - \alpha_2(x)t)}$$

$$= \frac{1}{\alpha_1(x) - \alpha_2(x)} \left[\frac{OF_{p,q,1}(x) - \alpha_2(x)OF_{p,q,0}(x)}{1 - \alpha_1(x)t} - \frac{OF_{p,q,1}(x) - \alpha_1(x)OF_{p,q,0}(x)}{1 - \alpha_2(x)t} \right].$$

□

Lemma 2.3. *Let $F_{p,q,n}(x)$ and $L_{p,q,n}(x)$ be the $(p,q)-$Fibonacci and Lucas polynomials respectively. We have*
1.

$$\begin{aligned} F_{p,q,k+1}(x)-\alpha_2(x)F_{p,q,k}(x) &= \alpha_1^k(x)\\ F_{p,q,k+1}(x)-\alpha_1(x)F_{p,q,k}(x) &= \alpha_2^k(x)\end{aligned}$$

2.

$$\begin{aligned} \frac{L_{p,q,k+1}(x)-\alpha_2(x)L_{p,q,k}(x)}{\alpha_1(x)-\alpha_2(x)} &= \alpha_1^k(x)\\ \frac{\alpha_1(x)L_{p,q,k}(x)-L_{p,q,k+1}(x)}{\alpha_1(x)-\alpha_2(x)} &= \alpha_2^k(x).\end{aligned}$$

Proof. 1.We prove it by induction. Let $k=1$

$$F_{p,q,2}(x)-\alpha_2(x)F_{p,q,1}(x)=p(x)-\alpha_2(x)=\alpha_1(x).$$

So the hypothesis is right for $k=1$. Let us assume that the equation is $F_{p,q,n}(x)-\alpha_2(x)F_{p,q,n-1}(x)=\alpha_1^{n-1}(x)$ for $k=n-1$. For $k=n$ it becomes

$$\begin{aligned}\alpha_1^n(x) &= \alpha_1^{n-1}(x)\alpha_1(x)\\ &= (F_{p,q,n}(x)-\alpha_2(x)F_{p,q,n-1}(x))\alpha_1(x)\\ &= \alpha_1(x)F_{p,q,n}(x)-\alpha_1(x)\alpha_2(x)F_{p,q,n-1}(x)\\ &= (p(x)-\alpha_2(x))F_{p,q,n}(x)-(-q(x))F_{p,q,n-1}(x)\\ &= p(x)F_{p,q,n}(x)+q(x)F_{p,q,n-1}(x)-\alpha_2(x)F_{p,q,n}(x)\\ &= F_{p,q,n+1}(x)-\alpha_2(x)F_{p,q,n}(x).\end{aligned}$$

So we get the desired result for the $(p,q)-$Fibonacci polynomials. 2. The $(p,q)-$Lucas polynomials can be proved similarly. □

To derive the Binet Formulas for $OF_{p,q,n}(x)$ and $OL_{p,q,n}(x)$, we can give the following theorems.

Theorem 2.4. *For $n\geq 0$, the Binet formula for the $(p,q)-$Fibonacci octonion polynomials $OF_{p,q,n}(x)$ and also $OL_{p,q,n}(x)$ is as follows*

$$\begin{aligned} OF_{p,q,n}(x) &= \frac{\alpha_1^*(x)\alpha_1^n(x)-\alpha_2^*(x)\alpha_2^n(x)}{\alpha_1(x)-\alpha_2(x)}\\ OL_{p,q,n}(x) &= \alpha_1^*(x)\alpha_1^n(x)+\alpha_2^*(x)\alpha_2^n(x)\end{aligned}$$

where $\alpha_1^*(x)=\sum\limits_{k=0}^{7}\alpha_1^k(x)e_k$ and $\alpha_2^*(x)=\sum\limits_{k=0}^{7}\alpha_2^k(x)e_k$.

Proof. From Lemma 2.1, we get

$$\begin{aligned} g_{OF}(t) &= \frac{1}{\alpha_1(x)-\alpha_2(x)}[(OF_{p,q,1}(x)-\alpha_2(x)OF_{p,q,0}(x))\\ &\quad \sum_{n=0}^{\infty}\alpha_1^n(x)t^n-(OF_{p,q,1}(x)-\alpha_1(x)OF_{p,q,0}(x))\sum_{n=0}^{\infty}\alpha_2^n(x)t^n]\\ &= \frac{1}{\alpha_1(x)-\alpha_2(x)}\left\{\begin{array}{c}\sum_{k=0}^{7}(F_{p,q,1+k}(x)-\alpha_2(x)F_{p,q,k}(x))e_k\sum_{n=0}^{\infty}\alpha_1^n(x)t^n\\ -\sum_{k=0}^{7}(F_{p,q,1+k}(x)-\alpha_1(x)F_{p,q,k}(x))e_k\sum_{n=0}^{\infty}\alpha_1^n(x)t^n\end{array}\right\}\\ &= \frac{1}{\alpha_1(x)-\alpha_2(x)}\left[\sum_{k=0}^{7}\alpha_1^k(x)e_k\sum_{n=0}^{\infty}\alpha_1^n(x)t^n-\sum_{k=0}^{7}\alpha_2^k(x)e^k\sum_{n=0}^{\infty}\alpha_2^n(x)t^n\right]\\ &= \sum_{n=0}^{\infty}\frac{\alpha_1^*(x)\alpha_1^n(x)-\alpha_2^*(x)\alpha_2^n(x)}{\alpha_1(x)-\alpha_2(x)}t^n.\end{aligned}$$

Similarly, it can be also proved that $OL_{p,q,n}(x)=\alpha_1^*(x)\alpha_1^n(x)+\alpha_2^*(x)\alpha_2^n(x)$. □

Theorem 2.5. *(Catalan identity) Let the $(p,q)-$Fibonacci and Lucas octonion polynomials $OF_{p,q,n}(x)$ and $OL_{p,q,n}(x)$. For n and α, nonnegative integer numbers, such that $\alpha\leq n$, we have*

$$\begin{aligned} OF_{p,q,n+r}(x)OF_{p,q,n-r}(x)-OF_{p,q,n}^2(x) &= \frac{(-1)^{r+n+1}\alpha_1^*(x)\alpha_2^*(x)q^{n-r}(x)(\alpha_1^r(x)-\alpha_2^r(x))^2}{(\alpha_1(x)-\alpha_2(x))^2}\\ OL_{p,q,n+r}(x)OL_{p,q,n-r}(x)-OL_{p,q,n}^2(x) &= (-1)^{r+n}\alpha_1^*(x)\alpha_2^*(x)q^{n-r}(x)(\alpha_1^r(x)-\alpha_2^r(x))^2.\end{aligned}$$

Proof. Using the identity (1.3), Lemma 2.2 and Theorem 2.2, we have

$$\begin{aligned}
&OF_{p,q,n+r}(x)OF_{p,q,n-r}(x)-OF^2_{p,q,n}(x)\\
&=\left(\frac{\alpha_1^*(x)\alpha_1^{n+r}(x)-\alpha_2^*(x)\alpha_2^{n+r}(x)}{\alpha_1(x)-\alpha_2(x)}\right)\left(\frac{\alpha_1^*(x)\alpha_1^{n-r}(x)-\alpha_2^*(x)\alpha_2^{n-r}(x)}{\alpha_1(x)-\alpha_2(x)}\right)\\
&-\left(\frac{\alpha_1^*(x)\alpha_1^{n}(x)-\alpha_2^*(x)\alpha_2^{n}(x)}{\alpha_1(x)-\alpha_2(x)}\right)^2\\
&=\frac{\left\{\begin{array}{c}-\alpha_1^*(x)\alpha_2^*(x)\alpha_1^{n-r}(x)\alpha_2^{n+r}(x)\\-\alpha_1^*(x)\alpha_2^*(x)\alpha_1^{n+r}(x)\alpha_2^{n-r}(x)\\+2\alpha_1^*(x)\alpha_2^*(x)\alpha_1^{n}(x)\alpha_2^{n}(x)\end{array}\right\}}{(\alpha_1(x)-\alpha_2(x))^2}\\
&=\frac{-\alpha_1^*(x)\alpha_2^*(x)\alpha_1^{n}(x)\alpha_2^{n}(x)\left[\left(-\frac{\alpha_2^2(x)}{q(x)}\right)^r+\left(-\frac{\alpha_1^2(x)}{q(x)}\right)^r-2\frac{(\alpha_1(x)\alpha_2(x))^r}{q^r(x)}\right]}{(\alpha_1(x)-\alpha_2(x))^2}\\
&=\frac{(-1)^{r+n+1}\alpha_1^*(x)\alpha_2^*(x)q^{n-r}(x)(\alpha_1^r(x)-\alpha_2^r(x))^2}{(\alpha_1(x)-\alpha_2(x))^2}.
\end{aligned}$$

The other case can be proved similarly. □

Theorem 2.6. *(Cassini identity) For the $(p,q)-$Fibonacci octonion polynomials $OF_{p,q,n}(x)$ and $(p,q)-$Lucas octonion polynomials $OL_{p,q,n}(x)$, we have*

$$\begin{aligned}
&OF_{p,q,n+1}(x)OF_{p,q,n-1}(x)-OF^2_{p,q,n}(x)=(-1)^n\alpha_1^*(x)\alpha_2^*(x)q^{n-1}(x)\\
&OL_{p,q,n+1}(x)OL_{p,q,n-1}(x)-OL^2_{p,q,n}(x)=(-1)^{1+n}\alpha_1^*(x)\alpha_2^*(x)q^{n-1}(x)(\alpha_1(x)-\alpha_2(x))^2
\end{aligned}$$

for any natural number n .

Theorem 2.7. *Let $OF_{p,q,n}(x)$ and $OL_{p,q,n}(x)$ be the $(p,q)-$Fibonacci and Lucas octonion polynomials respectively. Then for $n\geq 0$, we have*

1.

$$\begin{aligned}
&q(x)(OF_{p,q,n}(x))^2+(OF_{p,q,n+1}(x))^2=\frac{(\alpha_1^*)^2(x)\alpha_1^{2n+1}(x)-(\alpha_2^*)^2(x)\alpha_2^{2n+1}(x)}{\alpha_1(x)-\alpha_2(x)}\\
&q(x)(OL_{p,q,n}(x))^2+(OL_{p,q,n+1}(x))^2=(\alpha_1(x)-\alpha_2(x))(\alpha_1^*)^2(x)\alpha_1^{2n+1}(x)-(\alpha_2^*)^2(x)\alpha_2^{2n+1}(x)
\end{aligned}$$

2.

$$\begin{aligned}
&OF_{p,q,1}(x)-\alpha_1(x)QF_{p,q,0}(x)=\alpha_2^*(x)\\
&OF_{p,q,1}(x)-\alpha_2(x)QF_{p,q,0}(x)=\alpha_1^*(x)
\end{aligned}$$

and

$$\begin{aligned}
&OL_{p,q,1}(x)-\alpha_1(x)OL_{p,q,0}(x)=(\alpha_1(x)-\alpha_2(x))\alpha_2^*(x)\\
&OL_{p,q,1}(x)-\alpha_2(x)OL_{p,q,0}(x)=(\alpha_1(x)-\alpha_2(x))\alpha_1^*(x).
\end{aligned}$$

Proof. Let us prove the identity 1.. From Theorem 2.2

$$\begin{aligned}
q(x)(OF_{p,q,n}(x))^2+(OF_{p,q,n+1}(x))^2&=q(x)\left(\frac{\alpha_1^*(x)\alpha_1^{n}(x)-\alpha_2^*(x)\alpha_2^{n}(x)}{\alpha_1(x)-\alpha_2(x)}\right)^2+\left(\frac{\alpha_1^*(x)\alpha_1^{n+1}(x)-\alpha_2^*(x)\alpha_2^{n+1}(x)}{\alpha_1(x)-\alpha_2(x)}\right)^2\\
&=\frac{\left\{\begin{array}{c}q(x)(\alpha_1^*)^2(x)\alpha_1^{2n}(x)-2q(x)\alpha_1^*(x)\alpha_1^{n}(x)\alpha_2^*(x)\alpha_2^{n}(x)\\+q(x)(\alpha_2^*)^2(x)\alpha_2^{2n}(x)+(\alpha_1^*)^2(x)\alpha_1^{2n+2}(x)\\-2\alpha_1^*(x)\alpha_1^{n+1}(x)\alpha_2^*(x)\alpha_2^{n+1}(x)+(\alpha_2^*)^2(x)\alpha_2^{2n+2}(x)\end{array}\right\}}{(\alpha_1(x)-\alpha_2(x))^2}\\
&=\frac{(\alpha_1^*)^2(x)\alpha_1^{2n}(x)\left(q(x)-q(x)\frac{\alpha_1(x)}{\alpha_2(x)}\right)+(\alpha_2^*)^2(x)\alpha_2^{2n}(x)\left(q(x)-q(x)\frac{\alpha_2(x)}{\alpha_1(x)}\right)}{(\alpha_1(x)-\alpha_2(x))^2}\\
&=\frac{(\alpha_1^*)^2(x)\alpha_1^{2n+1}(x)-(\alpha_2^*)^2(x)\alpha_2^{2n+1}(x)}{\alpha_1(x)-\alpha_2(x)}.
\end{aligned}$$

Also the proof of the identity 2. is similar to 1.. □

Theorem 2.8. *For the $(p,q)-$Fibonacci and Lucas octonion polynomials $OF_{p,q,n}(x)$ and $OL_{p,q,n}(x)$, $n\geq 0$ we have following binomial sum formula for odd and even terms,* *1.*

$$OF_{p,q,2n}(x) = \sum_{m=0}^{n} \binom{n}{m} q(x)^{n-m} p(x)^m OF_{p,q,m}(x)$$
$$OF_{p,q,2n+1}(x) = \sum_{m=0}^{n} \binom{n}{m} q(x)^{n-m} p(x)^m OF_{p,q,m+1}(x)$$

2.

$$OL_{p,q,2n}(x) = \sum_{m=0}^{n} \binom{n}{m} q(x)^{n-m} p(x)^m OL_{p,q,m}(x)$$
$$OL_{p,q,2n+1}(x) = \sum_{m=0}^{n} \binom{n}{m} q(x)^{n-m} p(x)^m OL_{p,q,m+1}(x).$$

Proof. For 1. from (1.3) and Binet formulas, we get

$$\begin{aligned}
&\sum_{m=0}^{n} \binom{n}{m} q(x)^{n-m} p(x)^m OF_{p,q,m}(x) \\
&= \sum_{m=0}^{n} \binom{n}{m} q(x)^{n-m} p(x)^m \frac{\alpha_1^*(x)\alpha_1^m(x) - \alpha_2^*(x)\alpha_2^m(x)}{\alpha_1(x)-\alpha_2(x)} \\
&= \frac{\alpha_1^*(x)}{\alpha_1(x)-\alpha_2(x)} \sum_{m=0}^{n} \binom{n}{m} q(x)^{n-m} (p(x)\alpha_1(x))^m \\
&- \frac{\alpha_2^*(x)}{\alpha_1(x)-\alpha_2(x)} \sum_{m=0}^{n} \binom{n}{m} q(x)^{n-m} (p(x)\alpha_2(x))^m \\
&= \frac{\alpha_1^*(x)}{\alpha_1(x)-\alpha_2(x)} (q(x)+p(x)\alpha_1(x))^n - \frac{\alpha_2^*(x)}{\alpha_1(x)-\alpha_2(x)} (q(x)+p(x)\alpha_2(x))^n \\
&= \frac{\alpha_1^*(x)\alpha_1^{2n}(x) - \alpha_2^*(x)\alpha_2^{2n}(x)}{\alpha_1(x)-\alpha_2(x)} \\
&= OF_{p,q,2n}(x).
\end{aligned}$$

Also the other cases for $OL_{p,q,n}(x)$ can be done similarly. □

Theorem 2.9. *The sums of the first $n-$terms of the sequences $OF_{p,q,n}(x)$ and $OL_{p,q,n}(x)$ are given by*

$$\sum_{m=0}^{n} OF_{p,q,m}(x) = \frac{-q(x)OF_{p,q,n}(x) - OF_{p,q,n+1}(x) + OF_{p,q,0}(x) - \frac{\alpha_1^*(x)\alpha_2(x)-\alpha_2^*(x)\alpha_1(x)}{\alpha_1(x)-\alpha_2(x)}}{(\alpha_1(x)-1)(\alpha_2(x)-1)}$$

and

$$\sum_{m=0}^{n} OL_{p,q,m}(x) = \frac{-q(x)OL_{p,q,n}(x) - OL_{p,q,n+1}(x) + OL_{p,q,0}(x) - \left[\alpha_1^*(x)\alpha_2(x)+\alpha_2^*(x)\alpha_1(x)\right]}{(\alpha_1(x)-1)(\alpha_2(x)-1)}$$

respectively.

Proof. Using Binet formulas and the roots $\alpha_1(x)$, $\alpha_2(x)$, we get

$$\begin{aligned}
\sum_{m=0}^{n} OF_{p,q,m}(x) &= \frac{\alpha_1^*(x)\alpha_1^m(x) - \alpha_2^*(x)\alpha_2^m(x)}{\alpha_1(x)-\alpha_2(x)} \\
&= \frac{1}{\alpha_1(x)-\alpha_2(x)} \sum_{m=0}^{n} (\alpha_1^*(x)\alpha_1^m(x) - \alpha_2^*(x)\alpha_2^m(x)) \\
&= \frac{1}{\alpha_1(x)-\alpha_2(x)} (\alpha_1^*(x) \sum_{m=0}^{n} \alpha_1^m(x) - \alpha_2^*(x) \sum_{m=0}^{n} \alpha_2^m(x)) \\
&= \frac{1}{\alpha_1(x)-\alpha_2(x)} (\alpha_1^*(x) \frac{\alpha_1^{n+1}(x)-1}{\alpha_1(x)-1} - \alpha_2^*(x) \frac{\alpha_2^{n+1}(x)-1}{\alpha_2(x)-1}) \\
&= \frac{\alpha_1^*(x)(\alpha_1^{n+1}(x)-1)(\alpha_2(x)-1) - \alpha_2^*(x)(\alpha_2^{n+1}(x)-1)(\alpha_1(x)-1)}{(\alpha_1(x)-\alpha_2(x))(\alpha_1(x)-1)(\alpha_2(x)-1)} \\
&= \frac{\left\{ \begin{array}{c} (\alpha_1^*(x)\alpha_1^{n+1}(x)\alpha_2(x)) - (\alpha_1^*(x)\alpha_1^{n+1}(x)) - (\alpha_1^*(x)\alpha_2(x)) + \alpha_1^*(x) \\ -\alpha_2^*(x) - \alpha_2^*(x)\alpha_2^{n+1}(x)\alpha_1(x) + \alpha_2^*(x)\alpha_2^{n+1}(x) + \alpha_2^*(x)\alpha_2(x) \end{array} \right\}}{(\alpha_1(x)-\alpha_2(x))(\alpha_1(x)-1)(\alpha_2(x)-1)} \\
&= \frac{-q(x)OF_{p,q,n}(x) - OF_{p,q,n+1}(x) + OF_{p,q,0}(x) - \frac{\alpha_1^*(x)\alpha_2(x)-\alpha_2^*(x)\alpha_1(x)}{\alpha_1(x)-\alpha_2(x)}}{(\alpha_1(x)-1)(\alpha_2(x)-1)}.
\end{aligned}$$

The other cases for $OL_{p,q,n}(x)$ can be done similarly. □

3. Conclusion

Octonions have great importance as they are used in quantum physics, applied mathematics, graph theory. In this work, we introduce the $(p,q)-$Fibonacci and Lucas octonion polynomials and formulate the Binet-style formula, the generating function and some identities of the $(p,q)-$Fibonacci octonion and Lucas octonion polynomial sequence. Thus, in our future studies we plan to examine different quaternion and octonion polynomials and their key features.

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Quantum metric spaces of quantum maps

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Abstract

We show that any quantum family of quantum maps from a noncommutative space to a compact quantum metric space has a canonical quantum pseudo-metric structure. Here by a 'compact quantum metric space' we mean a unital C*-algebra together with a Lipschitz seminorm, in the sense of Rieffel, which induces the weak* topology on the state space of the C*-algebra. Our main result generalizes a classical result to noncommutative world.

1. Introduction

One of the basic ideas of *Noncommutative Geometry* is that any unital C*-algebra A can be considered as the algebra of *continuous functions* on a (symbolic) *compact quantum (noncommutative) space* $\mathfrak{Q}A$. From this point of view, any unital *-homomorphism $\Phi : B \to A$ between unital C*-algebras can be interpreted as a *quantum map* $\mathfrak{Q}\Phi$ from $\mathfrak{Q}A$ into $\mathfrak{Q}B$. There are many notions in Topology and Geometry that can be translate into NC language. The notion of *quantum family of (quantum) maps*, defined by Woronowicz [16] and Sołtan [15] (see also [10, 11, 12]), conclude from the following fact: "Every map f from X to the set of all maps from Y to Z (or in other word, any family of maps from Y to Z parameterized by f with parameters x in X) can be considered as a map $\tilde{f} : X \times Y \to Z$ defined by $\tilde{f}(x,y) = f(x)(y)$." A translation of this to noncommutative language is as follows.

Definition 1.1. *([10, 11, 12, 15, 16] Let B,C be unital C*-algebras. A quantum family of morphisms from B to C (or, a quantum family of maps from $\mathfrak{Q}C$ to $\mathfrak{Q}B$) is a pair (A,Φ) consisting of a unital C*-algebra A and a unital *-homomorphism $\Phi : B \to C \otimes A$, where $\otimes$ denotes the spatial tensor product of C*-algebras.*

Another concept that can be translate from Geometry into NC Geometry, is *distance* or *metric*. Marc Rieffel, by using the notion of *order unite spaces*, has developed the notion of *quantum metric space* in a series of papers [5, 6, 7, 8, 9]. For two other different notions of quantum metric see [3, 13, 14]. Here, we deals with special examples of Rieffel's quantum metric spaces, stated in the C*-algebraic formalism. The aim of this note is to show that any quantum family of maps from a quantum space to a compact quantum metric space has a canonical quantum pseudo-metric structure. We are motivated by the following trivial fact: Let (Z,d) be a metric space and $f : X \times Y \to Z$ be a family of maps from Y to Z, then X has a pseudo-metric ρ defined by

$$\rho(x,x') = \sup_{y\in Y} d(f(x,y), f(x',y)).$$

In Section 2 we introduce the notion of *compact quantum pseudo-metric space*. In Section 3 we define a natural compact quantum pseudo-metric space structure on any quantum family of maps from a quantum space to a compact quantum metric space. In Section 4 we examine our definition in the classical case.

2. Compact quantum pseudo-metric spaces

By a pseudo-metric d on a set X we mean a positive valued function on $X \times X$ which is symmetric, satisfies triangle inequality, and $d(x,x) = 0$ for every $x \in X$. For any topological space X with topology τ (resp. pseudo-metric space (X,d)) $\mathbf{C}(X,\tau)$ (resp. $\mathbf{C}(X,d)$) denotes the

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C*-algebra of all continuous bounded complex valued maps on X with the uniform norm. For a pseudo-metric d, τ_d denotes the topology induced by d. Let (X,d) be a pseudo-metric space. For every $f \in \mathbf{C}(X,d)$, the Lipschitz semi norm $\|f\|_d$ is defined by

$$\|f\|_d = \sup\{\frac{|f(x)-f(x')|}{d(x,x')} : x,x' \in X, d(x,x') \neq 0\}.$$

Also, the Lipschitz algebra of (X,d) is defined by,

$$\mathbf{Lip}(X,d) = \{f \in \mathbf{C}(X,d) : \|f\|_d < \infty\}.$$

We need the following simple lemma.

Lemma 2.1. *Let (X,d) be a pseudo-metric space and a be a complex valued map on X. Then $a \in \mathbf{Lip}(X,d)$ and $\|a\|_d \leq 1$ if and only if $|a(x)-a(x')| \leq d(x,x')$ for every $x,x' \in X$. In particular, if $b \in \mathbf{C}(X,d)$, then $\|b\|_d = 0$ if and only if b is a constant map.*

Proof. Let $a \in \mathbf{Lip}(X,d)$ and $\|a\|_d \leq 1$. Suppose that $x,x' \in X$. If $d(x,x') = 0$, then $a(x) = a(x')$, since a is continuous with τ_d. If $d(x,x') \neq 0$, then $1 \geq \|a\|_d \geq \frac{|a(x)-a(x')|}{d(x,x')}$, and thus $|a(x)-a(x')| \leq d(x,x')$. The other direction is trivial. □

For any C*-algebra $\mathfrak{A}$, $S(\mathfrak{A})$ denotes the state space of $\mathfrak{A}$ with w* topology. If $\mathfrak{A}$ is unital, $1_{\mathfrak{A}}$ denotes the unit element of $\mathfrak{A}$.
Let $\mathscr{A}$ be a self adjoint linear subspace of the C*-algebra $\mathfrak{A}$, and let $L : \mathscr{A} \to [0,\infty)$ be a semi norm on $\mathscr{A}$. Connes has pointed out [1], [2], that one can define a pseudo-metric ρ_L on $S(\mathfrak{A})$ by

$$\rho_L(\mu,\nu) = \sup\{|\mu(a)-\nu(a)| : a \in \mathscr{A}, L(a) \leq 1\} \qquad (\mu,\nu \in S(\mathfrak{A})). \tag{2.1}$$

Note that ρ_L can take values $+\infty$ and 0 for different states of $\mathfrak{A}$. Conversely, let d be a pseudo-metric on $S(A)$ (such that the topology induced by d on $S(\mathfrak{A})$ is not necessarily w* topology). Define a semi norm $L_d : \mathfrak{A} \to [0,+\infty]$ by

$$L_d(a) = \sup\{\frac{|\mu(a)-\nu(a)|}{d(\mu,\nu)} : \mu,\nu \in S(\mathfrak{A}), d(\mu,\nu) \neq 0\} \qquad (a \in \mathfrak{A}).$$

Note that $L_d(a) = L_d(a^*)$ for every $a \in \mathfrak{A}$.
Let (X,d) be a compact metric space. Consider the Lipschitz semi norm

$$\|\cdot\|_d : \mathbf{Lip}(X,d) \subset \mathbf{C}(X,d) \to [0,+\infty).$$

Then it is easily checked that the semi norm $\rho_{\|\cdot\|_d}$ on the state space of $\mathbf{C}(X,d)$ is a metric, called Monge-Kantorovich metric [4]. It is well known that the topology induced by $\rho_{\|\cdot\|_d}$, is the w* topology, and for every $x,y \in X$, $d(x,y) = \rho_{\|\cdot\|_d}(\delta_x,\delta_y)$, where $\delta : X \to \mathbf{C}(X,d)^*$ is the point mass measure map.

Proposition 2.2. *Let (X,τ) be a compact Hausdorff space and d be a pseudo-metric on X such that the topology induced by d on X is weaker than τ, i.e. $\tau_d \subset \tau$. Consider the Lipschitz semi norm $\|\cdot\|_d : \mathbf{Lip}(X,d) \subset \mathbf{C}(X,\tau) \to [0,+\infty)$ and let $\rho = \rho_{\|\cdot\|_d}$. Then the following are satisfied.*

i) $d(x,y) = \rho(\delta_x,\delta_y)$, for every $x,y \in X$.
ii) $L_\rho = \|\cdot\|_d$ on $\mathbf{C}(X,d) \subset \mathbf{C}(X,\tau)$.
iii) Let $a \in \mathbf{C}(X,\tau)$, then $a \in \mathbf{C}(X,d)$ if and only if the map $\nu \longmapsto \nu(a)$ on $S(\mathbf{C}(X,\tau))$ is continuous with ρ.
iv) the topology induced by ρ on $S(\mathbf{C}(X,\tau))$ is weaker than the w topology.*

Proof. i) Let x,y be in X. Suppose that $a \in \mathbf{Lip}(X,d)$ and $\|a\|_d \leq 1$. Then by Lemma 2.1, $|\delta_x(a)-\delta_y(a)| = |a(x)-a(y)| \leq d(x,y)$, and thus by definition of ρ, we have $\rho(\delta_x,\delta_y) \leq d(x,y)$. Conversely, let $a_x \in \mathbf{C}(X,d)$ be defined by $a_x(z) = d(x,z)$ $(z \in X)$; then for every $x',y' \in X$, $|a_x(x')-a_x(y')| = |d(x,x')-d(x,y')| \leq d(x',y')$, and thus by lemma 2.1, $a \in \mathbf{Lip}(X,d)$ and $\|a\|_d \leq 1$. Now, we have

$$\rho(\delta_x,\delta_y) \geq |\delta_x(a_x)-\delta_y(a_x)| = |a_x(x)-a_x(y)| = d(x,y).$$

ii) By i) and definitions of L_ρ and $\|\cdot\|_d$, it is clear that $\|\cdot\|_d \leq L_\rho$ on $\mathbf{C}(X,\tau)$.
Let $a \in \mathbf{C}(X,d)$. If $\|a\|_d = 0$, then by Lemma 2.1, a is a constant map and thus $L_\rho(a) = 0$. If $\|a\|_d = \infty$ then $L_\rho(a) = \infty$ since $\|a\|_d \leq L_\rho(a)$. Thus suppose that $0 < \|a\| < \infty$. Then for every $\mu,\nu \in S(\mathbf{C}(X,\tau))$, we have

$$\rho(\mu,\nu) \geq |\mu(\frac{a}{\|a\|_d})-\nu(\frac{a}{\|a\|_d})| = \frac{|\mu(a)-\nu(a)|}{\|a\|_d}$$

and thus if $\rho(\mu,\nu) \neq 0$ then $\|a\|_d \geq \frac{|\mu(a)-\nu(a)|}{\rho(\mu,\nu)}$. Therefore,

$$\|a\|_d \geq \sup\{\frac{|\mu(a)-\nu(a)|}{\rho(\mu,\nu)} : \quad \mu,\nu \in S(\mathbf{C}(X,\tau)), \rho(\mu,\nu) \neq 0\} = L_\rho(a).$$

iii) The 'if' part is an immediate consequence of i). For the other direction, we need some notations: Let $\sim$ be the equivalence relation on X defined by $x \sim x' \Leftrightarrow d(x,x') = 0$. Let $Y = X/\sim$ and let $\hat{}: X \to Y$ be the canonical projection. Then $\hat{d}$, defined by $\hat{d}(\hat{x}_1,\hat{x}_2) = d(x_1,x_2)$, is a well defined metric on Y, and $\hat{}$ is an isometry between (X,d) and $(Y,\hat{d})$. Thus the C*-algebras $\mathbf{C}(X,d)$ and $\mathbf{C}(Y,\hat{d})$, and the Lipschitz algebras $(\mathbf{Lip}(X,d),\|\cdot\|_d)$ and $(\mathbf{Lip}(Y,\hat{d}),\|\cdot\|_{\hat{d}})$ are isometric isomorph. In particular, the topology induced by ρ on $S(\mathbf{C}(X,d))$ is the w* topology, since as mentioned above the Monge-Kantorovich metric $\rho_{\|\cdot\|_{\hat{d}}}$ induces the w* topology on $S(\mathbf{C}(Y,\hat{d}))$. Consider the canonical embedding $\Phi : \mathbf{C}(X,d) \to \mathbf{C}(X,\tau)$. For every $\nu,\nu' \in S(\mathbf{C}(X,\tau))$, $\nu \circ \Phi$ and $\nu' \circ \Phi$ are in $S(\mathbf{C}(X,d))$ and

$$\rho(\nu,\nu') = \rho(\nu \circ \Phi, \nu' \circ \Phi). \tag{2.2}$$

Now, let $a \in \mathbf{C}(X,d)$ and $v_i \to v$ be a convergent net in $S(\mathbf{C}(X,\tau))$ with ρ. Then $v_i \circ \Phi \to v \circ \Phi$ is a convergent net in $S(\mathbf{C}(X,d))$ with ρ, and since the topology induced by ρ agrees with the w* topology on $S(\mathbf{C}(X,d))$, we have

$$v_i(a) = v_i \circ \Phi(a) \to v \circ \Phi(a) = v(a).$$

Thus we get the desired result.
iv) Let $v_i \to v$ be a convergent net in $S(\mathbf{C}(X,\tau))$ with w* topology. Thus as in the proof of iii), $v_i \circ \Phi \to v \circ \Phi$ with ρ, and by (2.2), $v_i \to v$ in $S(\mathbf{C}(X,\tau))$ with the topology induced by ρ. This completes the proof of iv). □

Definition 2.3. *By a compact quantum pseudo-metric space (QSM space, for short) we mean a triple $(\mathfrak{A},\mathscr{A},L)$, where $\mathfrak{A}$ is a unital C*-algebra, $\mathscr{A}$ is a self adjoint linear subspace of $\mathfrak{A}$ with $1_{\mathfrak{A}} \in \mathscr{A}$, and $L : \mathscr{A} \to [0,+\infty)$ is a semi norm such that*

(a) $L(a) = L(a^*)$ for every $a \in \mathscr{A}$,
(b) for every $a \in \mathscr{A}$, $L(a) = 0$ if and only if $a \in \mathbb{C}1_{\mathfrak{A}}$, and
(c) the topology induced by the pseudo-metric ρ_L on $S(\mathfrak{A})$ is weaker than the w topology.*

As an immediate corollary of the definition, for any compact quantum pseudo-metric space $(\mathfrak{A},\mathscr{A},L)$, the topology induced by ρ_L on $S(\mathfrak{A})$ is compact and in particular the diameter of $S(\mathfrak{A})$ under ρ_L is finite.

Proposition 2.4. *Let $(\mathfrak{A},\mathscr{A},L)$ be a QSM space. Then, for every $a \in \mathscr{A}$, the map $\mu \longmapsto \mu(a)$ on $S(\mathfrak{A})$ is continuous with topology induced by ρ_L.*

Proof. Straightforward. □

Definition 2.5. *A QSM space $(\mathfrak{A},\mathscr{A},L)$ is called a compact quantum metric space (QM space, for short) if $\mathscr{A}$ is a dense subspace of $\mathfrak{A}$.*

Let $(\mathfrak{A},\mathscr{A},L)$ be a QM space and μ,v be two different states of $\mathfrak{A}$. Then since $\mathscr{A}$ is dense in $\mathfrak{A}$, there is $a \in \mathscr{A}$ such that $\mu(a) \neq v(a)$. Thus (by (2.1)) ρ_L is a metric on $S(\mathfrak{A})$. It is an elementary result in Topology that any Hausdorff topology τ weaker than a compact Hausdorrf topology τ' on a set X, is equal to the same topology τ'. Using this, we conclude that the topology induced by ρ_L on $S(\mathfrak{A})$ is the w* topology.

Example 2.6. *Let (X,d) be a compact metric space. Then*

$$(\mathbf{C}(X,d),\mathbf{Lip}(X,d),\|\cdot\|_d)$$

is a compact quantum metric space.

Example 2.7. *Let (X,τ) be a compact Hausdorff space and let d be a pseudo-metric on X such that $\tau_d \subset \tau$. Then Proposition 2.2 and Lemma 2.1, show*

$$(\mathbf{C}(X,\tau),\mathbf{Lip}(X,d),\|\cdot\|_d)$$

is a compact quantum pseudo-metric space.

Remark 2.8. *Let $(\mathfrak{A},\mathscr{A},L)$ be a QM space and $A \subset \mathscr{A}$ be the linear subspace of all self-adjoint elements of $\mathscr{A}$. Then A is an order unite space and $(A,L|_A)$ is a compact quantum metric space in the sense of Rieffel's definition [7].*

Lemma 2.9. *Let $\mathfrak{A}$ be a C*-algebra with the C*-norm $\|\cdot\|$, $\mathscr{A}$ be a self adjoint linear subspace of $\mathfrak{A}$ containing $1_{\mathfrak{A}}$ and $L : \mathscr{A} \to [0,+\infty)$ be a semi norm such that for every $a \in \mathscr{A}$, $L(a) = 0$ if and only if $a \in \mathbb{C}1_{\mathfrak{A}}$. Let $\tilde{L}$ and $\|\cdot\|\tilde{}$ denote the quotient norm of L and $\|\cdot\|$ on $\frac{\mathscr{A}}{\mathbb{C}1_{\mathfrak{A}}}$ and $\frac{\mathfrak{A}}{\mathbb{C}1_{\mathfrak{A}}}$, respectively. Suppose that the image of $\{a \in \mathscr{A} : L(a) \leq 1\}$ in $\frac{\mathfrak{A}}{\mathbb{C}1_{\mathfrak{A}}}$ is totally bounded for $\|\cdot\|\tilde{}$. Then the topology induced by ρ_L on $S(\mathfrak{A})$ is weaker than the w* topology.*

Proof. See Theorem 1.8 of [5]. □

Example 2.10. *Let $\mathfrak{A}$ be a finite dimensional C*-algebra and N be a Banach space norm on $\mathfrak{A}$ such that $N(a) = N(a^*)$ for every $a \in \mathfrak{A}$. Let the semi norm $N_0 : \mathfrak{A} \to [0,\infty)$ be defined by*

$$N_0 = \inf\{N(a+\lambda 1_{\mathfrak{A}}) : \quad \lambda \in \mathbb{C}\}.$$

Since $\mathfrak{A}$ is finite dimensional, the C-norm of $\mathfrak{A}$ and N are equivalent. Thus the image K of $\{a \in \mathfrak{A} : N_0(a) \leq 1\}$ is closed and bounded in $\frac{\mathfrak{A}}{\mathbb{C}1_{\mathfrak{A}}}$. Again, since $\mathfrak{A}$ is finite dimensional, K is compact and thus totally bounded for the quotient norm of the C*-norm. Thus by Lemma 2.9, $(\mathfrak{A},\mathfrak{A},N_0)$ is a QM space.*

Example 2.11. *Let G be a compact Hausdorff group with identity element e. Let ℓ be a length function on G, i.e. ℓ is a continuous non negative real valued function on G such that*

(i) $\ell(gg') \leq \ell(g)+\ell(g')$, for every $g,g' \in G$,
(ii) $\ell(g) = \ell(g^{-1})$ for every $g \in G$, and
(iii) $\ell(g) = 0$ if and only if $g = e$.

Let $\mathfrak{A}$ be a unital C-algebra with a strongly continuous action $\cdot : G \times \mathfrak{A} \to \mathfrak{A}$ of G by automorphisms of $\mathfrak{A}$, i.e.*

*(a) for every $g \in G$ the map $a \longmapsto g \cdot a$ is a *-automorphism of $\mathfrak{A}$,*
(b) $e \cdot a = a$ for every $a \in \mathfrak{A}$,
(c) $g \cdot (g' \cdot a) = (gg') \cdot a$, for every $g,g' \in G, a \in A$, and
(d) if $g_i \to g$ is a convergent net in G and $a \in \mathfrak{A}$, then $g_i \cdot a \to g \cdot a$ with the C-norm of $\mathfrak{A}$.*

Define a semi norm L on $\mathfrak{A}$ by

$$L(a)=\sup\{\frac{\|g\cdot a-a\|}{\ell(g)}:\quad g\in G, g\neq e\}\qquad (a\in\mathfrak{A}).$$

*Let $\mathscr{A}=\{a\in\mathfrak{A}: L(a)<+\infty\}$. Then by Proposition 2.2 of [5], $\mathscr{A}$ is a dense *-subalgebra of $\mathfrak{A}$. Now, suppose that the action of G is ergodic, i.e. if $a\in\mathfrak{A}$ and for every $g\in G$, $g\cdot a=a$, then $a\in\mathbb{C}1_{\mathfrak{A}}$. Then it is trivial that $L(a)=0$ if and only if $a\in\mathbb{C}1_{\mathfrak{A}}$. Rieffel has proved [5, Theorem 2.3], that the topology induced by ρ_L on $S(\mathfrak{A})$ agrees with the w* topology. Thus $(\mathfrak{A},\mathscr{A},L)$ is a QM space.*

For some other examples that completely match our notion of QM space, see [5]. As we will see in the next section, using quantum family of morphisms we can construct many QSM spaces from a QSM space.

3. The main definition

We need the following simple topological lemma.

Lemma 3.1. *Let Y be a compact space, X be an arbitrary space and (Z,ρ) be a pseudo-metric space. Also, let $\mathbf{C}(Y,Z)$ be the space of all continuous maps from Y to Z, with the pseudo-metric $\hat{\rho}$ defined by*

$$\hat{\rho}(f,g)=\sup\{\rho(f(y),g(y)):\qquad y\in Y\}\qquad (f,g\in\mathbf{C}(Y,Z)).$$

Suppose that $F:Y\times X\to Z$ is a continuous map. Then the map $\tilde{F}:X\to\mathbf{C}(Y,Z)$, defined by $\tilde{F}(x)(y)=F(y,x)$ is continuous.

Proof. Let $x_0\in X$ and $\varepsilon>0$ be arbitrary. Since F is continuous, for every $y\in Y$, there are open sets U_y,V_y in X and Y respectively, such that $(y,x_0)\in V_y\times U_y$ and $\rho(F(y,x_0),F(y',x))<\varepsilon/2$ for every $(y',x)\in V_y\times U_y$. Since Y is compact, there are $y_1,\cdots,y_n\in Y$ such that $Y=\cup_{i=1}^n V_{y_i}$. Let W be the open set $\cap_{i=1}^n U_{y_i}$. Let $x\in W$ and $y\in Y$ be arbitrary. Then for some i $(i=1,\cdots,n)$, y belongs to V_{y_i} and we have,

$$\rho(F(y,x),F(y,x_0))\leq\rho(F(y,x),F(y_i,x_0))+\rho(F(y_i,x_0),F(y,x_0))<\varepsilon.$$

Thus we have $\hat{\rho}(\tilde{F}(x),\tilde{F}(x_0))<\varepsilon$ for every $x\in W$. The proof is complete. □

Let $(\mathfrak{A},\mathscr{A},L)$ be a QSM space, $\mathfrak{B}$ be a unital C*-algebra, and $(\mathfrak{C},\Phi)$ be a quantum family of morphisms from $\mathfrak{A}$ to $\mathfrak{B}$, $\Phi:\mathfrak{A}\to\mathfrak{B}\otimes\mathfrak{C}$. Let d be a pseudo-metric on $S(\mathfrak{C})$, defined by

$$d(\nu,\nu')=\sup\{\rho_L((\mu\otimes\nu)\Phi,(\mu\otimes\nu')\Phi):\quad \mu\in S(\mathfrak{B})\}\qquad (\nu,\nu'\in S(\mathfrak{C})).$$

Proposition 3.2. *With the above assumptions, let $\mathscr{C}$ be the linear space of all $c\in\mathfrak{C}$ such that the map $\nu\longmapsto\nu(c)$ on $S(\mathfrak{C})$ is continuous with the topology induced by d, and $L_d(c)<\infty$. Then the following are satisfied.*

- *i) $\mathscr{C}$ is a self adjoint linear subspace of $\mathfrak{C}$ and $1_{\mathfrak{C}}\in\mathscr{C}$.*
- *ii) For every $c\in\mathscr{C}$, $L_d(c)=0$ if and only if $c\in\mathbb{C}1_{\mathfrak{C}}$.*
- *iii) The topology induced by d on $S(\mathfrak{C})$ is weaker than the w* topology.*
- *iv) With the restriction of the domain of L_d to $\mathscr{C}$, $\rho_{L_d}\leq d$.*
- *v) The topology induced by ρ_{L_d} on $S(\mathfrak{C})$ is weaker than the w* topology.*

Proof. i) is easily checked.
ii) Let c be in $\mathscr{C}$ and $L_d(c)=0$. By Lemma 2.1, the map $\nu\longmapsto\nu(c)$ on $S(\mathfrak{C})$ is constant, and thus $c\in\mathbb{C}1_{\mathfrak{C}}$.
iii) Apply Lemma 3.1, with $X=S(\mathfrak{C})$, $Y=S(\mathfrak{B})$, $Z=S(\mathfrak{A})$, $\rho=\rho_L$ and $F:Y\times X\to Z$ defined by

$$F(\mu,\nu)=(\mu\otimes\nu)\Phi\qquad (\mu\in Y,\nu\in X).$$

We get $\tilde{F}:X\to\mathbf{C}(Y,Z)$ is continuous with the metric $\hat{\rho}$ on $\mathbf{C}(Y,Z)$. On the other hand, for every ν,ν' we have $d(\nu,\nu')=\hat{\rho}(\tilde{F}(\nu),\tilde{F}(\nu'))$. Thus, if $\nu_i\to\nu$ is a convergent net in X with w* topology, then

$$d(\nu_i,\nu)=\hat{\rho}(\tilde{F}(\nu_i),\tilde{F}(\nu))\to 0.$$

This implies that the topology induced by d is weaker than the w* topology.
iv) Let ν,ν' be in $S(\mathfrak{C})$. If $d(\nu,\nu')=0$ then for every $c\in\mathscr{C}$, $\nu(c)=\nu'(c)$ (since the map $\mu\longmapsto\mu(c)$ is continuous with d) and thus by the definition of ρ_{L_d}, $\rho_{L_d}(\nu,\nu')=0$. Thus suppose that $d(\nu,\nu')\neq 0$. Let $c\in\mathscr{C}$ with $L_d(c)\leq 1$. Then $1\geq L_d(c)\geq\frac{|\nu(c)-\nu'(c)|}{d(\nu,\nu')}$, and thus $|\nu(c)-\nu'(c)|\leq d(\nu,\nu')$. Therefore

$$\rho_{L_d}(\nu,\nu')\leq d(\nu,\nu').$$

v) follows directly from iv) and iii). □

Definition 3.3. *With the above assumptions, Proposition 3.2, shows that $(\mathfrak{C},\mathscr{C},L_d)$ is a QSM space that is called QSM space induced by the QSM space $(\mathfrak{A},\mathscr{A},L)$ and quantum family of maps $(\mathfrak{C},\Phi)$.*

Lemma 3.4. *With the above assumptions, let $a\in\mathscr{A}$ and let $\mu\in S(\mathfrak{B})$. Then $c=(\mu\otimes id_{\mathfrak{C}})\Phi(a)$ is in $\mathscr{C}$, and $L_d(c)\leq L(a)$.*

Proof. We first show that $L_d(c) \leq L(a) (< \infty)$. If $L(a) = 0$ then $a \in \mathbb{C}1_{\mathfrak{A}}$ and thus $c \in \mathbb{C}1_{\mathfrak{C}}$ and $L_d(c) = 0$. Suppose that $L(a) \neq 0$. We prove that for every $\nu, \nu' \in S(\mathfrak{C})$ with $d(\nu, \nu') \neq 0$,

$$\frac{|\nu(c) - \nu'(c)|}{d(\nu, \nu')} \leq L(a). \tag{3.1}$$

Let $\nu, \nu' \in S(\mathfrak{C})$ be such that $d(\nu, \nu') \neq 0$. If $|\nu(c) - \nu'(c)| = 0$, then (3.1) is satisfied. Suppose that

$$|\nu(c) - \nu'(c)| = |(\mu \otimes \nu)\Phi(a) - (\mu \otimes \nu')\Phi(a)| \neq 0.$$

By the definition of d, we have $d(\nu, \nu') \geq \rho_L((\mu \otimes \nu)\Phi, (\mu \otimes \nu')\Phi)$. On the other hand, by the definition of ρ_L,

$$\begin{aligned}\rho_L((\mu \otimes \nu)\Phi, (\mu \otimes \nu')\Phi) &\geq |(\mu \otimes \nu)\Phi(\frac{a}{L(a)}) - (\mu \otimes \nu')\Phi(\frac{a}{L(a)})| \\ &= \frac{|(\mu \otimes \nu)\Phi(a) - (\mu \otimes \nu')\Phi(a)|}{L(a)}.\end{aligned}$$

Thus, (3.1) is satisfied and $L_d(c) \leq L(a)$.
Now, we show that the map $\nu \longmapsto \nu(c)$ on $S(\mathfrak{C})$ is continuous with τ_d. Let $\nu_n \to \nu$ be a convergent sequence in $S(\mathfrak{C})$ with the metric d. Thus, by the definition of d, we have

$$\rho_L((\mu \otimes \nu_n)\Phi, (\mu \otimes \nu)\Phi) \to 0.$$

Therefore, by Proposition 2.4,

$$\nu_n(c) = (\mu \otimes \nu_n)\Phi(a) \to (\mu \otimes \nu)\Phi(a) = \nu(c).$$

□

Proposition 3.5. *With the above assumptions, suppose that $(\mathfrak{A}, \mathscr{A}, L)$ is a QM space and the linear span of*

$$G = \{(\mu \otimes id_{\mathfrak{C}})\Phi(a) : \quad \mu \in S(\mathfrak{B}), a \in \mathfrak{A}\}$$

is dense in $\mathfrak{C}$ (for example Φ is surjective). Then $(\mathfrak{C}, \mathscr{C}, L_d)$ is a QM space.

Proof. Since $\mathscr{A}$ is dense in $\mathfrak{A}$ and the linear span of G is dense in $\mathfrak{C}$, we have

$$G_0 = \{(\mu \otimes id_{\mathfrak{C}})\Phi(a) : \quad \mu \in S(\mathfrak{B}), a \in \mathscr{A}\}$$

is dense in $\mathfrak{C}$. On the other hand, by Lemma 3.4, $G_0 \subset \mathscr{C}$. Thus $\mathscr{C}$ is dense in $\mathfrak{C}$ and $(\mathfrak{C}, \mathscr{C}, L_d)$ is a QM space. □

Example 3.6. *Let $\mathfrak{A}$ and $\mathfrak{C}$ be unital C*-algebras. Suppose that $\mathfrak{A} \otimes \mathfrak{C}$ has a QSM structure. Consider *-homomorphisms*

$$id : \mathfrak{A} \otimes \mathfrak{C} \to \mathfrak{A} \otimes \mathfrak{C} \quad \textit{and} \quad F : \mathfrak{A} \otimes \mathfrak{C} \to \mathfrak{C} \otimes \mathfrak{A},$$

where F is the flip map, i.e. $F(a \otimes c) = c \otimes a$ for $a \in \mathfrak{A}, c \in \mathfrak{C}$. Then

$$(\mathfrak{C}, id_{\mathfrak{A} \otimes \mathfrak{C}}) \qquad \textit{and} \qquad (\mathfrak{A}, F)$$

are quantum families of morphisms. Thus $\mathfrak{A}$ and $\mathfrak{C}$ have naturally QSM structures. Also, by Proposition 3.5, if $\mathfrak{A} \otimes \mathfrak{C}$ has a QM structure then so are $\mathfrak{A}$ and $\mathfrak{C}$.

Example 3.7. *Let $\mathfrak{A}$ be a unital C*-algebra and suppose that $\mathfrak{A}$ has a QSM structure. Let $\Phi : \mathfrak{A} \to \mathfrak{B}$ be a unital *-homomorphism. Then $(\mathfrak{B}, \Phi)$ can be considered as a quantum family of morphisms from $\mathfrak{A}$ to $\mathbb{C}$. Thus $\mathfrak{B}$ naturally has a QSM structure. Also, if Φ is surjective and $\mathfrak{A}$ has a QM structure, then by Proposition 3.5, $\mathfrak{B}$ has a QM structure.*

4. The commutative case

In this last section we study induced metric structures on ordinary families of maps.

Lemma 4.1. *Let (X, τ) be a compact Hausdorff space and let d be a pseudo-metric on $S(\mathbf{C}(X, \tau))$ such that τ_d is weaker than the w^* topology. Let $\mathscr{C}$ be the space of all $c \in \mathbf{C}(X, \tau)$ such that the map $\nu \longmapsto \nu(c)$ is continuous on $S(\mathbf{C}(X, \tau))$ and $L_d(c) < \infty$. Consider the semi norm $L_d : \mathscr{C} \to [0, +\infty)$. Then for every $x, x' \in X$, $d(\delta_x, \delta_{x'}) = \rho_{L_d}(\delta_x, \delta_{x'})$.*

(We remark that Lemma 4.1 is different from part i) of Proposition 2.2.)

Proof. Let x, x' be in X. By the definition of ρ_{L_d}, we have

$$\rho_{L_d}(\delta_x, \delta_{x'}) = \sup\{|a(x) - a(x')| : \quad a \in \mathscr{C}, L_d(a) \leq 1\}. \tag{4.1}$$

Let $a \in \mathscr{C}$ and $L_d(a) \leq 1$. If $d(\delta_x, \delta_{x'}) = 0$, then $a(x) = a(x')$ since the map $\delta_x \longmapsto \delta_x(a) = a(x)$ is continuous with d, thus (4.1) implies that

$$\rho_{L_d}(\delta_x, \delta_{x'}) = d(\delta_x, \delta_{x'}) = 0.$$

Now, suppose that $d(\delta_x, \delta_{x'}) \neq 0$. Since $1 = L_d(a) \geq \frac{|a(x) - a(x')|}{d(\delta_x, \delta_{x'})}$, we have $d(\delta_x, \delta_{x'}) \geq |a(x) - a(x')|$, thus (4.1) implies that $\rho_{L_d}(\delta_x, \delta_{x'}) \leq d(\delta_x, \delta_{x'})$. Now, define a map b_x on X by $b_x(y) = d(\delta_x, \delta_y)$. Then $b_x \in \mathscr{C}$ and $L_d(b_x) \leq 1$. Thus

$$\rho_{L_d}(\delta_x, \delta_{x'}) \geq |b_x(x) - b_x(x')| = d(\delta_x, \delta_{x'}).$$

This completes the proof. □

Theorem 4.2. *Let (X,τ), (Y,τ'), (Z,τ'') be compact Hausdorff spaces and let d_0 be a pseudo-metric on X such that $\tau_{d_0} \subset \tau$. Let*

$$F : Y \times Z \to X$$

be a continuous map with τ, τ', τ'', and define a pseudo-metric d_1 on Z by

$$d_1(z,z') = \sup_{y\in Y} d_0(F(y,z),F(y,z')).$$

With the canonical identification $\mathbf{C}(Y\times Z,\tau'\times\tau'') \cong \mathbf{C}(Y,\tau')\otimes\mathbf{C}(Z,\tau'')$ let

$$\hat{F} : \mathbf{C}(X,\tau) \to \mathbf{C}(Y,\tau')\otimes\mathbf{C}(Z,\tau'')$$

be defined by $\hat{F}(a) = aF$, for $a \in \mathbf{C}(X,\tau)$. Let

$$(\mathbf{C}(Z,\tau''),\mathscr{C},N)$$

be the QSM space induced by QSM space $(\mathbf{C}(X,\tau),\mathbf{Lip}(X,d_0),\|\cdot\|_{d_0})$ and quantum family of morphisms $(\mathbf{C}(Z,\tau''),\hat{F})$. Then the following are satisfied.

i) $d_1(z,z') = \rho_N(\delta_z,\delta_{z'})$ *for every* $z,z' \in Z$.
ii) $\mathscr{C} \subset \mathbf{Lip}(Z,d_1)$.
iii) $\|\cdot\|_{d_1} \le N$.

Proof. i) Let $L = \|\cdot\|_{d_0}$. Let us recall the definition of $(\mathbf{C}(Z,\tau''),\mathscr{C},N)$. Let d be the pseudo-metric on $S(\mathbf{C}(Z,\tau''))$ defined by

$$d(\nu,\nu') = \sup\{\rho_L((\mu\otimes\nu)\hat{F},(\mu\otimes\nu')\hat{F}) : \quad \mu \in S(\mathbf{C}(Y,\tau'))\}.$$

Then $N = L_d$ and $\mathscr{C}$ is the space of all $c \in \mathbf{C}(Z,\tau'')$ such that the map $\nu \longmapsto \nu(c)$ on $S(\mathbf{C}(Z,\tau''))$ is continuous with d and $N(c) < \infty$. By Lemma 4.1, we have,

$$d(\delta_z,\delta_{z'}) = \rho_N(\delta_z,\delta_{z'}), \tag{4.2}$$

for every $z,z' \in Z$. Now, we explain the relation between d_1 and d.
Let $z,z' \in Z$ and $y \in Y$. Then

$$(\delta_y\otimes\delta_z)\hat{F} = \delta_{F(y,z)} \qquad \text{and} \qquad (\delta_y\otimes\delta_{z'})\hat{F} = \delta_{F(y,z')}.$$

On the other hand, by Proposition 2.2, for every $x,x' \in X$, $d_0(x,x') = \rho_L(\delta_x,\delta_{x'})$. Thus

$$\rho_L((\delta_y\otimes\delta_z)\hat{F},(\delta_y\otimes\delta_{z'})\hat{F}) = d_0(F(y,z),F(y,z')).$$

This formula together with the definitions of d and d_1, show that

$$d_1(z,z') \le d(\delta_z,\delta_{z'}). \tag{4.3}$$

Let $\mu \in S(\mathbf{C}(Y,\tau'))$ be arbitrary. We consider μ as a probability Borel regular measure on (Y,τ'). Then for every $a \in \mathbf{Lip}(X,d_0)$ with $\|a\|_{d_0} \le 1$, we have,

$$\begin{aligned} |(\mu\otimes\delta_z)\hat{F}(a) - (\mu\otimes\delta_{z'})\hat{F}(a)| &= |\int_Y (aF(y,z) - aF(y,z'))d_\mu(y)| \\ &\le \int_Y |a(F(y,z)) - a(F(y,z'))|d_\mu(y). \end{aligned} \tag{4.4}$$

For every $y \in Y$, by Lemma 2.1,

$$|a(F(y,z)) - a(F(y,z'))| \le d_0(F(y,z),F(y,z')).$$

Therefore, we have

$$|a(F(y,z)) - a(F(y,z'))| \le d_1(z,z'). \tag{4.5}$$

(4.5) and (4.4) implies that

$$|(\mu\otimes\delta_z)\hat{F}(a) - (\mu\otimes\delta_{z'})\hat{F}(a)| \le d_1(z,z').$$

Therefore, by the definition of d,

$$d(\delta_z,\delta_{z'}) \le d_1(z,z'). \tag{4.6}$$

Now, by (4.6) and (4.3), $d(\delta_z,\delta_{z'}) = d_1(z,z')$, and thus by (4.2),

$$d_1(\delta_z,\delta_{z'}) = \rho_N(\delta_z,\delta_{z'})$$

for every $z,z' \in Z$, and i) is satisfied. ii) and iii) are immediate consequence of i) and definitions of $\mathscr{C}$, $\|\cdot\|_{d_1}$ and N. $\square$

5. Conclusion

In this note, we introduced the new concept of *compact quantum pseudo-metric space* as a generalization of the concept of *compact quantum metric space*. The C*-algebraic examples of the latter concept, which has been introduced by Rieffel, are very restricted. But, by using the concept of *quantum family of maps*, it was denoted that the source of examples for (C*-algebraic) quantum pseudo-metric spaces are very wider than those for (C*-algebraic) quantum metric spaces.

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Geodesics on the momentum phase space with metric $^C g$

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Abstract

In this paper, a system of the differential equations giving geodesics on the momentum phase space with pseudo Riemann metric $^C g$ of a Hamilton space is found by using the Euler Lagrange equations. Then, space like geodesics on pseudo hyperbolic 2-space H_1^2 are obtained. Finally, a system of the differential equations giving geodesics on the cotangent bundle with pseudo Riemann metric $^C g$ of H_1^2 is get.

1. Introduction

The geometry of the cotangent bundle is one of the most important subjects attracted the attention of mathematicians as well as physicists.
A Lagrange mechanical system consists of a configuration space which contains the trajectory of each of the moving particles of a system with n-particles and Lagrangian which gives the difference between the kinetic and potential energy at any stage of the each particle in system [1].
The position and velocity coordinates depended on time parameter t of the motion of the each particle of a system with n-particles are represented by the coordinates of any point on the tangent bundle of a (pseudo) Riemann manifold. Lagrangian is defined as a real valuable and differentiable function on the tangent bundle [7].
The position and velocity coordinates of a moving particle in the system at any instant t are found by the Euler Lagrange equations. The Euler Lagrange equations are second order ordinary differential equations which depend on position and velocity of each particle in system [7].
Gravitational field is given by a (pseudo) Riemann metric. The movement depended on time t of all particles in system by the only effect of the gravitational field is described by a (pseudo) Riemann manifold. The trajectory of a moving particle by the effect of the gravitational field describes a geodesic, the shortest one among the curves passing from one point to another, in a (pseudo) Riemann manifold [8].
In different place of space, the effect of the gravitational field on moving particles is different. As metric is change, (pseudo) Riemann manifold must change. Moreover, in different (pseudo) Riemann manifold, the trajectory of moving particles by the only effect of the gravitational field, geodesics, must change. So, we examined geodesics on different space forms such as ${H_1}^2$, ${S_1}^2$, S^3 and its tangent sphere bundle from [[2],[3],[4]].
From the references [1], [5], [7], [8], [9] and [10], we searched some concept such as, the movement of particles in the gravitational field, total energy function in a Hamilton mechanic system, kinetic and potential energies of a moving particle, the conservation of energy and the conservation of momentum.
Then, we found out that how obtained space like, time like and null geodesics of the unit 2-sphere with index one by using the Euler Lagrange equations from [6].
Moreover, we analyzed the general equations of geodesics on the cotangent bundle with pseudo Riemann metric $^C g$ which is called as Riemann extension of the symmetric affine connection [11].
In this paper, we defined Lagrangian on the cotangent bundle and we calculated geodesics on the cotangent bundle with a pseudo Riemann metric $^C g$ by using the Euler Lagrange equations.
Then, we obtained space-like geodesics on the pseudo hyperbolic 2-space by using the Euler Lagrange equations. Finally, we calculated the general equations of geodesics on the cotangent bundle of pseudo hyperbolic 2-space with the pseudo Riemann metric $^C g$.

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2. The motion of the particles in the Hamilton Space

In this section, we obtain the geometrical interpretations of some concepts, well known in classic mechanic such as the movement of the particles by the effect of the gravitational field, the conservation of energy and momentum.
In classical mechanic, the motion of a system with n-particles is described by the position coordinates and the momentum coordinates of each of the particles in system. The position coordinates of a particle depend on the position vector in three-dimensional space. The momentum coordinates of a particle depend on a one form associated with the tangent vector of the curve determined by trajectory of the moving particle [10].
If the configuration space represents by an n=3N dimensional manifold whose the local coordinate functions give the positions coordinates of each of the particles at any instant t, the phase space of the configuration space can represent by an 2n dimensional manifold whose the local coordinate functions give the position and the momentum coordinates of each of the particles at any instant t. In other words, the phase space of the configuration space must represent 2n-dimensional the cotangent bundle as the configuration space of moving particles by the only effect of the gravitational field represents a n-dimensional differentiable manifold M [10]. The arc length between the position coordinates of infinitely close two points on the trajectory of a particle in the configuration space is determined by

$$ds^2 = g_{kj}(x)dx^k dx^j, \tag{2.1}$$

where $g_{kj}(x)$ is a pseudo Riemann metric on M. The trajectory of a moving particle in M is represented by a curve $\gamma : I \subset R \to M$. For any time t, the position coordinates of γ is given by $x^i(t) = x^i \circ \gamma(t)$, $i = 1,...,n$ and the velocity and momentum coordinates of γ are given by $y^i = \frac{dx^i}{dt}$, and $p_i = g_{ij}y^j$, respectively.
The curve γ must be called as the space-like, the time-like or the light-like curve if the value under the pseudo Riemann metric g of the unit tangent vector v at every point of γ, $ds^2 = g(v,v) = \varepsilon$ is provided $\varepsilon = 1$, $\varepsilon = -1$ or $\varepsilon = 0$, respectively [3].
The sum of the kinetic and the potential energies of all moving particles in the system are represented by the function $H : T^*M \to R$ in the phase space and it is called Hamiltonian.
Assuming that all particles in the system act only the effect of the gravitational field. Then the Hamiltonian H equals to the sum of the total energies of the moving all particles in system and the Hamiltonian H of the test particles i.e. unit mass particles are described by

$$H(x^i, p_i) = \frac{1}{2} g^{ij}(x) p_i p_j, \tag{2.2}$$

where $g^{ij}(x)$ is a metric tensor with type (2,0) given by $g^{ik}g_{kj} = \delta^i_j$ [5]. The Hamilton space M consists of n-dimensional differentiable manifold M and a Hamiltonian H on the cotangent bundle of M. Assuming that $\eta = p_i dx^i$ is the basic 1-form on T^*M. The exterior differential $\vartheta = -d\eta$ of the basic 1-form η is the 2-form given by $\vartheta = dx^i \wedge dp_i$ with respect to the induced local coordinates of T^*M. Since ϑ is closed (i.e. $d\vartheta = 0$), non degenerate 2-form on T^*M, ϑ is called as the canonic symplectic structure on T^*M. The cotangent bundle T^*M with the symplectic structure ϑ is called as a symplectic manifold [1]. A vector field $X_H : T^*M \to TT^*M$ is called as Hamilton vector field if there is a function $H : T^*M \to R$ such that $i_{X_H}\vartheta = dH$. The condition $i_{X_H}\vartheta = dH$ is equivalent to $\vartheta(X_H, Y) = dH(Y)$ for $Y \in TT^*M$. The local expression of the Hamilton vector field is

$$X_H = \frac{\partial H}{\partial p_i}\frac{\partial}{\partial x^i} - \frac{\partial H}{\partial x^i}\frac{\partial}{\partial p_i}, \tag{2.3}$$

with respect to the induced coordinates $(x^i, p_i); i = 1,...,n$ on T^*M [1].
The curve $\phi : t \to (x^i(t), p_i(t))$ is called as the integral curve of the Hamilton vector field X_H since the equality $X_H(\phi(t)) = \dot{\phi}(t)$ provides. The condition $X_H(\phi(t)) = \dot{\phi}(t)$ is equivalent to

$$\frac{dx^i}{dt} = \frac{\partial H}{\partial p_i}, \quad \frac{dp_i}{dt} = -\frac{\partial H}{\partial x^i}. \tag{2.4}$$

The equations in (2.4) are called as Hamilton equations. The solution curves of these 2n first order differential equations describe a symplectic transformation called as phase flow in phase space [1]. The Hamilton equations are also obtained by the Legendre transformation of the Euler Lagrange equations described as follows:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial y^i}\right) - \frac{\partial L}{\partial x^i} = 0, \tag{2.5}$$

where $y^i = \frac{dx^i}{dt}$. Legendre transformation is differentiable transformation from TM to T^*M defined by

$$\pounds : (x^i, y^i) \to (x^i, p_i = \frac{\partial L}{\partial y^i}). \tag{2.6}$$

There is a relation between H and L as follows:

$$H(x^i, p_i) = p_i y^i - L(x^i, y^i), \tag{2.7}$$

where L is a differentiable function on TM called as Lagrangian with local expression as follows:

$$L(x^i, y^i) = \frac{1}{2} g_{ij}(x) y^i y^j. \tag{2.8}$$

The Lagrangian of all particles in the system is equal to the difference between kinetic and potential energies of these particles [1, 7].

The Hamiltonian H is a differentiable function on T^*M given in (2.2). Let ϕ be a curve in T*M. The curve γ is called as the projected curve of ϕ by canonic projection π from T^*M to M i.e. $\pi \circ \phi = \gamma$. If the curve ϕ is an integral curve of the Hamilton vector field X_H, the curve γ must be geodesic [1]. In other words, the particles in the system exposed gravitational field act along geodesic curves in the configuration space [8].
The sum of kinetic and potential energies of each of the moving particles by the only effect of the gravitational field is equal to Hamiltonian H which has constant value along the integral curves of the Hamilton vector field. In other words, the Hamiltonian must be constant on every point of the curve ϕ as the particles act along the integral curve of the Hamilton vector field. In classical mechanic, this fact is known as the conservation of energy [1].
Since the Hamiltonian H is constant along the integral curves of the Hamilton vector field, the Lagrangian L to be real valuable a function on the tangent bundle TM of the configuration space M must be constant along the integral curve of a vector field called as geodesic spray, which is the horizontal vector field in TTM [1].
The Lagrangian of the system with n-particles acting the only effect of the gravitational field is defined by (2.8). In addition, the momentum $p_i = \partial L/dy^i$ of each particle in the system must be constant. In classical mechanic, this fact is known as the conservation of momentum [7].

3. The Euler Lagrange equations on (T*M,^{C}g)

In this section, the Lagrangian on $(T^*M,\ {}^Cg)$ is obtained and the general equations of geodesics of $\left(T^*M,\ {}^Cg\right)$ in terms of the Euler Lagrange equations are found.
Definition 3.1. The disjoint union of each tangent vector space at all point of T^*M is called as the tangent bundle of T^*M denoted by

$$TT^*M = \bigcup_{\forall (p,\omega)\in T^*M} T_{(p,\omega)}T^*M. \tag{3.1}$$

Any point on TT^*M is represented by $(p,\ \omega,\ \tilde{X}_{(p,\omega)})$ where p is any point in a neighborhood U of M, ω is a cotangent vector at a point p of M and $\tilde{X}_{(p,\omega)}$ is a tangent vector at a point (p,ω) of T^*M. $\tau_{T^*M} : TT^*M \to T^*M$ is called as the canonical projection map. Let $\tilde{\tilde{p}} = (p,\ \omega,\ \tilde{X}_{(p,\omega)})$ be a point on $(\pi_M \circ \tau_{T^*M})^{-1}(U) \subset TT^*M$. Then $\left(x^1,...,x^n,p_1,...,p_n,\dot{x}^1,...,\dot{x}^n,\dot{p}_1,...\dot{p}_n\right)$ is a the induced local coordinates of a point $\tilde{\tilde{p}}$ on TT^*M where $\{x^i\}, i = 1,...,n$ is the local components of a point $\pi_M \circ \tau_{T^*M}(\tilde{\tilde{p}}) = p$ of M and $\{p_i\},\ i = 1,...,n$ is the local components of the cotangent vector ω providing $p_i = \omega(\frac{\partial}{\partial x^i})$. $\{\dot{x}^i\},\ i = 1,...,n$ is the local coordinate function of the tangent vector $\tilde{X}_{(p,\omega)}$ providing $\dot{x}^i = dx^i(\tilde{X}_{(p,\omega)}) = \tilde{X}_{(p,\omega)}[x^i]$ and $\{\dot{p}_i\},\ i = 1,...,n$ is the local coordinate functions of the tangent vector $\tilde{X}_{(p,\omega)}$ providing $\dot{p}_i = dp_i(\tilde{X}_{(p,\omega)}) = \tilde{X}_{(p,\omega)}[p_i]$. Thus TT^*M has 4n dimensional manifold structure.
Definition 3.2. The pseudo Riemann metric ${}^Cg$ is a tensor field of type (0, 2) in the cotangent bundle T^*M, whose components $\tilde{g}_{AB}$ are given by

$$\tilde{g}_{AB} : \begin{pmatrix} -2p_a\Gamma^a_{ij} & \delta^j_i \\ \delta^i_j & 0 \end{pmatrix}_{2n\times 2n}, \tag{3.2}$$

where $A,\ B \in \{1,...,2n\}$ and $i,\ j \in \{1,...,n\}$. The line element of a pseudo Riemann metric ${}^Cg$ is given by

$${}^Cg = -2p_a\Gamma^a_{ij}dx^idx^j + 2\delta^j_i dp_i dx^j, \tag{3.3}$$

with respect to the induced coordinates (x^i, p_i) of T^*M. ${}^Cg$ is called as the Riemann extension of the symmetric affine connection ∇ of M [11].

Theorem 3.1. *The difference between kinetic and potential energies of the moving particles by the only effect of the gravitational field in the momentum phase space with a pseudo Riemann metric ${}^Cg$ is given as follows:*

$$\tilde{L}(x^i, p_i, \dot{x}^i, \dot{p}_i) = -p_a\Gamma^a_{ij}\dot{x}^i\dot{x}^j + \dot{p}_i\dot{x}^i. \tag{3.4}$$

Proof. The difference between kinetic and potential energy of the moving particles by the only effect of the gravitational field at any stage $\left(\tilde{x}^A, \tilde{y}^A\right)$ on TT^*M, $\tilde{L}$, is calculated by

$$\tilde{L}(\tilde{x}^A, \tilde{y}^A) = \frac{1}{2}\tilde{g}_{AB}\tilde{y}^A\tilde{y}^B; A,B,D = 1,...2n,$$

where g^{ik} , $A = 1,...,2n$ corresponds to the local coordinates of a point $\tilde{p} = (x^i, p_i),\ i = 1,...,n$ on T^*M and $\tilde{y}^A$ corresponds to the coordinate components $(\dot{x}^i, \dot{p}_i)$ of a vector field $\tilde{X} : T^*M \to T(T^*M)$, $\tilde{X} = \tilde{X}^A\frac{\partial}{\partial \tilde{x}^A}$ such that $d\tilde{x}^i[\tilde{X}] = \dot{x}^i,\ d\tilde{x}^{n+i}[\tilde{X}] = \dot{p}_i$. The Lagrangian $\tilde{L}$ is a differentiable function from TT^*M to IR. The local expression of $\tilde{L}$ is

$$\tilde{L}(x^i, p_i, \dot{x}^i, \dot{p}_i) = \frac{1}{2}\begin{pmatrix} \dot{x}^i \\ \dot{p}_i \end{pmatrix}^T \begin{pmatrix} -2p_a\Gamma^a_{ij} & \delta^j_i \\ \delta^i_j & 0 \end{pmatrix}\begin{pmatrix} \dot{x}^j \\ \dot{p}_j \end{pmatrix} = -p_a\Gamma^a_{ij}\dot{x}^i\dot{x}^j + \dot{p}_i\dot{x}^i.$$

$\square$

Definition 3.3. The one with minimum arc length of among the curves given by $\tilde{\gamma} : t \in [t_0, t_1] \subset R \to \left(x^i(t), p_i(t)\right) \in T^*M$ is described by integral

$$\varphi(\tilde{\gamma}) = \int_{t_0}^{t_1} \tilde{L}(x^i(t), p_i(t), \dot{x}^i, \dot{p}_i))\ dt \tag{3.5}$$

where $\varphi(\tilde{\gamma})$ is called as functional.

Theorem 3.2. *A curve $\tilde{\gamma}: t \to \left(x^i(t), p_i(t)\right)$ in $(T^*M, {}^C g)$ is geodesic iff $\tilde{\gamma}$ satisfies the Euler-Lagrange equations given by*

$$\frac{d}{dt}\left(\frac{\partial \tilde{L}}{\partial \dot{x}^i}\right) - \frac{\partial \tilde{L}}{\partial x^i} = 0, \quad \frac{d}{dt}\left(\frac{\partial \tilde{L}}{\partial \dot{p}_i}\right) - \frac{\partial \tilde{L}}{\partial p_i} = 0, \tag{3.6}$$

where $\tilde{L}$ is defined as (3.4).

Proof. By solving the differential equations in (3.6), we get the general equations of geodesics of T^*M with the pseudo Riemann metric ${}^C g$ as follows:

$$\frac{d\dot{x}^h}{dt} + \Gamma^h_{ij}\dot{x}^i\dot{x}^j = 0, \tag{3.7}$$

$$\frac{d\dot{p}_h}{dt} + p_a\frac{\partial \Gamma^a_{ij}}{\partial x^h}\dot{x}^i\dot{x}^j - 2p_a\frac{\partial \Gamma^a_{hj}}{\partial x^i}\dot{x}^j\dot{x}^i - 2\Gamma^a_{hj}\dot{p}_a\dot{x}^j + 2p_a\Gamma^a_{hk}\Gamma^k_{ji}\dot{x}^j\dot{x}^i = 0, \tag{3.8}$$

which is also obtained by classical method in Yano and Ishihara' s book [11]. □

4. Geodesics on H_1^2

In this section, the space-like geodesics on pseudo hyperbolic 2-space are obtained by using the Euler Lagrange differential equations.
Definition 4.1. Non-degenerate, symmetric, bilinear form g is called as a semi Riemann metric in semi-Euclidean space E_1^3 and g is defined by

$$g(u,v) = -u_1v_1 + u_2v_2 + u_3v_3, \tag{4.1}$$

for any vectors $u, v \in E_1^3$.
Definition 4.2. H_1^2 is a surface in E_1^3 given by

$$H_1^2 = \{u = (x_1,x_2,x_3) : \|u\|^2 = g(u,u) = -1,\ u \in E_1^3\}. \tag{4.2}$$

H_1^2 is called as pseudo hyperbolic 2-space. H_1^2 may be considered as hyperboloid of two sheet in Euclidean space.
The representation of H_1^2 with respect to Cartesian coordinate system is given as follows:

$$-x_1^2 + x_2^2 + x_3^2 = -1. \tag{4.3}$$

The parametric representation of H_1^2 is given as follows:

$$\begin{aligned} x_1(a,\theta) &= \cosh a, \\ x_2(a,\theta) &= \sinh a \sin\theta, \\ x_3(a,\theta) &= \sinh a \cos\theta. \end{aligned} \tag{4.4}$$

Any curve on the surface H_1^2 is described by giving the following coordinates as functions of a single parameter t

$$\begin{aligned} a &= a(t) \\ \theta &= \theta(t). \end{aligned} \tag{4.5}$$

Theorem 4.1. *The length between infinitely close two points on H_1^2 is determined by the following metric:*

$$ds^2 = da^2 + \sinh^2 a d\theta^2. \tag{4.6}$$

Proof. In order to find length of a one parameter curve in H_1^2, we use the covariant differentiations of x_1, x_2, x_3 as follow:

$$\begin{aligned} dx_1 &= \sinh a da \\ dx_2 &= \cosh a \sin\theta da + \sinh a \cos\theta d\theta \\ dx_3 &= \cosh a \cos\theta da - \sinh a \sin\theta d\theta. \end{aligned} \tag{4.7}$$

The length between infinitely close two points on H_1^2 is calculated with

$$\begin{aligned} ds^2 &= g((dx_1,dx_2,dx_3),(dx_1,dx_2,dx_3)) \\ &= -dx_1^2 + dx_2^2 + dx_3^2. \end{aligned} \tag{4.8}$$

By using the covariant differentiations of x_1, x_2, x_3 in (4.4), we get

$$ds^2 = da^2 + \sinh^2 a d\theta^2$$

and also the matrix representation of this metric is given as follows:

$$g_{ij} : \begin{pmatrix} 1 & 0 \\ 0 & \sinh^2 a \end{pmatrix}. \tag{4.9}$$

□

Definition 4.3. A differentiable function from TH^2_1 to R is defined as follows:

$$L(a,\theta,\dot{a},\dot{\theta})=\frac{1}{2}(\dot{a}^2+\sinh^2 a\dot{\theta}^2) \tag{4.10}$$

where L is called as Lagrangian of H^2_1.

Definition 4.4. The one with minimum arc length of among the curves γ in H^2_1 is described by

$$\varphi(\gamma)=\int_{t_0}^{t_1} L(a,\theta,\dot{a},\dot{\theta})\,dt, \tag{4.11}$$

where $\varphi(\gamma)$ describes a map from family of curves passing through different two point in H^2_1 to real numbers and γ is also a curve such that $\varphi(\gamma)$ has minimum arc length on H^2_1. $\varphi(\gamma)$ is called as functional. To find γ, the Euler-Lagrange equations are used.

Definition 4.5. The trajectories of moving test particles on H^2_1 are determined by the following equations:

$$\frac{d}{dt}\left(\frac{\partial L}{d\dot{a}}\right)-\frac{\partial L}{da}=0, \tag{4.12}$$

$$\frac{d}{dt}\left(\frac{\partial L}{d\dot{\theta}}\right)-\frac{\partial L}{d\theta}=0. \tag{4.13}$$

These equations are called as the Euler Lagrange equations in H^2_1. The particular solution providing the initial value of differential equations in (4.12) and (4.13) is a geodesic γ passing through initial point $(a(t_0),\theta(t_0))$ and the end point $(a(t_1),\theta(t_1))$. The curve γ may be visualized as the trajectory of moving a test particle by the effect of gravitational field on throat of hyperboloid of two sheets.

Definition 4.6. The line element of H^2_1 is given by

$$ds^2=(\dot{a})^2+\sinh^2 a\big(\dot{\theta}\big)^2=\varepsilon. \tag{4.14}$$

The curve connecting different two point to be infinitely close on H^2_1 is called as the space-like curve of H^2_1 for $\varepsilon=1$.

Theorem 4.2. *The general equation of geodesics on H^2_1 is given by*

$$\frac{da}{d\theta}=\frac{\sqrt{\varepsilon\sinh^4 a-k_1^2\sinh^2 a}}{k_1}. \tag{4.15}$$

Proof. In order to obtain the general equation of geodesics, we should consider the Euler Lagrange equations in (4.12), (4.13) together with metric on H^2_1. From the solving of the differential equations in (4.13), we obtain

$$\frac{d}{dt}\left(\dot{\theta}\sinh^2 a\right)=0\Rightarrow\dot{\theta}=\frac{k_1}{\sinh^2 a},\ \mathrm{k}_1-\text{real constant.} \tag{4.16}$$

If we put the value of $\dot{\theta}$ in (4.16) into at (4.14)

$$\left(\frac{da}{dt}\right)^2+\sinh^2 a\left(\frac{d\theta}{dt}\right)^2=\varepsilon\Rightarrow\left(\frac{da}{d\theta}\frac{d\theta}{dt}\right)^2+\sinh^2 a\left(\frac{d\theta}{dt}\right)^2=\varepsilon,$$

we can obtain the general equation of geodesics on H^2_1 as follow:

$$\frac{da}{d\theta}=\frac{\sqrt{\varepsilon\sinh^4 a-k_1^2\sinh^2 a}}{k_1}.$$

□

Since the gradient vector field to be normal to tangent vector space at any point of H^2_1 is time like, H^2_1 must be called as a space like surface. So, H^2_1 has only space like geodesics. The general equation of these geodesics on H^2_1 is given by the following theorem.

Theorem 4.3. *The space-like geodesics on H^2_1 are given by the following equations:*

$$\frac{\sqrt{1-k_1^2\cos ech^2 a}}{\sqrt{k_1^2+1}}=\frac{\sin\theta}{k_2},\ \frac{x_1^2}{k_1^2+1}-\frac{x_2^2}{k_2^2}=1$$

with respect to generalized and cartesian coordinates.

Proof. If we chose $\varepsilon=1$ in (4.15), we can obtain following surface:

$$\frac{\sqrt{1-k_1^2\cos ech^2 a}}{\sqrt{k_1^2+1}}=\frac{\sin\theta}{k_2},\ \frac{x_1^2}{k_1^2+1}-\frac{x_2^2}{k_2^2}=1$$

with respect to generalized coordinates (a,θ) and cartesian coordinates. □

The space-like geodesic characterized by trajectory of a moving test particle by the effect of gravitational field acts faster than speed of light on the surface H^2_1. This mechanical interpretation was inspired from [9].

5. Geodesics on $\left(\mathbf{T^{*}H_{1}{}^{2}, g^{C}}\right)$

In this section, the general equations of geodesics on the cotangent bundle of pseudo hyperbolic 2-space with the pseudo Riemann metric ${}^{C}g$ are calculated.

Definition 5.1. Let H_1^2 be pseudo hyperbolic 2-space and $T^*H_1^2$ be its cotangent bundle. As any point q on H_1^2 has the generalized coordinates $(a,\ \theta)$, the point (q,η_q) on $T^*H_1^2$ has the local coordinates (a,θ,p_1,p_2) where p_1,p_2 are the local coordinate function of the cotangent vector η_q at q. In addition, the point $\left(q,\eta_q,X_{(q,\eta_q)}\right)$ on $TT^*H_1^2$ has the local coordinates $(a,\theta,p_1,p_2,\dot{a},\dot{\theta},\dot{p}_1,\dot{p}_2)$ and the point $\left(q,\eta_q,\omega_{(q,\eta_q)}\right)$ on $T^*T^*H_1^2$ has the local coordinates $(a,\theta,p_1,p_2,\omega_a,\omega_\theta,\omega_{p_1},\omega_{p_2})$ where $\omega_a,\omega_\theta,\omega_{p_1},\omega_{p_2}$ are the local coordinate functions of the cotangent vector at (q,η_q) i.e. $\omega_{(q,\eta_q)}\in T^*_{(q,\eta_q)}T^*H_1^2$.

Theorem 5.1. *The general equations of geodesics of H_1^2 are represented by the following equations:*

$$\ddot{a}-\cosh a\sinh a\ (\dot{\theta})^2=0,\ \ddot{\theta}+2\coth a\ \dot{a}\ \dot{\theta}=0.$$

Proof. To find the general equations of geodesics on H_1^2, we need to the general formula of the geodesic equations on Riemann manifolds given by

$$\ddot{x}^{\alpha}+\Gamma^{\alpha}_{\gamma\beta}\dot{x}^{\gamma}\dot{x}^{\beta}=0, \tag{5.1}$$

and the Christoffel symbols given by

$$\Gamma^{\alpha}_{\beta\gamma}=\frac{1}{2}g^{\alpha\mu}\left(g_{\mu\gamma,\beta}+g_{\beta\mu,\gamma}-g_{\beta\gamma,\mu}\right), \tag{5.2}$$

and where $g_{\alpha\beta}$ is the components of the matrix representation of the metric tensor of H_1^2 and $g^{\alpha\beta}$ is inverse of $g_{\alpha\beta}$. The non-zero components of the Christoffel symbols of H_1^2 are obtained as follows:

$$\Gamma^1_{22}=-\sinh a\cosh a,\quad \Gamma^2_{12}=\Gamma^2_{21}=\coth a. \tag{5.3}$$

If we put the non-zero Christoffel symbols in (5.3) into (5.1) , we get the general equations of geodesics on H_1^2 as follows:

$$\ddot{a}-\cosh a\sinh a\ (\dot{\theta})^2=0,\ \ddot{\theta}+2\coth a\ \dot{a}\ \dot{\theta}=0. \tag{5.4}$$

□

Theorem 5.2. *The pseudo Riemann metric ${}^{C}g$ on $T^*H_1^2$ has components*

$${}^{C}g=-4p_2\coth a da d\theta+2p_1\sinh a\cosh a(d\theta)^2+2dadp_1+2d\theta dp_2. \tag{5.5}$$

Proof. By using the equations in (3.3) and (5.3), the pseudo Riemann metric ${}^{C}g$ on $T^*H_1^2$ has following component

$${}^{C}g=\left(-2p_1\Gamma^1_{11}-2p_2\Gamma^2_{11}\right)(da)^2+2\left(-2p_1\Gamma^1_{12}-2p_2\Gamma^2_{12}\right)dad\theta+\left(-2p_1\Gamma^1_{22}-2p_2\Gamma^2_{22}\right)(d\theta)^2+2dadp_1+2d\theta dp_2.$$

The matrix representation of ${}^{C}g$ is as follows:

$$g_{AB}:\begin{pmatrix}0 & -2p_2\coth a & 1 & 0\\ -2p_2\coth a & 2p_1\sinh a\cosh a & 0 & 1\\ 1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\end{pmatrix},$$

and the matrix representation of inverse of ${}^{C}g$ is as follows:

$$\left({}^{C}g\right)^{-1}:\begin{pmatrix}0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\\ 1 & 0 & 0 & 2p_2\coth a\\ 0 & 1 & 2p_2\coth a & -2p_1\sinh a\cosh a\end{pmatrix}.$$

□

Theorem 5.3. *The non-zero components of the Christoffel symbols of $T^*H_1^2$ with the pseudo Riemann metric ${}^{C}g$ has the following components:*

$$\begin{aligned}
&\Gamma^1_{22}=-\sinh a\cosh a\\
&\Gamma^2_{12}=\Gamma^2_{21}=\coth a\\
&\Gamma^3_{12}=\Gamma^3_{21}=2p_2\coth^2 a,\Gamma^3_{22}=-p_1\cosh 2a,\Gamma^3_{24}=\Gamma^3_{24}=-\coth a\\
&\Gamma^4_{11}=2p_2\cos ech^2 a,\Gamma^4_{12}=\Gamma^4_{21}=-p_1,\Gamma^4_{22}=-2p_2\cosh^2 a,\\
&\Gamma^4_{14}=\Gamma^4_{41}=-\coth a,\Gamma^4_{23}=\Gamma^4_{32}=\sinh a\cosh a.
\end{aligned} \tag{5.6}$$

Proof. By using the general formula of the Christoffel symbols in (5.2), it is seen to be correct of theorem. □

Theorem 5.4. *Any curve $\tilde{c}: t \to \tilde{c}(t) = (a, \theta, p_1, p_2)$ on $(T^*H_1^2, {}^Cg)$ is geodesic iff following differential equations system must provide:*

$$\ddot{a} - \sinh a \cosh a \left(\dot{\theta}\right)^2 = 0,$$

$$\ddot{\theta} + 2\coth a\ \dot{a}\ \dot{\theta} = 0,$$

$$\ddot{p}_1 + 4p_2 \coth^2 a\ \dot{a}\ \dot{\theta} - p_1 \cosh 2a \left(\dot{\theta}\right)^2 - 2\coth a\ \dot{\theta}\ \dot{p}_2 = 0,$$

$$\ddot{p}_2 + 2p_2 \cos ech^2 a\ (\dot{a})^2 - 2p_1\ \dot{a}\ \dot{\theta} - 2p_2 \cosh^2 a \left(\dot{\theta}\right)^2 - 2\coth a\ \dot{a}\ \dot{p}_2 + 2\sinh a \cosh a\ \dot{\theta}\ \dot{p}_2 = 0.$$

Proof. By using the general formula of the geodesic equation in (5.1) and components of the Christoffel symbol in (5.6), it is seen easily to be correct of theorem. □

Theorem 5.5. *The general equations of geodesics of $(T^*H_1^2, {}^Cg)$ obtained in Theorem 3.2 by using the Euler Lagrange equations provide the geodesic equations obtained by classical method at Theorem 5.1.*

Proof. If we rewrite with respect to the components of the general equations of geodesics of $(T^*M, {}^Cg)$ obtained by Theorem 3.2 for $(T^*H_1^2, {}^Cg)$, we get the following equations:

$$\frac{d\dot{x}^1}{dt} + \Gamma_{22}^1 \dot{x}^2 \dot{x}^2 = 0,$$

$$\frac{d\dot{x}^2}{dt} + 2\Gamma_{12}^1\ \dot{x}^1 \dot{x}^2 = 0,$$

$$\frac{d\dot{p}_1}{dt} + p_1 \frac{\partial \Gamma_{22}^1}{\partial x^1} \dot{x}^2 \dot{x}^2 - 2\Gamma_{12}^2\ \dot{p}_2 \dot{x}^2 + 2p_2 \Gamma_{12}^2 \Gamma_{21}^2 \dot{x}^1 \dot{x}^2 = 0,$$

$$\frac{d\dot{p}_2}{dt} - 2p_2 \frac{\partial \Gamma_{21}^2}{\partial x^1} \dot{x}^1 \dot{x}^1 - 2p_1 \frac{\partial \Gamma_{22}^1}{\partial x^1} \dot{x}^2 \dot{x}^1 + 2p_1 \Gamma_{22}^1 \Gamma_{12}^2 \dot{x}^2 \dot{x}^1 + 2p_2 \Gamma_{21}^2 \Gamma_{22}^1 \dot{x}^2 \dot{x}^2 - 2\Gamma_{21}^2 \dot{p}_2 \dot{x}^1 - 2\Gamma_{22}^1 \dot{p}_1 \dot{x}^2 = 0.$$

When we use the coefficients of the Christoffel symbol in (5.1), it is seen to be the above equations is equal to the geodesic equations obtained by classical method at Theorem 5.4. □

6. Conclusion

This study contains six sections. In the second section, the geometrical interpretation of the mechanic concepts are considered such as the movement of the particles by the effect of the gravitational field, the conservation of momentum or energy, well known in classic mechanic. In the third section, the general equations of geodesics on the cotangent bundle $\left(\mathrm{T}^*\mathrm{H}_1{}^2, \mathrm{g}^\mathrm{C}\right)$ are found by using the Euler Lagrange equations. In the fourth section, the space-like geodesics on pseudo hyperbolic 2-space are obtained by using the Euler Lagrange equations.
Finally, in the fifth section, the general equations of geodesics on the cotangent bundle of pseudo hyperbolic 2-space $\left(\mathrm{T}^*\mathrm{H}_1{}^2, \mathrm{g}^\mathrm{C}\right)$ are obtained by using general formula of geodesics on a Riemann manifold and then it is shown that the general equations of geodesics on $\left(\mathrm{T}^*\mathrm{H}_1{}^2, \mathrm{g}^\mathrm{C}\right)$ are equal to the geodesic equations obtained by using the Euler Lagrange equations in the second section.

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Boundedly solvable degenerate differential operators for first order

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Abstract

Using the methods of operator theory all boundedly solvable extensions of the minimal operator generated by degenerated type differential-operator expression in the weighted Hilbert space of vector-functions in finite interval in terms of boundary conditions are described. Later on, the structure of spectrum of these type extensions will be investigated.

1. Introduction

The general information on the degenerate differential equations in Banach spaces can be found in book of A. Favini and A. Yagi [1]. The fundamental interest to such equations are motivated by applications in different fields of life sciences.
Recall that an operator

$$A : D(A) \subset H \to H \tag{1.1}$$

in a Hilbert space H is called boundedly solvable, if A is one-to-one

$$AD(A) = H \text{ and } A^{-1} \in L(H). \tag{1.2}$$

In this work using the methods of operator theory, all boundedly solvable extensions of minimal operator generated by linear degenerate type differential-operator expression in the weighted Hilbert space of vector-functions in finite interval in terms of boundary conditions have been defined (see Sec.2). In Section 3 the geometry of spectrum set of these type extensions has been investigated.

Let H be a separable Hilbert space and $\alpha : (0,1) \to (0,\infty),\ \alpha \in C(0,1)$ and $\int\limits_0^1 \frac{dt}{\alpha(t)} < \infty$. In the weighted Hilbert space $L^2_\alpha(H,(0,1))$ of $H-$ valued vector-functions defined at the interval $(0,1)$ consider the following degenerate type differential expression with operator coefficient for first order in a form

$$l(u) = (\alpha u)'(t) + A(t)u(t), \tag{1.3}$$

where:
(1) operator-function $A(\cdot) : (0,1) \to L(H)$ is continuous on the uniform operator topology;
(2) $\frac{\|A(t)\|}{\alpha(t)} \in L^1(0,1)$.
By the standard way the minimal L_0 and maximal L operators corresponding differential expression $l(\cdot)$ in $L^2_\alpha(H,(0,1))$ can be defined [3]. In this case $KerL_0 = \{0\}$ and $\overline{Im(L_0)} \neq L^2_\alpha(H,(0,1))$ (see Sec.3).
In this work, firstly all boundedly solvable extensions of the minimal operator generated by first order linear degenerate type differential-operator expression in the weighted Hilbert space of vector-functions in (0,1) in terms of boundary conditions are described. Later on, the structure of spectrum of these type extensions will be investigated.

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2. Description of boundedly solvable extensions

In this section using the Vishik's methods all boundedly solvable extensions of the minimal operator generated by linear degenerate type differential-operator expression $l(\cdot)$ in weighted Hilbert space $L^2_\alpha(H,(0,1))$ are represented.
Before of all note that using the knowing standard way the minimal M_0 and the maximal M operators generated by differential expression

$$m(v) = (\alpha v)'(t) \tag{2.1}$$

in Hilbert space $L^2_\alpha(H,(0,1))$ can be defined [3].
Later on, by $U(t,s),\ t,s \in [0,1)$ will be defined the family of evolution operators corresponding to the homogeneous differential-operator equation

$$\alpha(t)\frac{\partial}{\partial t}U(t,s)f + A(t)U(t,s)f = 0,\ t,s \in (0,1) \tag{2.2}$$

with boundary condition

$$U(s,s)f = f,\ f \in H. \tag{2.3}$$

The operator $U(t,s),\ t,s \in (0,1)$ is linear continuous and boundedly solvable in H. And also for any $t,s \in (0,1)$ there is the following equation:

$$U^{-1}(t,s) = U(s,t) \tag{2.4}$$

(for detail analysis see [2]).
If introduce the following operator

$$\begin{aligned} &Uz(t) = U(t,0)z(t),\\ &U : L^2_\alpha(H,(0,1)) \to L^2_\alpha(H,(0,1)), \end{aligned}$$

then it is easily to check that

$$\begin{aligned} l(Uz) &= (\alpha Uz)'(t) + A(t)Uz(t)\\ &= U(\alpha z)'(t) + U'_t(\alpha z)(t) + A(t)Uz(t)\\ &= U(\alpha z)'(t) + \left[\alpha(t)U'_t z(t) + A(t)Uz(t)\right]\\ &= U(\alpha z)'(t)\\ &= Um(z). \end{aligned}$$

Therefore it can be obtained

$$U^{-1}l(Uz) = m(z). \tag{2.5}$$

Hence it is clear that if $\widetilde{L}$ is some extension of the minimal operator L_0, that is, $L_0 \subset \widetilde{L} \subset L$, then $U^{-1}L_0U = M_0,\ M_0 \subset U^{-1}\widetilde{L}U = \widetilde{M} \subset M,\ U^{-1}LU = M$.
Now we prove the following assertion.

Theorem 2.1. *$KerL_0 = \{0\}$ and $\overline{Im(L_0)} \neq L^2_\alpha(H,(0,1))$.*

Proof. Consider the following boundary value problem in $L^2_\alpha(H,(0,1))$

$$\begin{aligned} &(\alpha u)'(t) + A(t)u(t) = 0,\ t \in (0,1),\\ &(\alpha u)(0) = (\alpha u)(1) = 0. \end{aligned} \tag{2.6}$$

Then the general solution of above differential equation is in form

$$(\alpha u)(t) = exp\left(-\int_0^t \frac{A(s)}{\alpha(s)}ds\right) f_0,\ f_0 \in H. \tag{2.7}$$

From (2.7) and boundary condition (2.6) we have following equation

$$u(t) = 0,\ t \in (0,1). \tag{2.8}$$

Consequently, following equality $Ker(L_0) = \{0\}$ hold.
On the other hand it is clear that the general solution of following differential equation in $L^2_\alpha(H,(0,1))$

$$-(\alpha v)'(t) + A^*(t)v(t) = 0 \tag{2.9}$$

in form

$$v(t) = \frac{1}{\alpha(t)}exp\left(\int_0^t \frac{A^*(s)}{\alpha(s)}ds\right) g,\ g \in H. \tag{2.10}$$

This means that

$$dimKerL_0^* = \infty. \tag{2.11}$$

So the following inequality is realized

$$\overline{Im(L_0)} \neq L_\alpha^2(H,(0,1)). \tag{2.12}$$

$\square$

Theorem 2.2. *Each solvable extension $\widetilde{L}$ of the minimal operator L_0 in $L_\alpha^2(H,(0,1))$ is generated by the differential-operator expression $l(\cdot)$ with boundary condition*

$$(B+E)(\alpha U^{-1}u)(0) = B(\alpha U^{-1}u)(1), \tag{2.13}$$

where $B \in L(H)$, E is a identity operator in H. The operator B is determined uniquely by the extension $\widetilde{L}$, i.e $\widetilde{L} = L_B$.
On the contrary, the restriction of the maximal operator L to the manifold of vector-functions satisfy the above boundary condition for some bounded operator $B \in L(H)$ is a boundedly solvable extension of the minimal operator L_0 in $L_\alpha^2(H,(0,1))$.

Proof. Firstly, all boundedly solvable extensions $\widetilde{M}$ of the minimal operator L_0 in $L_\alpha^2(H,(0,1))$ in terms of boundary conditions will be described.
Consider the following so-called Cauchy extension M_c,

$$\begin{aligned}
&M_c u = (\alpha u)'(t),\\
&M_c : D(M_c) \subset L_\alpha^2(H,(0,1)) \to L_\alpha^2(H,(0,1)),\\
&D(M_c) = \{u \in D(L) : (\alpha u)(0) = 0\}
\end{aligned}$$

of the minimal operator M_0. It is clear that M_c is a boundedly solvable extension of minimal operator M_0 and

$$\begin{aligned}
&M_c^{-1} f(t) = \frac{1}{\alpha(t)} \int_0^t f(s)ds,\ f \in L_\alpha^2(H,(0,1)),\\
&M_c^{-1} : L_\alpha^2(H,(0,1)) \to L_\alpha^2(H,(0,1)).
\end{aligned}$$

Indeed, for any $f \in L_\alpha^2(H,(0,1))$ we have

$$\begin{aligned}
\left\| \frac{1}{\alpha(t)} \int_0^t f(s)ds \right\|_{L_\alpha^2(H,(0,1))}^2 &= \int_0^1 \alpha(t) \frac{1}{\alpha^2(t)} \left\| \int_0^t f(s)ds \right\|_H^2 dt\\
&\leq \int_0^1 \frac{1}{\alpha(t)} \left(\int_0^t \frac{1}{\sqrt{\alpha(s)}} \sqrt{\alpha(s)} \|f(s)\|_H ds \right)^2 dt\\
&\leq \int_0^1 \frac{dt}{\alpha(t)} \left(\int_0^1 \frac{ds}{\alpha(s)} \right) \left(\int_0^1 \|f(s)\|_H^2 \alpha(s) ds \right)\\
&= \left(\int_0^1 \frac{dt}{\alpha(t)} \right)^2 \|f\|_{L_\alpha^2(H,(0,1))}^2.
\end{aligned}$$

Now assumed that $\widetilde{M}$ is a solvable extension of the minimal operator M_0 in $L_\alpha^2(H,(0,1))$. In this case it is known that the domain of $\widetilde{M}$ can be written as a direct sum

$$D(\widetilde{M}) = D(M_0) \oplus \left(M_c^{-1} + K \right) V, \tag{2.14}$$

where $V = KerM_0^*$, $K \in L(H)$ (see [4], [5]).
It is easily to see that

$$KerM_0^* = \left\{ \frac{1}{\alpha(t)} f :\ f \in H \right\}. \tag{2.15}$$

Therefore each function $u \in D(\widetilde{M})$ can be written in following form

$$u(t) = u_0(t) + M_c^{-1} \left(\frac{1}{\alpha(t)} f \right) + \frac{1}{\alpha(t)} Kf,\ u_0 \in D(M_0),\ f \in H.$$

And from this we have

$$(\alpha u)(t) = (\alpha u_0)(t) + \int_0^t \frac{ds}{\alpha(s)} f + Kf,\ f \in H. \tag{2.16}$$

Hence, following equalities

$$(\alpha u)(0) = Kf,$$
$$(\alpha u)(1) = \left(\int_0^1 \frac{ds}{\alpha(s)} + K \right) f.$$

From these relations it is obtained that

$$\left(\int_0^1 \frac{ds}{\alpha(s)} + K \right) (\alpha u)(0) = K(\alpha u)(1). \tag{2.17}$$

Then the last equality can be written in form

$$(B+E)(\alpha u)(0) = B(\alpha u)(1), \tag{2.18}$$

where

$$B = \left(\int_0^1 \frac{ds}{\alpha(s)} \right)^{-1} K. \tag{2.19}$$

On the other hand note that the uniquenses of the operator $B \in L(H)$ is clear from [4], [5]. Therefore, $\widetilde{M} = M_B$. This completes of necessary part of assertion.
On the contrary, if M_B is a operator generated by $m(\,\cdot\,)$ and boundary condition

$$(B+E)(\alpha u)(0) = B(\alpha u)(1), \tag{2.20}$$

then M_B is boundedly invertible and

$$M_B^{-1} : L^2_\alpha(H,(0,1)) \to L^2_\alpha(H,(0,1)),$$
$$M_B^{-1} f(t) = \frac{1}{\alpha(t)} \int_0^t f(s)ds + B \int_0^1 f(s)ds, \; f \in L^2_\alpha(H,(0,1)).$$

Consequently, assertion of theorem for the boundedly solvable extension of the minimal operator M_0 is true.
The extension $\widetilde{L}$ of the minimal operator L_0 is boundedly solvable in $L^2_\alpha(H,(0,1))$ if and only if the operator $\widetilde{M} = U^{-1}\widetilde{L}U$ is a boundedly solvable extension of the minimal operator M_0 in $L^2_\alpha(H,(0,1))$. Then $u \in D(\widetilde{L})$ if and only if $U^{-1}u \in D(\widetilde{M})$.
Since $\widetilde{M} = M_B$ for some $B \in L(H)$, then we have

$$(B+E)(\alpha U^{-1}u)(0) = B(\alpha U^{-1}u)(1). \tag{2.21}$$

This completes the proof of theorem. □

3. Spectrum of boundedly solvable extensions

In this section the structure of spectrum of boundedly solvable extensions of the minimal operator L_0 in $L^2_\alpha(H,(0,1))$ will be investigated. Firstly, prove the following result.

Theorem 3.1. *If L_B is a boundedly solvable extension of the minimal operator L_0 and $M_B = U^{-1}L_BU$ corresponding boundedly solvable extension of the minimal operator M_0, then it is true $\sigma(L_B) = \sigma(M_B)$.*

Proof. Consider the following problem to spectrum for any boundedly solvable extension L_B in $L^2_\alpha(H,(0,1))$, that is

$$L_B u = \lambda u + f, \; \lambda \in \mathbb{C}, \; f \in L^2_\alpha(H,(0,1)). \tag{3.1}$$

From this it is obtained that

$$(L_B - \lambda E)u = f \text{ or } (UM_BU^{-1} - \lambda E)u = f. \tag{3.2}$$

Then we have

$$U(M_B - \lambda)U^{-1}(u) = f. \tag{3.3}$$

Therefore, the validity of the theorem is clear. □

Now prove the main theorem on the structure of spectrum.

Theorem 3.2. *The spectrum of the boundedly solvable extension L_B of the minimal operator L_0 in $L^2_\alpha(H,(0,1))$ has the form*

$$\sigma(L_B)=\left\{\lambda\in\mathbb{C}:\lambda=\left(\int_0^1\frac{ds}{\alpha(s)}\right)^{-1}\left(\ln\left|\frac{\mu+1}{\mu}\right|+i\arg\left(\frac{\mu+1}{\mu}\right)+2n\pi i\right),\right.$$
$$\left.\mu\in\sigma(B)\backslash\{0,-1\},\ n\in\mathbb{Z}\right\}.$$

Proof. By Theorem 3.1. for this it is sufficiently the investigate the spectrum of the corresponding boundedly solvable extension $M_B=U^{-1}L_BU$ of the minimal operator M_0 in $L^2_\alpha(H,(0,1))$.
Now consider the following problem to spectrum for the extension M_B, that is,

$$M_Bu=\lambda u+f,\ \lambda\in\mathbb{C},\ f\in L^2_\alpha(H,(0,1)). \tag{3.4}$$

Then

$$(\alpha u)'(t)=\lambda u(t)+f(t),\ t\in(0,1) \tag{3.5}$$

with boundary condition

$$(B+E)(\alpha u)(0)=B(\alpha u)(1). \tag{3.6}$$

It is clear that a general solution of the above differential equation has the form

$$u(t;\lambda)=\frac{1}{\alpha(t)}exp\left\{\lambda\int_0^t\frac{ds}{\alpha(s)}\right\}f_0+\frac{1}{\alpha(t)}\int_0^t exp\left\{\lambda\int_s^t\frac{d\tau}{\alpha(\tau)}\right\}f(s)ds,\ f_0\in H. \tag{3.7}$$

From this and boundary condition it is obtained that

$$\left(E+B\left(1-exp\left\{\lambda\int_0^1\frac{ds}{\alpha(s)}\right\}\right)\right)f_0=B\left(\int_0^1 exp\left\{\lambda\int_s^1\frac{d\tau}{\alpha(\tau)}\right\}f(s)ds\right). \tag{3.8}$$

In case when $\lambda_m\int_0^1\frac{ds}{\alpha(s)}=2m\pi i,\ m\in\mathbb{Z}$, from the last relation it is established that

$$f_0^{(m)}=B\left(\int_0^1 exp\left\{\lambda_m\int_s^1\frac{d\tau}{\alpha(\tau)}\right\}f(s)ds\right),\ m\in\mathbb{Z}. \tag{3.9}$$

Consequently, in this case the resolvent operator of M_B is in form

$$R_{\lambda_m}(M_B)f(t)=B\left(\frac{1}{\alpha(t)}exp\left\{\lambda_m\int_0^t\frac{ds}{\alpha(s)}\right\}\int_0^1 exp\left\{\lambda_m\int_s^1\frac{d\tau}{\alpha(\tau)}\right\}f(s)ds\right)$$
$$+\frac{1}{\alpha(t)}\int_0^t exp\left\{\lambda_m\int_s^t\frac{d\tau}{\alpha(\tau)}\right\}f(s)ds,\ m\in\mathbb{Z}.$$

Now assumed that $\lambda\neq 2m\pi i,\ m\in\mathbb{Z}$. Then from the mentioned above equation for $f_0\in H$ we have

$$\left(B-\left(1-exp\left\{\lambda\int_0^1\frac{ds}{\alpha(s)}\right\}\right)^{-1}E\right)f_0$$
$$=\left(1-exp\left\{\lambda\int_0^1\frac{ds}{\alpha(s)}\right\}\right)^{-1}B\left(\int_0^1 exp\left\{\lambda\int_0^1\frac{d\tau}{\alpha(\tau)}\right\}f(s)ds\right),$$
$$f_0\in H,\ f\in L^2_\alpha(H,(0,1)).$$

Then $\lambda\in\sigma(M_B)$ if and only if

$$\mu=\left(1-exp\left\{\lambda\int_0^1\frac{ds}{\alpha(s)}\right\}\right)^{-1}\in\sigma(B). \tag{3.10}$$

In this case since $\mu\neq 0$,

$$exp\left\{\lambda\int_0^1\frac{ds}{\alpha(s)}\right\}=\frac{\mu+1}{\mu},\ \mu\in\sigma(B),\ \mu\neq -1. \tag{3.11}$$

Then

$$\lambda=\left(\int_0^1\frac{ds}{\alpha(s)}\right)^{-1}\left(\ln\left|\frac{\mu+1}{\mu}\right|+i\arg\left(\frac{\mu+1}{\mu}\right)+2n\pi i\right),\ n\in\mathbb{Z}. \tag{3.12}$$

□

Example 3.1. All boundedly solvable extensions L_k of the minimal operator L_0 in $L^2_\alpha(0,1),\ \alpha(t)=t^p,\ p<1,\ 0<t<1$, generated by differential expression

$$l(u)=(t^pu(t))'+a(t)u(t),\ p<1,\ 0<t<1,\ \frac{a(t)}{t^p}\in L^1(0,1) \tag{3.13}$$

are generated by differential expression $l(\cdot)$ and boundary condition

$$(k+1)\left(\alpha U^{-1}u\right)(0)=k\left(\alpha U^{-1}u\right)(1),\ k\in\mathbb{C}, \tag{3.14}$$

where $U(\cdot,\cdot)$ are the corresponding evolution operators. In this case the spectrum $\sigma(L_k)$ of the extension $L_k,\ k\neq 0,-1$ is in form

$$\sigma(L_k)=\left\{\lambda\in\mathbb{C}:\lambda=(1-p)\ln\left|\frac{k+1}{k}\right|+i\arg\left(\frac{k+1}{k}\right)+2n\pi i,\ n\in\mathbb{Z}\right\}.$$

4. Conclusion

It is known that problem on the solvability of the degenerate differential equations with corresponding boundary conditions in finite and infinite regions is main subject in mathematical literature always (for detail informations see [1]).
It is noted that the general form of boundedly solvable extensions of some densely defined closed operator in Hilbert space has been found by M. I. Vishik. In our work using the techniques of mentioned above theory a parametrization of boundedly solvable extensions of the minimal operator generated by degenerate differential-operator expression for first order in the weighted Hilbert space of vector-functions at finite interval is investigated. Lastly, the structure of spectrum of these type extensions is given.
Point out that the general form and spectral analysis of subclasses of differential operators in Banach spaces are main research topics in operator theory.

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